Higher order perturbation treatment of three-nucleon forces in the Faddeev equations

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(Received 21 February 1986)

The modification of the Faddeev equations including three-nucleon forces is presented in a perturbational approach for arbitrary orders in the three-nucleon force. We report our results for the triton bound state energy using the Reid-soft-core and Paris potential and the Tucson-Melbourne two-pion-exchange three-body force. Furthermore, we show that only a small number of the N^2 channel to channel contributions are important for the energetic shift caused by the three-nucleon potential.

I. INTRODUCTION

The inclusion of three-nucleon forces has gained much attention in explaining the difference between the experimental binding energy of the triton (−8.48 MeV) (Ref. 1) and the theoretical value using realistic two-body potentials which ranges between about −7.0 and −7.5 MeV (Ref. 2). Since the two-pion-exchange three-nucleon potential (2PE-3BP) should be most important among the three-body forces because of its long range, we use this potential in a first attempt to explain the aforementioned discrepancy. The basic ingredient of the 2PE-3BP is the π-N scattering amplitude which is needed for off-shell pions (Figs. 1 and 2). We use the description of the π-N scattering amplitude developed by Coon *et al.*[3] which uses current algebra and PCAC and fulfills the soft pion theorems by construction. This procedure has its main advantage in being almost completely model independent, whereas the model making approach[4] suffers from uncertainties, such as, for instance, the ambiguity of the off-shell Δ propagator.[5]

Since first calculations using three-nucleon forces in first order perturbation theory gave results[6] which are small compared to the expectation value of realistic two-body forces,[7] one could expect that first order results represent the main part of the total energetic shift caused by the three-body force. Unfortunately, this is not the case, as recent nonperturbational treatments[8] of the three-nucleon force in the Faddeev scheme have shown. In fact, first order results make up only about half of the total energetic shift. Therefore we extended our method of incorporating three-nucleon potentials into the Faddeev equations[9] to arbitrary order.

In Sec. II of this paper we describe our perturbation theory with three-body forces in the Faddeev formalism and discuss the solvability of the resulting set of integral equations. In Sec. III we present some details of the numerical treatment of these equations. Finally, Sec. IV contains our numerical results and a comparison with the results of other groups.

II. PERTURBATION THEORY OF THREE-BODY FORCES WITHIN THE FADDEEV SCHEME

The standard method of solving the quantum mechanical three body problem is the solution of the Faddeev equations,[10] originally developed for two-body forces only. There exists a variety of ways to include three-body potentials in this scheme.[11] The basic assumption for our way is the decomposition of the three-body potential W into three parts,

$$W=\sum_{i=1}^{3} W_i \ , \tag{1}$$

which appears naturally for the 2PE-3BP we used. With $G_0 \equiv (E-H_0)^{-1}$, the free resolvent operator, and the pair interactions $V_i = V_{jk}$ $(j,k \neq i)$, the Schrödinger equation for the three-nucleon wave function Ψ in integral form reads

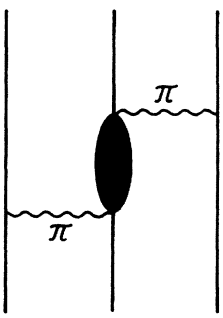

FIG. 1. The two-pion exchange three-nucleon force.

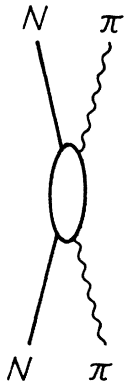

FIG. 2. The pion-nucleon scattering amplitude.

$$\Psi = G_0 \sum_{i=1}^{3} (V_i + W_i)\Psi \ . \tag{2}$$

The usual decomposition of the total wave function Ψ splits Ψ into three Faddeev components ψ_i with

$$\psi_i = G_0(V_i + W_i)\Psi \tag{3}$$

and

$$\Psi = \sum_{i=1}^{3} \psi_i \ . \tag{4}$$

This leads to the modified form of the Faddeev equations used in[12]

$$\psi_i = G_0 T_i \sum_{j \neq i} \psi_j \ . \tag{5}$$

Here the three-body T matrices obey the following Lippmann-Schwinger equations,

$$T_i = (V_i + W_i) + (V_i + W_i) G_0 T_i \ , \tag{6}$$

which can be rewritten by use of the Lippmann-Schwinger equations for the two-body t matrices,

