On-shell equivalent transformations for eliminating energy dependence from coupled channel equations with nonlocal linearly energy-dependent potentials

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A general method is developed for constructing on-shell equivalent transformations which eliminate the energy dependence of coupled channel equations with nonlocal linearly energy-dependent potentials. Our derivation shows that such transformations are not unique because of ambiguities of the half-off-shell t matrix compatible with the original Hermitian on-shell equivalent Hamiltonian. A class of such transformations for single and coupled channel cases is explicitly constructed. Our method is applied to a constituent quark-cluster model for nucleon-nucleon scattering.

I. INTRODUCTION

In many cases of the scattering problems involving composite particle systems, microscopic formulations lead to two-cluster coupled channel equations with effective optical potentials which are both energy dependent and nonlocal. Some examples are formalisms including resonating group methods[1] (RGM's) for nuclear reactions, generator coordinate methods[2] (GCM's), extended optical potential models,[3] various integral equation methods, and recently the RGM equations applied to quark-cluster models.

In this paper, we present a general method for constructing explicitly a class of on-shell equivalent transformations which eliminate the energy dependence of two-cluster coupled channel equations with nonlocal linearly energy dependent potentials. In case of the RGM coupled channel equations, the energy dependence of the corresponding potentials indicates the fact that the microscopic channel spaces are not orthogonal to each other.[1] Because of this, and since the Hamiltonian with energy dependent potentials is non-Hermitian, it is desirable to look for on-shell equivalent transformations which transform the non-Hermitian Hamiltonian into an energy independent Hermitian Hamiltonian by eliminating the energy dependence. Such transformations are also very helpful in studying the channel coupling potentials and related phenomena, such as threshold behavior, inelasticities, and resonances in the framework of a familiar Hermitian potential model.

A transformed Hermitian Hamiltonian is needed for generalizing the two-cluster formulation to the three-cluster or multicluster case, since an appropriate continuation of a two-cluster t matrix to negative energies needed for the latter cases can be done uniquely with a Hermitian two-cluster Hamiltonian. Furthermore, investigation of such on-shell equivalent transformations can provide us with a deeper understanding of short-range nodes in wave functions generated from the coupled channel RGM equations with linearly energy dependent effective potentials. The nodes in the wave function at short distances have attracted much attention in the past[4-17] and most recently in the framework of the nonrelativistic quark-cluster coupled-channel model for the nucleon-nucleon scattering problem.[15,18,19]

In Sec. II, after briefly describing the single-channel method proposed previously by one of us (M.O.),[20] we remove some of the constraints used for the construction of the off-shell transformations derived in Ref. 20. We show that for a general case we can construct a class of off-shell transformations which turns out to be infinite in both its continuous and discrete parameters. Transformations of special interest of this class are discussed in more detail. In Sec. III the method of determining off-shell transformations which lead to on-shell equivalent Hermitian Hamiltonians is generalized to coupled channel equations. The role of coupled channels as compared with a simple channel is discussed. In Sec. IV our method is compared with a recently proposed recursion method[21] for the RGM equations. We also discuss the nature of the nodes in the wave function at short distances. A brief summary and concluding remarks are presented in Sec. V.

II. SINGLE CHANNEL CASE

In this section we show how we can remove some of the restrictions imposed in Ref. 20 on the on-shell equivalent transformations which eliminate linear energy dependent potentials.

In Ref. 20 it has been shown that one can explicitly construct a class of off-shell transformations which transform an energy dependent Hamiltonian,

$$(h+EW-E)\chi=0\ , \tag{2.1}$$

to a Schrödinger equation with energy-independent interaction,

$$(\tilde{h}-E)\bar{\chi}=0\ . \tag{2.2}$$

Here, W is a compact operator,

$$h(r,r')=[K_l(r)+U(r)]\delta(r-r')+V(r,r')\ ,$$

and

$$\tilde{h}(r,r')=[K_l(r)+U(r)]\delta(r-r')+\bar{V}(r,r')\ ,$$

$K_l(r)$ being the kinetic energy operator. The potentials $V(r,r')$ and $\bar{V}(r,r')$ are supposed to be Hermitian and compact, or Hilbert-Schmidt operators. The wave functions $\bar{\chi}$ and χ are interrelated by the transformation T,

$$\bar{\chi}=T\chi \ , \tag{2.3}$$

where

$$T=\mathbb{1}_{\rm id}-\sum_i |w_i\rangle\alpha_i\langle w_i| \ , \tag{2.4}$$

(id denotes identity) with

$$\alpha_{i(1)}=1-(1-\gamma_i)^{1/2} \ , \tag{2.5}$$

or

$$\alpha_{i(2)}=1+(1-\gamma_i)^{1/2} \ . \tag{2.6}$$

$|w_i\rangle$ and γ_i are the eigenvectors and eigenvalues of the Hermitian operator W, i.e., $W=\sum |w_i\rangle\gamma_i\langle w_i|$, with

$$\langle w_i|w_j\rangle=\int_0^\infty w_i(r)w_j(r)r^2dr=\delta_{ij} \ .$$

It is seen from Eq. (2.5) that for $\gamma_i>1$ the energy dependent Hamiltonian (2.1) cannot be transformed to an energy-independent Hamiltonian $\tilde{h}=T^{-1}hT^{-1}$.[20]

We now discuss a method for removing the initial restriction[20] that $\gamma_i<1$. We consider now that a finite number of $\gamma_k^*>1$. Note that only a finite number of $\gamma_k^*>1$ is allowed; otherwise the Hilbert-Schmidt norm $||W||_{\rm HS}$ of the operator W will diverge (the asterisk is merely symbolic and does not represent complex conjugation):

$$||W||_{\rm HS}=\left[\sum_i \gamma_i^2\right]^{1/2} \ . \tag{2.7}$$

It is sufficient to demonstrate our method for one $\gamma^*>1$, since the corresponding eigenstates are orthonormal to each other for all γ's. We consider the following Schrödinger equation with an energy-dependent potential:

$$(h+E|u^*\rangle\gamma^*\langle u^*|-E)\bar{\psi}=0 \ . \tag{2.8}$$

Let us now define a transformation

$$U=\mathbb{1}_{\rm id}-|u^*\rangle\beta^*\langle u^*| \ . \tag{2.9}$$

The inverse operator U^{-1} is then given by

$$U^{-1}=\mathbb{1}_{\rm id}+|u^*\rangle\frac{\beta^*}{1-\beta^*}\langle u^*| \ . \tag{2.10}$$