$$t_i = V_i + V_i G_0 t_i \ , \tag{7}$$

to give

$$T_i = t_i + (1 + t_i G_0) W_i (1 + G_0 T_i) \ . \tag{8}$$

This is obviously suitable for a perturbational treatment in W. Instead of this procedure, which leads to problems with the storage requirements to be discussed later, we prefer to introduce a fourth Faddeev component:

$$\begin{aligned} \Psi &= G_0 \sum_{i=1}^{3} V_i \Psi + G_0 W \Psi \\ &\equiv \sum_{i=1}^{3} \psi_i + \Psi_4 \ . \end{aligned} \tag{9}$$

Taking into account Eq. (7) and the Lippmann-Schwinger equations for the scattering by W alone,

$$T_4 = W + W G_0 T_4 \ , \tag{10}$$

we end up with the following set of coupled integral equations:

$$\psi_i = G_0 t_i (P\psi_i + \Psi_4) \ , \tag{11a}$$

$$\Psi_4 = G_0 T_4 (1 + P)\psi_i \ . \tag{11b}$$

Here the permutation operator P, for instance, related to the choice of subsystem 1, is

$$P = P_{12} P_{23} + P_{13} P_{23} \ . \tag{12}$$

Because of the particles' identity, we will restrict our attention to one subsystem, leaving the indices i behind. In this way the modified set (11) reads

$$\psi = G_0 t (P\psi + \Psi_4) \ , \tag{13a}$$

$$\Psi_4 = G_0 T_4 (1 + P)\psi \ . \tag{13b}$$

We want to perform a perturbational expansion around the unperturbed solution ($W = 0$) of the Faddeev equations $\psi^{(0)}$ with energy $E^{(0)}$,

$$\psi^{(0)}(E^{(0)}) = G_0(E^{(0)}) t(E^{(0)}) P \psi^{(0)}(E^{(0)}) \ . \tag{14}$$

We insert Eq. (13b) into (13a) and indicate the energy dependence explicitly,

$$\psi(E) = G_0(E) t(E) [P + G_0(E) T_4(E)(1 + P)] \psi(E) \ . \tag{15}$$

To make the notations more readable, we define the following abbreviations:

$$\psi^{(0)}(E^{(0)}) \equiv \psi^{(0)} \ , \tag{16a}$$

$$G_0(E^{(0)}) \equiv G_0 \ , \tag{16b}$$

$$t(E^{(0)}) \equiv t_0 \ . \tag{16c}$$

For our perturbational ansatz, we write

$$E \equiv E^{(0)} + \Delta E \ , \tag{17a}$$

$$\psi \equiv \psi(E) \equiv \psi^{(0)} + \psi' \ . \tag{17b}$$

The perturbational expansion of the free resolvent G_0 and the two-body t matrix around the unperturbed energy $E^{(0)}$ is

$$G_0(E^{(0)} + \Delta E) = G_0 - G_0 \Delta E G_0 + G_0 \Delta E G_0 \Delta E G_0 \mp \cdots \ , \tag{18}$$

$$\begin{aligned} t(E^{(0)} + \Delta E) = t_0 &- \Delta E t_0 G_0 G_0 t_0 + (\Delta E)^2 \\ &\times t_0 G_0 (G_0 + G_0 t_0 G_0) G_0 t_0 \\ &- (\Delta E)^3 t_0 G_0 (G_0 + G_0 t_0 G_0)^2 G_0 t_0 \pm \cdots \ , \end{aligned} \tag{19}$$

which leads to the product's expansion occurring in (15):

$$\begin{aligned} &G_0(E^{(0)} + \Delta E) t(E^{(0)} + \Delta E) \\ &\quad = G_0 t_0 - \Delta E (G_0 + G_0 t_0 G_0) G_0 t_0 \\ &\qquad + (\Delta E)^2 (G_0 + G_0 t_0 G_0)^2 G_0 t_0 \mp \cdots \ . \end{aligned} \tag{20}$$