U^{-1} exists for all $\beta^*\neq 1$. Hence we have

$$U^{-1}(h+E|u^*\rangle\gamma^*\langle u^*|-E)U^{-1}=U^{-1}hU^{-1}+E|u^*\rangle\left[\gamma^*+(\gamma^*-1)\frac{\beta^*(2-\beta^*)}{(1-\beta^*)^2}\right]\langle u^*|-E \ . \tag{2.11}$$

We have to determine β^* such that

$$\Gamma\equiv\gamma^*+(\gamma^*-1)\frac{\beta^*(2-\beta^*)}{(1-\beta^*)^2}\leq 1 \ . \tag{2.12}$$

It can be seen for any finite β^* the inequality (2.12) is equivalent to $\gamma^*\leq 1$, but this is in contradiction with our assumption $\gamma^*>1$. This means that for no finite β^* will $\Gamma\leq 1$. However, one observes that

$$\frac{\beta^*(2-\beta^*)}{(1-\beta^*)^2}>-1$$

for all finite β^* and is equal to -1 for $\beta^*\to\infty$. In this limit Γ is independent of γ^* and is $\Gamma=1$. In this limit

$$U^{-1}=\mathbb{1}_{\rm id}-|u^*\rangle\langle u^*| \ , \tag{2.13}$$

and the transformed Hamiltonian reads now

$$U^{-1}hU^{-1}+E|u^*\rangle\langle u^*| \ . \tag{2.14}$$

As it is well known from the Kukulin[22] method, $|u^*\rangle$ is a forbidden or spurious state, which can be present in the wave function with any weight. An on-shell equivalent Hamiltonian which does not depend on the energy is then given[22] by

$$U^{-1}hU^{-1}+\lim_{\lambda\to\infty}\lambda|u^*\rangle\langle u^*| \ . \tag{2.15}$$

The nature of the forbidden state explains why the transformation $U=\mathbb{1}_{\rm id}-|u^*\rangle\beta^*\langle u^*|$ is allowed to diverge for $\beta^*\to\infty$. It is seen from Eq. (2.14) that the corresponding Hamiltonian is well behaved and determines $\bar{\chi}$ uniquely, except for the arbitrary admixture of the redundant state u^*. This property has been studied extensively within the RGM (Ref. 1) and related models.[22–24] An alternative to the energy-independent Hamiltonian (2.15) is the Saito Hamiltonian,[23] given in our context by

$$U^{-1}hU^{-1}-|u^*\rangle\langle u^*|U^{-1}hU^{-1} \ , \tag{2.16}$$

or by its symmetrized form.[23]

We see thus that the original condition $\gamma_i\leq 1$ can be removed for a finite number of eigenvalues $\gamma_k^*>1$, because the corresponding Hamiltonian can be transformed into a Hamiltonian for which $|u_k^*\rangle$ become forbidden states and which can be treated by the Kukulin[22] or Saito[23] method in order to make the Hamiltonian energy independent. Based on this finding, we can generalize the definition of a (Pauli) forbidden state: Any eigenstate of the kernel W with an eigenvalue larger or equal to 1 is a forbidden state with respect to the corresponding wave function.

It is useful to recall at this point the definition of the forbidden or spurious or redundant state. The forbidden states are the relative states between two clusters ϕ_1 and ϕ_2 which give a null norm under the full antisymmetrization. As a result they enter into the RGM—or related equations of the form (2.1) with arbitrary admixture. Therefore they are also called redundant or spurious states. Let u_0 be a forbidden state. Then, we have

$$\langle\phi_1\phi_2u_0|A|\phi_1\phi_2u_0\rangle=0 \ , \tag{2.17}$$

where A denotes the antisymmetrization operator of the entire system. Equation (2.17) can be rewritten as follows

$$\begin{aligned}\langle\phi_1\phi_2 u_0 \mid A \mid \phi_1\phi_2 u_0\rangle &= \langle\phi_1\phi_2 u_0 \mid 1-A' \mid \phi_1\phi_2 u_0\rangle \\ &= \langle u_0 \mid u_0\rangle - \langle u_0 \mid W \mid u_0\rangle \\ &= \langle u_0 \mid u_0\rangle - \sum_i \langle u_0 \mid u_i\rangle \\ &\quad \times \lambda_i \langle u_i \mid u_0\rangle = 0 \, .\end{aligned} \tag{2.18}$$

The last two equalities in Eq. (2.18) have been obtained by integration over the cluster-internal coordinates. The operator W here denotes the so-called RGM norm kernel. Obviously, in order to fulfill the above equation, u_0 has to be one of $\{u_i\}$ with an eigenvalue $\lambda_i = 1$. From Eq. (2.18) it can be seen that the radial wave function χ is not unique because we have

$$A \mid \phi_1\phi_2\chi\rangle = A \mid \phi_1\phi_2(\chi + \alpha u_0)\rangle \tag{2.19}$$

for arbitrary α, i.e., u_0 satisfies Eq. (2.1) irrespective of the energy and does not determine the on-shell properties. Therefore the forbidden state is called the redundant or spurious state.

We wish now to show that all off-shell transformed energy-dependent Hamiltonians of the form (2.1) have the same class of on-shell equivalent energy-independent Hamiltonians. We consider again

$$(h + EW - E)\chi = 0 \, , \tag{2.20}$$

with $W = \sum \mid w_i\rangle\gamma_i\langle w_i \mid$ and $\gamma_i < 1$ with no loss of generality as demonstrated above. We are considering arbitrary off-shell transformation θ of χ. Any such transformation can be represented by

$$\theta = \mathbb{1}_{\text{id}} - \sum_i \mid w_i\rangle\lambda_i\langle w_i \mid \, . \tag{2.21}$$

As a proof, we take an arbitrary $\bar{\chi}$ which is asymptotically equal to χ. Since the eigenstates of W form a complete basis for $L^2(r)$, we can expand $\chi - \bar{\chi}$, which is square integrable, in terms of $\mid w_i\rangle$:[25]

$$\chi - \bar{\chi} = \sum_i \mid w_i\rangle(\langle w_i \mid \chi\rangle - \langle w_i \mid \bar{\chi}\rangle) \tag{2.22a}$$

or

$$\bar{\chi} = \chi - \sum_i \mid w_i\rangle(\langle w_i \mid \chi\rangle - \langle w_i \mid \bar{\chi}\rangle) \, . \tag{2.22b}$$

Using Eq. (2.21) and comparing $\bar{\chi} = \theta\chi$ with Eq. (2.22b), we find

$$\lambda_i = 1 - \frac{\langle w_i \mid \bar{\chi}\rangle}{\langle w_i \mid \chi\rangle} \, . \tag{2.23}$$

Thus given an arbitrary $\bar{\chi}$ at short distances, Eq. (2.23) determines the coefficients of the pertinent transformation. If $\bar{\chi}$ is orthogonal to $\mid w_i\rangle$, then $\mid w_i\rangle$ is a forbidden state and has to be treated separately as mentioned above. If $\langle w_i \mid \bar{\chi}\rangle \neq 0$, then $\lambda_i \neq 1$ and the inverse operator θ^{-1} exists

$$\theta^{-1} = \mathbb{1}_{\text{id}} + \sum_i \mid w_i\rangle \frac{\lambda_i}{1-\lambda_i} \langle w_i \mid \, . \tag{2.24}$$

We now transform Eq. (2.20) and obtain

$$(\theta^{-1} h \theta^{-1} + E\tilde{W} - E)\bar{\chi} = 0 \, , \tag{2.25}$$

where $\bar{\chi} = \theta\chi$ and

$$\tilde{W} = \sum_i \mid w_i\rangle \left[\gamma_i + (\gamma_i - 1)\frac{\lambda_i(2-\lambda_i)}{(1-\lambda_i)^2}\right] \langle w_i \mid \, . \tag{2.26}$$

According to our previous discussions, Eq. (2.25) can be transformed into an energy-independent on-shell equivalent Hamiltonian if

$$\pi_i \equiv \left[\gamma_+ (\gamma_i - 1)\frac{\lambda_i(2-\lambda_i)}{(1-\lambda_i)^2}\right] < 1 \, . \tag{2.27}$$

Note that since $\gamma_i < 1$. π_i will be smaller than 1 only if

$$\lambda_i(2-\lambda_i)/(1-\lambda_i)^2 > -1 \, .$$

This is satisfied for any λ_i. This means that for $\gamma_i < 1$ also $\pi_i < 1$, irrespective of the transformation θ. There is one interesting point about the relation (2.27) with respect to the treatment of the so called almost forbidden states.[24] An almost forbidden state is such a state whose eigenvalue is close to 1. In fact, the forbidden states are fully forbidden states in the oscillator model approximation of the cluster wave functions. Usually the almost forbidden states have been treated on the same footing as the forbidden states, i.e., they have been projected out. However, this might be a dangerous procedure, since for any γ_i we can choose λ_i such as to make π_i as close to 1 as we wish. Therefore, according to the conventional definition, the corresponding state w_i would have to be called an almost forbidden state, although it would not have been identified, as such, within the original Hamiltonian given in Eq. (2.20).

We like to point out that for $\lambda_i = 2$ the transformation θ is self-inverse, i.e., $\theta_i^{-1} = \theta_i$ and $\theta_i^{-1}\theta_i = \mathbb{1}_{\text{id}}$, where $\theta_i = \mathbb{1}_{\text{id}} - \mid w_i\rangle 2\langle w_i \mid$. This is an interesting transformation for which $\langle w_i \mid \chi\rangle = -\langle w_i \mid \bar{\chi}\rangle$, χ and $\bar{\chi}$ being asymptotically identical.

Our final step is the transformation of the Hamiltonian,

$$\theta^{-1} h \theta^{-1} + E \sum \mid w_i\rangle\pi_i\langle w_i \mid \, , \tag{2.28}$$

subject to Eq. (2.27), to an energy independent Hamiltonian,

$$\hat{T}^{-1}\theta^{-1} h \theta^{-1}\hat{T}^{-1} \, , \tag{2.29}$$

where

$$\hat{T} = \hat{T}(\theta) = \mathbb{1}_{\text{id}} - \sum_i \mid w_i\rangle\hat{\alpha}_i\langle w_i \mid \tag{2.30}$$

with two possible solutions for α_i,

$$\hat{\alpha}_{i(1)} = 1 - (1-\pi_i)^{1/2} \tag{2.31a}$$

or

$$\hat{\alpha}_{i(2)} = 1 + (1-\pi_i)^{1/2} \, , \tag{2.31b}$$

where

$$\pi_i = \gamma_i + (\gamma_i - 1)\frac{\lambda_i(2-\lambda_i)}{(1-\lambda_i)^2} \, . \tag{2.32}$$

The overall transformation can be written as

$$\tilde{\chi}=\hat{T}\theta\chi=\hat{T}(\theta)\theta\chi \ . \tag{2.33}$$

By writing $\hat{T}=\hat{T}(\theta)$, we indicate that transformation $\hat{T}$ depends on the transformation θ. The transformation is given explicitly by

$$\hat{T}\theta=\mathbb{1}_{\rm id}-\sum |w_i\rangle\mu_i\langle w_i| \ . \tag{2.34}$$

It is now straightforward to show that

$$\mu_{i(1,2)}=\hat{\alpha}_{i(1,2)}(\lambda_i,\gamma_i)+\lambda_i-\hat{\alpha}_{i(1,2)}(\lambda_i,\gamma_i)\lambda_i \equiv\alpha_{i(1,2)} \ , \tag{2.35}$$

where $\alpha_{i(1,2)}$ is given by Eqs. (2.5) and (2.6), or $\hat{T}(\theta)\theta\equiv T$. This identity completes our proof. We have found that all Hamiltonians $\theta^{-1}h\theta^{-1}+E\tilde{W}$ are on-shell equivalent with the class of energy independent Hamiltonians (2.20). Finally, we question whether the chosen diagonal ansatz for the transformation T in Eq. (2.4) restricts the off-shell transformations which eliminate the energy dependence of the Hamiltonian (2.1) to a subclass. In order to answer this question, we investigate the existence of solutions for a nondiagonal ansatz,

$$T=\mathbb{1}_{\rm id}-\sum_{i,j}^{\infty}|w_i\rangle\alpha_{ij}\langle w_j| \ , \tag{2.36}$$

where at least one nondiagonal element α_{ij} $(j\neq i)$ does not vanish. We consider the operator equation which must be satisfied by T,[20]

$$T^2=\mathbb{1}_{\rm id}-W \ . \tag{2.37}$$

As it is very difficult to solve Eq. (2.37) for the most general ansatz (2.36), here we study some special cases. Consider a rank-2 transformation:

$$T=1-\sum_{i,j}^{2}|w_i\rangle\alpha_{ij}\langle w_j| \ .$$

Then, according to Eq. (2.37), we have to solve

$$-2\times\begin{bmatrix}\alpha&\beta\\ \gamma&\delta\end{bmatrix}+\begin{bmatrix}\alpha&\beta\\ \gamma&\delta\end{bmatrix}\begin{bmatrix}\alpha&\beta\\ \gamma&\delta\end{bmatrix}=-\begin{bmatrix}\lambda_1&0\\ 0&\lambda_2\end{bmatrix} , \tag{2.38}$$

where

$$(\alpha_{ij})=\begin{bmatrix}\alpha&\beta\\ \gamma&\delta\end{bmatrix}$$

or, more explicitly,

$$\begin{aligned}&(\alpha-1)^2+\beta\gamma=1-\lambda_1 \ ,\\ &\beta[(\alpha-1)+(\delta-1)]=0 \ ,\\ &\gamma[(\alpha-1)+(\delta-1)]=0 \ ,\\ &(\delta-1)^2+\beta\gamma=1-\lambda_2 \ .\end{aligned} \tag{2.39}$$

It can be seen that for $\lambda_1\neq\lambda_2$ only diagonal solutions exist:

$$(\alpha_{ij})=\begin{bmatrix}1\pm\sqrt{1-\lambda_1}&0\\ 0&1\pm\sqrt{1-\lambda_2}\end{bmatrix} , \tag{2.40}$$

where the signs for both elements are independent of one another. The nondiagonal solution exists only for the degenerate spectrum of W, i.e., $\lambda_1=\lambda_2=\lambda_0$ in our case:

$$(\alpha_{ij})=\begin{bmatrix}2&a\sqrt{\lambda_0}\\ -\frac{1}{a}\sqrt{\lambda_0}&0\end{bmatrix} , \tag{2.41}$$

where a is any real number not equal to zero. For the degenerate case we thus obtain an infinitely continuous class of off-shell transformations in addition to the discrete ones obtained above. The inverse operator T^{-1} is given by

$$T^{-1}=\mathbb{1}_{\rm id}+\sum_{i,j}^{2}|w_i\rangle x_{ij}\langle w_j| \ ,$$

where

$$(x_{ij})=\begin{bmatrix}\frac{2-\lambda_0}{\lambda_0-1}&\frac{1}{a}\sqrt{\lambda_0}/(\lambda_0-1)\\ -\frac{1}{a}\sqrt{\lambda_0}/(\lambda_0-1)&\frac{\lambda_0}{\lambda_0-1}\end{bmatrix} . \tag{2.42}$$

A similar analysis with similar results also holds for the rank-3 case. Although we have no proof, it can be assumed from the above findings that nondiagonal solutions exist only for the degenerate spectrum of the operator W for any rank. In that case the class of infinite discrete off-shell transformations has to be enlarged by an infinite continuous off-shell transformation of the type (2.41). For the single channel case in realistic applications, a degenerate spectrum of W will not occur. However, this might be the case for coupled channels. (See the next section.)

III. COUPLED CHANNEL CASE

We consider coupled two-body equations of the following form,

$$\sum_j(H_{ij}+EN_{ij})\chi_j=E\chi_i \ , \tag{3.1}$$

where

$$H_{ij}=K_i\delta_{ij}+V_i\delta_{ij}+W_{ij} \ ,$$

and K_i represents the kinetic operator for the ith channel. $W_{ij}^{\dagger}=W_{ji}$ and $N_{ij}=N_{ji}^{\dagger}$ [or, more explicitly, $N_{ij}(r_i,r_j')=N_{ji}(r_j',r_i)$] are the spatial parts of the energy-independent and energy-dependent channel coupling potentials, respectively. For every channel, the relative energy E_i is given by $E_i=E-E_{i1}-E_{i2}$, where E_{i1} and E_{i2} are the bound state energies of the two clusters ϕ_{i1} and ϕ_{i2} in the ith channel. If $E_i>0$, we deal with the case of open channel; if $E_i<0$, the channel is called a closed channel. Formally, a closed channel with fixed energy $E_i>0$ can appear, but in this case it is treated as a distortion.[1] It will become clear in the following that Eq. (3.1) covers both the coupled channel RGM and GCM equations. In case of the RGM, Eq. (3.1) arises from projecting the true microscopic solution Ψ onto the model space (A is the antisymmetrization operator):

$$\psi = P\Psi = \sum_i A\{\phi_{i1}\phi_{i2}\chi_i\} \ . \tag{3.2}$$

Note that, in general, the channel spaces are not orthogonal to each other:

$$\langle A\phi_{i1}\phi_{i2}\chi_i \mid A\phi_{ji}\phi_{j2}\chi_j\rangle \neq 0 \ , \tag{3.3}$$

for $i \neq j$. The use of Eq. (3.2) gives rise to the energy-dependent coupling potentials N_{ij}. However, the potentials N_{ij} in Eq. (3.1) might not be necessarily the ones generated by the RGM prescription, and therefore Eq. (3.1) is more general. We only require that operators N_{ij} are such that they can be approximated by a separable expansion with sufficient accuracy. We also require that there is only a finite number of eigenvalues of the operator $\underline{N} = (N_{ij})$ which are larger or equal to 1.

In the following, it will be shown that there are infinite number of off-shell transformations which transform Eq. (3.1) into the form

$$\sum_j (\tilde{H}_{ij} - \delta_{ij}E)\tilde{\chi}_j = 0 \ , \tag{3.4}$$

i.e., those transformations which eliminate the energy dependence in Eq. (3.1) while preserving all on-shell properties. In view of Eq. (3.4), the elimination of energy dependence is used as a method for the orthogonalization of channel spaces in the case of the RGM. Our main result from the explicit construction of a class of such transformations indicates that the elimination of energy dependence (or orthogonalization of channel spaces) is not unique as far as the behavior of the wave functions at short distances is concerned. The degree of ambiguity is defined by a space of continuous and discrete parameters.