For the three-body T matrix T_4 the equivalent procedure leads to

$$\begin{aligned} &G_0(E^{(0)} + \Delta E) T_4(E^{(0)} + \Delta E) \\ &\quad = 0 + G_0 W + G_0 (W - \Delta E) G_0 W \\ &\qquad + G_0 [(W - \Delta E) G_0]^2 W + \cdots \ . \end{aligned} \tag{21}$$

It is convenient to introduce the abbreviations

$$\epsilon \equiv -\Delta E (G_0 + G_0 t_0 G_0) \ , \tag{22a}$$

$$\omega \equiv (W - \Delta E) G_0 \ . \tag{22b}$$

In these terms the fourfold product of energy dependent operators occurring in (15) an be rewritten as

$$G_0(E) t(E) G_0(E) T_4(E) = \sum_{m \geq 0} \sum_{n=0}^{m} \epsilon^n G_0 t_0 G_0 \omega^{m-n} W \ . \tag{23}$$

Now we are able to write exact but, up to the moment, still formal equations for our basic equation (15). To do this we expand the perturbed part of the Faddeev component ψ' corresponding to powers of W,

$$\psi' = \sum_{n \geq 1} \psi^{(n)} , \tag{24}$$

and obtain

$$\sum_{m \geq 0} \psi^{(m)} = \sum_{m \geq 0} \sum_{n=0}^{m} \epsilon^n G_0 t_0 P \psi^{(m-n)} + \sum_{m \geq 0} \sum_{l=0}^{n} \sum_{n=0}^{l} \epsilon^n G_0 t_0 G_0 \omega^{l-n} W (1+P) \psi^{(m-l)} . \tag{25}$$

This equation can be successively reduced, up to Nth order in W and ΔE, respectively, to the following equations, ignoring errors of higher orders,

$$\sum_{m=0}^{N} \psi^{(m)} = G_0 t_0 P \sum_{m=0}^{N} \psi^{(m)} + \epsilon \sum_{m=0}^{N-1} \psi^{(m)} + G_0 t_0 G_0 \sum_{m=0}^{N-1} \sum_{n=0}^{m} \omega^n W (1+P) \psi^{(m-n)} , \tag{26a}$$

$$\Psi^{(N)} = (1+P) \sum_{m=0}^{N} \psi^{(m)} + G_0 \sum_{m=0}^{N-1} \sum_{n=0}^{m} \omega^n W (1+P) \psi^{(m-n)} , \tag{26b}$$

$$\Delta E^{(N)} = \frac{\langle \psi^{(0)} \mid W \mid \Psi^{(N-1)} \rangle}{\langle \psi^{(0)} \mid \Psi^{(N-1)} \rangle} , \tag{26c}$$

with the normalization condition

$$\langle \Psi^{(0)} \mid \Psi^{(N)} \rangle = 1 . \tag{27}$$

For $N=0$ the given formulas are trivial; for the first order equations the formulas are shown in Ref. 9. The proof for arbitrary orders can be done by induction and is given in Appendix A. Looking at the perturbed part of Eq. (26a) ($m \neq 0$), we recognize the same integral kernel K as in the unperturbed problem

$$K = G_0 t_0 P , \tag{28}$$

but now we face an inhomogeneous integral equation. It is well known from Fredholm theory that it is sufficient and necessary for the inhomogeneous equation

$$\psi' = \phi + K\psi' \tag{29}$$

to have a solution that ϕ is orthogonal to the left handed solution of the homogeneous equation

$$\tilde{\psi}^{(0)} = \tilde{\psi}^{(0)} K . \tag{30}$$

The left handed eigenfunction $\tilde{\psi}^{(0)}$ can be shown to be

$$\langle \tilde{\psi}^{(0)} \mid = \langle \psi^{(0)} \mid P t_0 P , \tag{31}$$

and the orthogonality to our driving terms ϕ follows immediately from the expression for the energetic shift (Eq. (26c)].

The solution of (29) is obviously given only up to an arbitrary admixture of the unperturbed solution $\psi^{(0)}$. However, it can be made unique by our normalization requirement (27). To make our formulas clearer to the reader, we give explicit representations of them for lower orders in Appendix B.