We consider a two coupled channel equation representatively in the following, because the generalization to many channels is then straightforward. The two-channel coupled equations are given as follows:

$$\begin{aligned}
&(T_1 + V_1^l - E)\chi_1 + \int [V_1^{\rm nl}(r_1,r_1') + EN_{11}(r_1,r_1')]\chi_1(r_1')r_1'^2 dr_1' + \int [W_{12}(r_1,r_2') + EN_{12}(r_1,r_2')]\chi_2(r_2')r_2'^2 dr_2' = 0 \ , \\
&(T_2 + V_2^l - E)\chi_2 + \int [V_2^{\rm nl}(r_2,r_2') + EN_{22}(r_2,r_2')]\chi_2(r_2')r_2'^2 dr_2' + \int [W_{21}(r_2,r_1') + EN_{21}(r_2,r_1')]\chi_1(r_1')r_1'^2 dr_1' = 0 \ ,
\end{aligned} \tag{3.5a}$$

or, in matrix notation,

$$\left[\begin{bmatrix} T_1 + V_1^l + V_1^{\rm nl} & W_{12} \\ W_{21} & T_2 + V_2^l + V_2^{\rm nl} \end{bmatrix} + E \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \right] \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = E \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} \ , \tag{3.5b}$$

which we write symbolically as

$$(\underline{H} + E\underline{N} - E)\boldsymbol{\chi} = 0 \ . \tag{3.5c}$$

V_i^l and $V_i^{\rm nl}$ are the local and nonlocal energy independent potentials respectively, for the ith channel. Since $\underline{N}$ is a Hermitian operator in the channel spaces, the following decomposition exists,[24]

$$\underline{N} = \sum_i |\mathbf{u}_i\rangle \gamma_i \langle \mathbf{u}_i| \ , \tag{3.6}$$

where

$$|\mathbf{u}_i\rangle = \begin{bmatrix} |u_{1i}\rangle \\ |u_{2i}\rangle \end{bmatrix} \ ,$$

$$\langle r_j | u_{ji}\rangle = u_{ji}(r_j), \quad j = 1,2 \ ,$$

and

$$\langle \mathbf{u}_i | \mathbf{u}_j\rangle = \langle u_{1i} | u_{1j}\rangle + \langle u_{2i} | u_{2j}\rangle = \delta_{ij} \ .$$

$|\mathbf{u}_i\rangle$ and γ_i are the eigenvectors and eigenvalues of the operator $\underline{N}$, respectively.

For the case in which the coupling potentials vanish, we can divide the eigenvalues γ_i in two groups, γ_{1i} and γ_{2j}, where the first group corresponds to the eigenvalues of N_{11} and the second group corresponds to the eigenvalues of N_{22}, with the respective eigenvector

$$\begin{bmatrix} |u_{1i}\rangle \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ |u_{2j}\rangle \end{bmatrix} \ .$$

Note that for $N_{12} = N_{21} = 0$ it holds that $\langle u_{jl} | u_{jk}\rangle = \delta_{lk}$ for $j = 1,2$ and $|u_{jl}\rangle \neq 0$. In the case of nonvanishing coupling, this is no longer true, and $|u_{jl}\rangle$, $l = 1, \ldots, \infty$, are not orthogonal and not even linearly independent, in general. However, the relation

$$\langle \mathbf{u}_i | \mathbf{u}_k\rangle = \langle u_{1i} | u_{1k}\rangle + \langle u_{2i} | u_{2k}\rangle = \delta_{ik}$$

holds always. Note also that the eigenvalues of $\underline{N}$ might be degenerate.

There is also an ill-conditioned problem when all N_{ij} have the same functional form reflecting linear dependence of the original cluster channels (in this case the W_{ij} naturally have also the same functional form). In this case it is sufficient to solve the eigenvalue problem for one kernel N_{ii}. The eigenvectors of $\underline{N}$ are then given by

$$|\mathbf{u}_\nu\rangle = \begin{bmatrix} |u_\nu\rangle \\ |u_\nu\rangle \end{bmatrix}$$

with eigenvalues $2\gamma_\nu$. Here, γ_ν and $|u_\nu\rangle$ are eigenvalues and eigenvectors of the operator N_{ii}.

We now construct a transformation $\underline{U}$:

$$\underline{U}\boldsymbol{\chi} = \bar{\boldsymbol{\chi}} = \begin{bmatrix} \bar{\chi}_1 \\ \bar{\chi}_2 \end{bmatrix} \ , \tag{3.7}$$

such that, for large r_i,

$$\begin{bmatrix}\chi_1(r_1)\\ \chi_2(r_2)\end{bmatrix}-\begin{bmatrix}\bar{\chi}_1(r_1)\\ \bar{\chi}_2(r_2)\end{bmatrix}\rightarrow\begin{bmatrix}0\\0\end{bmatrix} \tag{3.8}$$

holds. As done in Sec. II, the transformation can be represented as follows:

$$\underline{U}=\mathbb{1}_{\mathrm{id}}-\sum|\mathbf{u}_\nu\rangle\alpha_\nu\langle\mathbf{u}_\nu| \ . \tag{3.9}$$

From Eqs. (3.7) and (3.9), one finds

$$\alpha_\nu=1-\frac{\langle\mathbf{u}_\nu|\bar{\chi}\rangle}{\langle\mathbf{u}_\nu|\chi\rangle}=1-\frac{\langle u_{1\nu}|\bar{\chi}_1\rangle+\langle u_{2\nu}|\bar{\chi}_2\rangle}{\langle u_{1\nu}|\chi_1\rangle+\langle u_{2\nu}|\chi_2\rangle} \ . \tag{3.10}$$

The inverse operator $\underline{U}^{-1}$ exists for $\alpha_\nu\neq1$ and is given by

$$\underline{U}^{-1}=\mathbb{1}_{\mathrm{id}}+\sum|\mathbf{u}_\nu\rangle\beta_\nu\langle\mathbf{u}_\nu| \ , \tag{3.11}$$

with

$$\beta_\nu=\alpha_\nu(1-\alpha_\nu)^{-1} \ . \tag{3.12}$$

Equations (3.5) are transformed under $\underline{U}$ into

$$\underline{U}^{-1}(\underline{H}+E\underline{N}-E)\underline{U}^{-1}\bar{\chi}=0 \ . \tag{3.13}$$

In order to eliminate the energy-dependent potential, the following operator equation has to be satisfied,

$$\underline{U}^{-1}\underline{U}^{-1}-\underline{U}^{-1}\underline{N}\underline{U}^{-1}=\mathbb{1}_{\mathrm{id}} \ , \tag{3.14}$$

in complete analogy to the single channel case.[20] The last equation is satisfied if

$$\beta_\nu^{(\pm)}=\frac{\pm1-(1-\gamma_\nu)^{1/2}}{(1-\gamma_\nu)^{1/2}} \ . \tag{3.15}$$