III. NUMERICAL TREATMENT

We focus our attention on the solution of the inhomogeneous (perturbed) integral equation (26a). We remind the reader[13] that in addition to the physical eigenstate corresponding to the eigenvalue $\lambda_0 = 1$,

$$\lambda_0 \psi^{(0)} = K \psi^{(0)} , \tag{32}$$

because of the strong short range repulsion there exists an unphysical eigenstate $\psi^{(-)}$ obeying

$$\lambda^{(-)} \psi^{(-)} = K \psi^{(-)}, \quad \lambda^{(-)} < -1 . \tag{33}$$

We transform our inhomogeneous equation

$$\psi' = \phi + K\psi' \tag{34}$$

into

$$\phi' \equiv \frac{1}{1-\lambda^{(-1)}} \phi , \tag{35a}$$

$$K' \equiv \frac{1}{1-\lambda^{(-)}} (K - \lambda^{(-)} \mathbb{1}) , \tag{35b}$$

$$\psi' = \phi' + K'\psi' . \tag{35c}$$

Since λ_0 and $\lambda^{(-)}$ are the largest positive and negative eigenvalues, respectively, it can be shown by simple estimates that the Neumann series of Eq. (35c) converges if the irrelevant part of ψ' proportional to $\psi^{(0)}$ is projected out.

To this end we introduce the following projection operator Λ,

$$\Lambda = \mathbb{1} - \frac{\mid \psi^{(0)} \rangle \langle \tilde{\psi}^{(0)} \mid}{\langle \tilde{\psi}^{(0)} \mid \psi^{(0)} \rangle} , \tag{36}$$

which obviously has the desired properties

$$\Lambda \phi = \phi , \tag{37a}$$

$$\Lambda \psi^{(0)} = 0 , \tag{37b}$$

$$[\Lambda, K] = 0 . \tag{37c}$$

Thus, instead of solving Eq. (34), we use

$$\Lambda\psi' = \phi' + K'\Lambda\psi', \quad \Lambda\psi' \equiv \psi'_{\perp} . \tag{38}$$

The resulting Neumann series converges in our actual case after about 40 iterations. Since we determined in this iteration scheme only the part of ψ' which has no admixture of $\psi^{(0)}$ in it, $\psi'_{\perp}$, we finally set

$$\psi' = \psi'_{\perp} + \alpha \psi^{(0)} , \tag{39}$$

where the parameter α is fixed by the normalization condition (27).

We emphasize the treatment in momentum space because relativistic effects[14] can easily be handled in it. Certainly then, one has to deal with nonlocal potential operators, which is the price which has to be paid. What helps, however, is that wave functions are smoother than in coordinate space.

We discretize our integral equations using $NP = 22$ meshpoints for the two-body subsystem's momentum and $NQ = 12$ meshpoints for the spectator's momentum. All

18 partial waves with the total angular momentum of the subsystem $j \leq 2$ are taken into account (see Ref. 13) for the definition of our channel states) The resulting dimension D of the three-body matrices which we have to work with on the computer is of the order $D \approx 5000$. Our treatment of the incorporation of three-nucleon forces into the Faddeev equations is advantageous compared to the equivalent usage of Eqs. (5) and (8) because it requires only the disk storage of one $D \times D$ matrix, whereas the solution of, for instance, Eq. (8) would need the disk space for two such matrices. Furthermore, the kernel for the perturbed part of the Faddeev component is the same as in the unperturbed problem, which reduces the programmer's expense.

IV. NUMERICAL RESULTS

For the definition of our Jacobian momenta and basis states, and for the explicit formulas for the angular momentum decomposition of the permutation operator and of the three-body force W, we refer to Refs. 9 and 13, where also the strength parameters and the cutoff parameter for the three-body force are cited.