For the coefficients α_ν, we obtain two solutions:

$$\alpha_\nu^{(+)}=1+(1-\gamma_\nu)^{1/2} \tag{3.16a}$$

and

$$\alpha_\nu^{(-)}=1-(1-\gamma_\nu)^{1/2} \ . \tag{3.16b}$$

Because of the above two solutions for any index ν, we have constructed an infinite class of off-shell transformations which lead to an infinite number of energy-independent (or channel orthogonalized) on-shell equivalent Hamiltonians. The transformed Schrödinger equation reads now

$$(\tilde{\underline{H}}-E)\bar{\chi}=0 \ , \tag{3.17}$$

where $\tilde{\underline{H}}=\underline{H}+\underline{H}_2$, $\underline{H}$ being the original Hamiltonian without $E\underline{N}$, and $\underline{H}_2$ is given by

$$\underline{H}_2=\sum_{l\kappa}|\mathbf{u}_l\rangle\langle\mathbf{u}_l|\underline{H}|\mathbf{u}_\kappa\rangle\lambda_{l\kappa}\langle\mathbf{u}_\kappa| \ , \tag{3.18}$$

with

$$\lambda_{\kappa l}=\frac{(\pm1)_\kappa(\pm1)_l-(1-\gamma_l)^{1/2}(1-\gamma_\kappa)^{1/2}}{(1-\gamma_l)^{1/2}(1-\gamma_\kappa)^{1/2}} \ . \tag{3.19}$$

The symbol $(\pm1)_l$ means that the sign for the different indices l and κ can be chosen independently from each other. This means that for every $\lambda_{\kappa l}$, there are four solutions, and $\lambda_{\kappa l}=\lambda_{l\kappa}$ holds. The diagonal elements λ_{ii} have naturally only two solutions, because for $(\pm1)_l$ and $(\pm1)_\kappa$ the same sign has to be chosen. If we denote the four solutions of $\lambda_{\kappa l}$, $\kappa\neq l$, with $\lambda_{\kappa l}^{(++)}$, $\lambda_{\kappa l}^{(+-)}$, $\lambda_{\kappa l}^{(-+)}$, and $\lambda_{\kappa l}^{(--)}$, we can write for the diagonal elements $\lambda_{ii}^{(++)}$ and $\lambda_{ii}^{(--)}$ the explicit formula

$$\lambda_{ii}^{(++)}=\frac{\gamma_i}{1-\gamma_i}=\lambda_{ii}^{(--)} \ . \tag{3.20}$$

Similarly, the formula for $\lambda_{l\kappa}^{(++)}$, $\lambda_{l\kappa}^{(+-)}$, $\lambda_{l\kappa}^{(-+)}$, and $\lambda_{l\kappa}^{(--)}$ can be worked out. The matrix element $\langle\mathbf{u}_l|\underline{H}|\mathbf{u}_\kappa\rangle$ is given by

$$\langle\mathbf{u}_l|\underline{H}|\mathbf{u}_\kappa\rangle=\langle u_{1l}|T_1+V_1^l+V_1^{\mathrm{nl}}|u_{1\kappa}\rangle+\langle u_{2l}|W_{21}|u_{1\kappa}\rangle+\langle u_{1l}|W_{12}|u_{2\kappa}\rangle+\langle u_{2l}|T_2+V_2^l+V_2^{\mathrm{nl}}|u_{2\kappa}\rangle \ . \tag{3.21}$$

Since the dyadic product $|\mathbf{u}_l\rangle\langle\mathbf{u}_\kappa|$ can be written in a matrix form

$$|\mathbf{u}_l\rangle\langle\mathbf{u}_\kappa|=\begin{bmatrix}|u_{1l}\rangle\langle u_{1\kappa}| & |u_{1l}\rangle\langle u_{2\kappa}|\\ |u_{2l}\rangle\langle u_{1\kappa}| & |u_{2l}\rangle\langle u_{2\kappa}|\end{bmatrix} \ , \tag{3.22}$$

one can see from Eqs. (3.18), (3.21), and (3.22) that the direct terms of the Hamiltonian $\tilde{\underline{H}}$ depend on the coupling terms of $\underline{H}$. The coupling terms of $\tilde{\underline{H}}$ depend on all direct terms of the Hamiltonian $\underline{H}$. As an example, we write down the complete noncoupling channel potential $\tilde{V}_1$ for channel 1, of the energy independent Hamiltonian:

$$\tilde{V}_1=V_1^l+V_1^{\mathrm{nl}}+\sum_{l,\kappa}\lambda_{l\kappa}|u_{1l}\rangle(\langle u_{1l}|T_1+V_1^l+V_1^{\mathrm{nl}}|u_{1\kappa}\rangle+\langle u_{2l}|W_{21}|u_{1\kappa}\rangle$$
$$+\langle u_{1l}|W_{12}|u_{2\kappa}\rangle+\langle u_{2l}|T_2+V_2^l+V_l^{\mathrm{nl}}|u_{2\kappa}\rangle)\langle u_{1\kappa}| \ . \tag{3.23}$$

A similar formula can be derived using the above equations for the coupling potential V_{12}. We like to stress that, in practice, we do not need to work out the off-shell equivalent Hamiltonian, because its solution $\bar{\chi}$ can be obtained easily be integration from Eqs. (3.7) and (3.9) once the original solution χ is known. The above formulae, however, allow one to study how the off-shell transformation reduces or enhances the strength of the direct and coupling potentials.

In analogy to the results of Sec. II due to the identical algebraic structure, we can repeat the proof for the case of coupled channels, namely that all on-shell equivalent

energy-dependent Hamiltonians of the form (3.1) have the same class of on-shell equivalent energy-independent Hamiltonians. Our previous findings concerning nondiagonal solutions for the off-shell transformation matrix apply directly to the coupled channel problem.

The forbidden states $|\mathbf{u}_i\rangle$ with $\gamma_i=1$ $(i=1,2,\ldots,n)$ can be treated by the Saito[24] or Kukulin[22] method for coupled channel equations. One can see that a forbidden state for a single channel k ($N_{kk}=\sum_i |u_{ki}\rangle\lambda_{ki}\langle u_{ki}|$ with $\gamma_1=\lambda_{k1}=1$), $|u_{k1}\rangle$ (which is the kth component of $|\mathbf{u}_1\rangle$) becomes a forbidden state even for the many channel case, since

$$N_{k'k}|u_{k1}\rangle=0 \quad \text{for } k'\neq k\ , \tag{3.24}$$

because $A|\phi_{k1}\phi_{k2}u_{k1}\rangle=0$ and thus also

$$\langle\phi_{k'1}\phi_{k'2}u_{k'j}|A|\phi_{k1}\phi_{k2}u_{k1}\rangle=0$$

for arbitrary j. This has some interesting consequences. If there is a forbidden state in a single channel, then the orthogonality condition to the wave function $\bar{\chi}_k$ within an energy-independent Hamiltonian formulation will generate an energy-independent node in the relative wave function $\bar{\chi}_k$. Because the corresponding eigenvalue is equal to 1, there is no off-shell transformation which can remove this node. Since the forbidden state will remain a forbidden state even in the coupled channel case [see Eq. (3.24)], this means that the energy independent node will be present in this relative wave function even if coupled to other channels. This holds only for energy-dependent potential $\underline{N}$, which is consistent with RGM derivation.