First, we want to present our results for the expectation value correcting an erroneous table in Ref. 9 which was caused by irregularities in handling tapes. We decompose both the Faddeev component ψ and the three-nucleon wave function Ψ into three-body partial waves:

$$\psi_{[n]} = \sum_{\alpha=1}^{n} \psi_\alpha \,, \tag{40a}$$

$$\Psi_{[n,N]} = \sum_{\alpha=1}^{N} \Psi_{\alpha[n]} \,. \tag{40b}$$

It should be noted that even a finite number n in (40a) introduces infinitely many states in (40b), which is truncated to N in our calculational scheme. $\psi_{[n]}$ is gained by Faddeev calculation with n channels and two-body forces. In first order perturbation theory in W, we get

$$\Delta E_{[n,N]} = 3\langle \Psi_{[n,N]} \mid W_1 \mid \Psi_{[n,N]} \rangle / \langle \Psi_{[n,N]} \mid \Psi_{[n,N]} \rangle \,. \tag{41}$$

Using the Reid soft core potential, we obtain the following:

$$\Delta E_{[5,18]} = -0.498 \text{ MeV} \,,$$

$$\Delta E_{[18,18]} = -1.000 \text{ MeV} \,.$$

The first value corrects that given in Ref. 9 and agrees very well with the results found in a coordinate treatment of the Faddeev equations and with the results of the Sendai group.[15] We also would like to present in Table I the corrected Table II of Ref. 9.

To test the accuracy of our values, we use the following identities,

$$(1+P)^2 = 3(1+P) \,, \tag{42}$$

$$W \mid \Psi^{(0)} \rangle = (1+P) W_1 \mid \Psi^{(0)} \rangle \,, \tag{43}$$

which are fulfilled only approximately after a finite expansion into three-body partial waves. It should be kept in mind, furthermore, that the appearance of the permutation operator makes an interpolation unavoidable if it is treated numerically. The following expressions for the expectation value should lead to the same value as Eq. (41):

$$\Delta E_a = \frac{\langle \psi^{(0)} \mid \chi \rangle}{\langle \psi^{(0)} \mid \Psi^{(0)} \rangle}, \quad \mid \chi \rangle \equiv (1+P) W_1 \mid \Psi^{(0)} \rangle \tag{44a}$$

$$\Delta E_b = \frac{\langle \Psi^{(0)} \mid W_1 \mid \Psi^{(0)} \rangle}{\langle \psi^{(0)} \mid \Psi^{(0)} \rangle} \,. \tag{44b}$$

The resulting numbers for $\Delta E_{a[18,18]}$ and $\Delta E_{b[18,18]}$ are

$$\Delta E_{a[18,18]} = -0.994 \text{ MeV} \,,$$

$$\Delta E_{b[18,18]} = -0.982 \text{ MeV} \,.$$

The difference between $\Delta E_{b[18,18]}$ and $\Delta E_{[18,18]}$ can be traced back to the scalar product of $\langle \psi^{(0)} \mid \Psi^{(0)} \rangle$, which should be exactly one-third. Because of the above mentioned reasons the actual value in our numerical treatment is given by

$$\langle \psi^{(0)} \mid \Psi^{(0)} \rangle = 0.339 \,,$$

which differs from the exact value by 1.8%. Furthermore, we checked the constituent of the first order driving term evaluating

$$\Delta E_c = \frac{\langle \tilde{\psi}^{(0)} \mid G_0 t_0 G_0 \mid \chi \rangle}{\langle \tilde{\psi}^{(0)} \mid G_0 + G_0 t_0 G_0 \mid \psi^{(0)} \rangle} \,, \tag{44c}$$

giving

$$\Delta E_{c[18,18]} = -0.986 \text{ MeV} \,.$$

Comparing these numbers, we conclude that our results should be correct up to about 10 keV.

Now we come to our results in higher orders of the per-

TABLE I. Contributions to the energy shift in MeV ($n=5$, $N=18$) resulting from channels 1 and 2.