If, however, there is no energy independent node in the single channel, the coupled-channel system may or may not introduce nodes in the wave functions, depending on the off-shell transformation chosen for orthogonalization of the channel spaces described above. The generalization of our derivation to many coupled channel equations can be read off directly from the above derivation by extending the matrix notation.

As a summary of Sec. III, we now list the advantages of this method.

(1) We have formulated the off-shell transformations directly in terms of the eigenvalues and eigenvectors of the spatial part of the energy dependent potential and in terms of additional parameters given in Eq. (2.41). Thus, in order to obtain a transformed wave function at short distances, it is not necessary to solve the entire system of coupled equations, but only to apply the transformation operator to the previously found solution.

(2) We can study explicitly which off-shell extensions of the relative wave function χ are allowed in order to be a solution of an energy-independent on-shell equivalent Hamiltonian. This constitutes a new method of studying possible half-off-shell extensions of the t matrix once the on-shell behavior is given within the framework of a Hermitian potential theory.

(3) The structure of the transformed direct and coupling potentials can be studied analytically, thus facilitating the investigation of the role of coupling for threshold and resonance phenomena.

A serious limitation of the method is the numerical capability of the computer to solve the eigenvalue problem for general $\underline{N}$. If the eigensolutions of $\underline{N}$ are known analytically,[26] as in some cases, the method is easily applied to a large number of channels. For numerical solutions it will be generally restricted to only a few channels.

IV. APPLICATIONS AND COMPARISON

In this section we discuss our method in some detail in the context of RGM equations and their recent applications to nucleon-nucleon scattering within a nonrelativistic quark model. We start with a brief derivation of RGM coupled equations[1] for two-cluster channels. The RGM ansatz for the trial wave function is

$$\psi=A\sum_i^n \phi_{i1}\phi_{i2}\chi_i=\sum_i A_i\phi_{i1}\phi_{i2}\chi_i\ , \tag{4.1}$$

where A_i denotes the antisymmetrization operator between the constituents of the clusters ϕ_{i1} and ϕ_{i2}, and the cluster functions ϕ_{i1} and ϕ_{i2} are supposed to be already antisymmetrized. The RGM equations are obtained by projecting the microscopic Hamiltonian H onto the space of relative wave functions χ_i:

$$\langle\delta\psi|H-E|\psi\rangle=0\ , \tag{4.2}$$

where

$$\delta\psi=\sum A_i\phi_{i1}\phi_{i2}\delta\chi_i\ , \tag{4.3}$$

ϕ_{i1} and ϕ_{i2} being kept fixed. Equation (4.2) can be written more explicitly in the following way,

$$\sum_{j=1}^n dr_j'\langle A\phi_{i1}\phi_{i2}r_i|H-E|A\phi_{j1}\phi_{j2}r_j'\rangle\langle r_j'|\chi_j\rangle=0\ , \tag{4.4}$$

for $i=1,2,\ldots,n$. Evaluating

$$\langle A|\phi_{i1}\phi_{i2}r_i|H-E|A\phi_{j1}\phi_{j2}r_j\rangle$$

by integrating over the internal coordinates of cluster coordinates, we obtain

$$\sum_{j=1}^n (H_{ij}-E\bar{N}_{ij})\chi_j=0\ , \tag{4.5}$$

where

$$\begin{aligned}\bar{N}_{ij}(r_i,r_j')&\equiv\langle\phi_{i1}\phi_{i2}r_i|A|\phi_{j1}\phi_{j2}r_j'\rangle\\&=\delta(r_i-r_i')\delta_{ij}-N_{ij}(r_i,r_j')\ ,\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}H_{ij}(r_i,r_j')&\equiv\langle\phi_{i1}\phi_{i2}r_i|AH|\phi_{j1}\phi_{j2}r_j\rangle\\&=T_i(r_i)\delta(r_i-r_i')\delta_{ij}\\&\quad+V_i(r_i,r_i')\delta_{ij}+W_{ij}(r_i,r_j')\ ,\end{aligned} \tag{4.7}$$

$T_i(r_i)$ denoting the kinetic energy operator.

The relative motion energy $E_{\text{rel},i}\equiv E_i$ is obtained by subtraction of the cluster eigenenergies [or, in practice, cluster expectation values $\langle\phi_{ij}|H^{\text{int}}_{(ij)}|\phi_{ij}\rangle$ $(i=1,\ldots,n:\ j=1,2)$] from the total energy E:

$$E_{\text{rel},i}=E_i=E-\langle\phi_{i1}\,|\,H_{i1}^{\text{int}}\,|\,\phi_{i1}\rangle-\langle\phi_{i2}\,|\,H_{i2}^{\text{int}}\,|\,\phi_{i2}\rangle\ , \tag{4.8}$$

where the internal Hamiltonians, $H_{i,j}^{\text{int}}$ $(j=1,2)$, are defined by the decomposition of the microscopic Hamiltonian H, $H=H_{i1}^{\text{int}}+H_{i2}^{\text{int}}+H_{i12}^{\text{rel}}$.[1] RGM coupled channel equations are the form of Eq. (3.1), and therefore our methods described in Secs. II and III also apply to the RGM case. Note, however, that in the case of the RGM the eigenvalues of the (norm) operator $\underline{N}$ cannot be larger than 1, because of the projection properties of the antisymmetrization operator A, $A^2=A$.

There is a controversy[19,27,28] as to the existence of a node in the N-N relative wave function as calculated on the basis of the nonrelativistic constituent quark model with the RGM method.[15,18] By analogy with a short-range repulsion between two α clusters, the short-range repulsion in the quark model N-N potential was expected to originate from the Pauli exclusion principle between quarks.[4–10] However, it was pointed out by Faessler *et al.*[18] that the situation in the quark model is slightly different from the case of α clusters because of the color degree of freedom. It has been argued that the totally symmetric [6] state would be rather unfavored compared to the symmetry [42] state, because of the spin and color dependent structure of the quark-quark interactions. If the orbital symmetry is [42], then the relative wave function has a node which would simulate a hard core. Recently, it has been found that under certain circumstances one can construct explicitly on-shell equivalent hard cores and structural cores.[20] The corresponding wave functions have different off-shell behavior, but have the common property that they disappear at some r_c which can be determined explicitly by Bauhoff's[29,30] method rather than by a fitting procedure of the corresponding phase shifts.[18] Note, however, that the on shell equivalence between a hard core potential with a core radius r_c and a structural core still persist even in the absence of a forbidden or almost forbidden state, or, in other words, even in the case when the relative wave function does not display an energy independent (or almost energy independent) node.[20,24]

Faessler *et al.*[18] have shown that a projection onto the [6] orbital symmetry from the two-nucleon system automatically introduces the so-called hidden color state.[13] In RGM calculations by Faessler *et al.*[18] including NN, $\Delta\Delta$, and CC (a hidden color channel), the relative motion function of the N-N channel had a node in the s wave at approximately $r_c=0.5$ fm. In a recent paper, Spitz and Schmid[19,27,28] pointed out that this node is due to a particular off-shell behavior of the wave function χ_{NN}. Applying their method,[21] in which the $\Delta\Delta$ and CC channels are orthogonal (in the sense of RGM overlap for different channels) corrections to the NN channel, i.e.,

$$\langle\phi_{\alpha 1}\phi_{\alpha 2}\chi_\alpha\,|\,A\,|\,\phi_{\text{N}1}\phi_{\text{N}2}\chi_{\text{NN}}\rangle=0\ , \tag{4.9}$$

$\alpha=\Delta\Delta$ and CC, they have found that this node is no longer present. Of course, in both formulations the asymptotic behavior is identical. It turns out[19] that the relative wave function χ_{NN} subject to Eq. (4.9) is similar to that obtained from a single-channel approximation. Spitz and Schmid conclude that, in the one- and three-channel approximations, the six-quark resonating group method calculations with energy-independent potentials do not support the existence of a short distance node in the N-N relative wave function, χ_{NN}. In view of our new results that there are an infinite number of transformations which orthogonalize channel spaces, their conclusion appears to be premature. In the single channel case, and in the coupled channel case as well, *even* with energy-independent potentials (or, in the case of the RGM, for orthogonalized channel spaces), the behavior of the relative wave function at short distances is not unique. Since the method given in Ref. 21 leads to only one specific off-shell extension, no firm conclusion can be reached.

The question of the existence of a node in the relative wave function is a matter of the off-shell ambiguity of the given Hermitian on-shell equivalent Hamiltonian. Furthermore, there is no compelling reason for emphasizing on energy-independent formulation of the Hamiltonian, because the solutions of both the energy-dependent and energy-independent Hamiltonians are on-shell equivalent and the off-shell behavior does not correspond directly to physical observables. From a theoretical point of view, the investigation of the off-shell behavior could be justified if the quark-quark forces would be known both on shell and off shell, and the nucleon cluster functions are described more realistically and not simply by Gaussian functions. The fact that the calculations of Faessler *et al.*[18] display a node, while those of Spitz and Schmid[19] do not, shows that appropriate off-shell transformations can introduce or remove a node in the wave function at short distances, reflecting our finding that the orthogonalization is not unique due to the off-shell ambiguity. There is only one special case in which the energy-independent node in the wave function can be discussed consistently within the framework of the RGM. This case applies to the situation in which, for the single channel case, a forbidden state exists. This forbidden state remains forbidden even in the coupled channel case, as shown in Sec. III. The orthogonality of the relative wave functions. The fact that the calculations of Faessler *et al.*[18] display a node, while those of Spitz and Schmid[19] do not, shows that appropriate off-shell transformations can introduce or remove a node in the wave function at

It has been suggested by Spitz and Schmid[19,27,28] that the node in the s-wave function of the deuteron can be investigated within a three-nucleon calculation. However, it is most unlikely that such a calculation will be conclusive since the different behavior of χ_{NN} determines only the half-off-shell behavior of the corresponding t matrix. The Faddeev equation, however, requires the knowledge of the fully-off-shell behavior, which includes the structure of the left hand cuts of the t matrix. Using different potentials, we are not only changing the half-off-shell behavior of the t matrix, but also the left-hand structure of the t matrix. Therefore, two different families of potentials which introduce a node into the wave function are likely to lead to conflicting predictions for a three-nucleon system,[31] and the Faddeev calculation can only lead to inconclusive results.

Finally, we comment on the method devised by Schmid and Spitz[21] to orthogonalize RGM channel spaces. Their method leads to only one specific off-shell transformation which eliminates the energy dependence. It consists of recursive elimination of the overlaps between a specific channel i and the remaining channels in terms of the eigenvalues and eigenvectors of the operator N_{ii}. This method is much more cumbersome than ours, but it may have some numerical advantages compared to our method in cases in which the number of channels is fairly large, the operator $\underline{N}$ has no analytical solutions, and the overlaps of only one particular channel are of interest.

V. SUMMARY

The elimination method of linearly energy dependent potentials given in Ref. 20 has been extended so that a class of half-off-shell extensions can be explicitly constructed for a large class of potential operators. The extended method is exact and straightforward to apply. Our method provides a systematic way of studying how large a half-off-shell variation of a t matrix can be. The class of allowed off-shell transformations is given in terms of an infinite number of parameters which can have discrete and continuous values. There also exists an infinite discrete subclass of these transformations which leave the energy dependent part of the original Hamiltonian unaltered. This might be helpful in studying the properties of RGM potentials while preserving the functional form of the normalization kernel.

The methods derived for the single channel case have been generalized to the coupled channel two-body equations. The method for the coupled channel case retains the same features as in the single channel case. The new coupling potentials after the transformation can be easily analyzed, because of their transparent algebraic structure. The constraints on the allowed off-shell transformations are determined explicitly. The role of the forbidden state in the single channel and coupled channel cases has been investigated and clarified.

We have employed our method to clarify the recent controversy as to the existence and role of a node in the nucleon-nucleon relative wave function at short distances obtained from RGM equations within a nonrelativistic quark model. We have shown that neither the RGM equations nor their interpretation can provide a unique conclusion or answer regarding the problem of nodes in the N-N quark model relative wave function.

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