α	$\alpha'=1$	$\alpha'=2$
1	−0.168	(+0.126)
2	+0.126	−0.148
3	−0.186	−0.991
4	−0.244	+1.506
5	+0.196	+0.000
6	−0.020	−0.108
7	−0.027	−0.054
8	0	+0.393
9	−0.305	+0.244
10	+0.138	+0.015
11	0	+0.018
12	0	−0.194
13	0	−0.236
14	0	+0.003
15	+0.004	−0.049
16	−0.025	−0.063
17	−0.127	+0.003
18	−0.074	−0.017
$\sum_\alpha$	−0.712	+0.449−(0.126)=0.323

turbational expansion. Using the Reid soft core potential,[16] we find the values given in Table II. They are based on an $n=N=18$ expansion. For the maximum momenta of the two-body subsystem (of the spectator particle) we used $P_{\max}=25.0$ fm^{-1} ($q_{\max}=10.0$ fm^{-1}). The values given in Table II signify the following: E_i is the energetic shift in ith order, $E^{(i)}$ the summation of these effects up to ith order, and $\Sigma^{(i)}$ denotes the resulting triton bound-state energy in ith order. A closer look at Table II reveals that the main contributions in our perturbational expansion come from first and second order; in fact, they make up more than 90% of the total energetic shift, whereas the first three orders comprise about 97% of the total effect of the three-body force in the triton. Hence we conclude that our perturbational approach is useful in getting the three-nucleon bound state energy, though the convergence is not as fast as we supposed before actually performing these calculations. The corresponding numbers for the Paris potential[17] can be found in Table III. Comparing our results with the latest numbers of the Sendai group, who found a triton binding energy of -9.11 MeV for the Reid soft core–Tucson-Melbourne (RSC-TM) potentials and the same 18 partial waves, we notice a satisfactory agreement with our high order perturbational result of -9.08 MeV. However, this small difference would certainly be increased if we both used the same cutoff parameter for the three-nucleon force. (We used $\Lambda^2=17.0$ fm^{-2}, or equivalently, $\Lambda=813.6$ MeV, whereas Ishikawa *et al.* took $\Lambda^2=16.44$ fm^{-2} or $\Lambda=800$ MeV.) As an estimate, we take their results for the two values of Λ, namely $\Lambda=700$ and 800 MeV (taken in a 34-channel calculation with the Paris and Argonne two-body forces) and extrapolate these numbers to $\Lambda=813.6$ MeV assuming linear dependence of the energetic shift from $\Lambda(\Lambda^2)$. Proceeding this way, we would find additional binding energy compared to $\Lambda=800$ MeV by an amount of -0.12 MeV (-0.13 MeV), thus shifting (by estimate) their number of -9.11 MeV to about -9.23 MeV. The agreement concerning the usage of the Paris potential between our results and those of the Sendai group is not so convincing. Our zeroth order binding energy (-7.33 MeV) differs from their number (-7.56 MeV) by more than 0.2 MeV, whereas the calculation of Hajduk *et al.* (see Ref. 7) yielded -7.38 MeV, which is much closer to our value. The total energetic shift for the Paris and Tucson-Melbourne potentials, however, gained by Ishikawa *et al.* (-1.93 MeV), is again very close to our result of -1.86 MeV.

Lastly, we can compare our values with those gained by the Los Alamos group, who arrived at a triton binding energy of -8.93 MeV for the above mentioned potentials (RSC + TM), 18 channels, and a cutoff parameter for the 2πE-3BP of $\Lambda^2=16.836$ fm^{-2} If we again take their values for the energetic shift using different cutoff parameters Λ, namely $\Lambda^2=8.413$, 16.836, and $\Lambda^2=25.229$ fm^{-2}, and interpolate to our value of $\Lambda^2=17.0$ fm^{-2}, we would get a triton binding energy of -8.97 MeV. This differs from our value only by 0.11 MeV, which seems very close if one takes into account the enormous complexity of such a calculation. The agreement between all three groups is satisfactory, though not yet on the same level as the agreement between calculations with two-body forces only.

We conclude that at the moment the theoretical binding energies with 18 channels and the Reid soft core –Tucson-Melbourne potentials lie in the range between -8.97 and -9.23 MeV for the cutoff value of $\Lambda^2=17.0$ fm^{-2}, which we propose to read as -9.08 ± 0.15 MeV.

In addition, we want to look more closely at the known chaotic collection of positive and negative values which builds up the energetic shift ΔE in first (higher) order:

$$\Delta E=\sum_{\alpha,\alpha'=1}^{N}\Delta E^{(\alpha,\alpha')}\,. \tag{45}$$

Since the channel states $\alpha=1$ and 2 alone (subsystem's and spectator's orbital angular momentum equal 0) comprise about 90% of the wave function's norm, and together with the state $\alpha=3$ [subsystem's (spectator's) orbital angular momentum equals 2 (0)], about 93%, we split up ΔE as follows:

$$\Delta E_{[n,N]}=\sum_{\substack{\alpha\le N_0\\ \text{or}\\ \alpha'\le N_0}}\Delta E^{(\alpha,\alpha')}+\sum_{\substack{\alpha>N_0\\ \text{and}\\ \alpha'>N_0}}\Delta E^{(\alpha,\alpha')}\equiv S_{N_0}+\text{rest}\,, \tag{46}$$

TABLE II. Perturbation theory results for the Reid soft core potential ($n=18$, $N=18$).

i	E_i/MeV	$E^{(i)}$/MeV	$\Sigma^{(i)}$/MeV
0			−7.24
1	−0.99	−0.99	−8.23
2	−0.69	−1.68	−8.92
3	−0.10	−1.78	−9.02
4	−0.06	−1.84	−9.08
5	−0.01	−1.84	−9.08

TABLE III. Perturbation theory results for the Paris potential ($n=18$, $N=18$).

i	E_i/MeV	$E^{(i)}$/MeV	$\Sigma^{(i)}$/MeV
0			−7.33
1	−0.72	−0.72	−8.05
2	−0.93	−1.66	−8.99
3	−0.07	−1.73	−9.06
4	−0.13	−1.86	−9.19
5	−0.00	−1.86	−9.19

TABLE IV. Splitting of $\Delta E_{[n,N]}$ in first order perturbation theory for the Reid softcore potential (all energies in MeV).

n	N	N_0	S_{N_0}	Rest	$\Delta E_{[n,N]}$
5	18	2	−0.389	−0.109	−0.498
5	18	3	−0.530	+0.032	−0.498
18	18	2	−0.907	−0.093	−1.000
18	18	3	−0.975	−0.025	−1.000

TABLE V. Splitting of ΔE in high order perturbation theory ($n=18$, $N=18$ all energies in MeV).

Potential	Order	N_0	ΔE	S_{N_0}	Rest
Reid	5	2	−1.84	−1.70	−0.14
Reid	5	3	−1.84	−1.78	−0.06
Paris	5	2	−1.86	−1.74	−0.12
Paris	5	3	−1.86	−1.81	−0.05

where N_0 equals 2 or 3. Because of the wave function's percentages, the first sum in (46) should build up the main contribution to ΔE. This assumption can be seen to be correct by regarding Table IV. The corresponding values for our highest order energetic shift with the Reid (Paris) potential are given in Table V.

The values in Table V refer to the perturbed wave function which was gained taking into account all channel to channel couplings of W. Though the three-nucleon force modifies the wave function it does not change the fact that the first two or three channels are by far the most important ones percentagewise. As a consequence, also the full energetic shift is very well approximated by S_{N_0} in (46).

Together with the Tucson-Melbourne three-body force, the results for the triton binding energy with the Reid soft core and Paris potentials are very similar, though the convergence of the perturbation series with the Paris potential is more irregular than the one with the Reid potential. We prefer usage of the Paris potential because it was derived from a more concrete theoretical basis than the essentially phenomenological Reid potential.

Summarizing our results, we state that the 2πE-3BP in the form used yields a substantial additional binding energy in the triton. Therefore it will be a worthwhile challenge to solidify or modify the building blocks in that force, the πNN form factor, and the πN off-shell amplitude, especially for pion momenta where the soft-pion requirements underlying the force are not fulfilled.

A recent nuclear matter calculation[18] has extended the inclusion of 3BP's to the π-ρ and ρ-ρ forces, likewise based on the current algebra program. The results for the contributions to the binding energy of nuclear matter have shown a repulsive effect of the π-ρ-3BP and an almost negligible attractive effect of the ρ-ρ-3BP. These additional potentials reduced the attractive energy shift of the 2πE-3BP in nuclear matter by a factor of about $\frac{1}{2}$. Therefore one can hope to reproduce or at least nearly reproduce the triton binding energy using these forces together with a realistic two-body interaction. Our calculation has demonstrated that the total energy shift arises essentially of the first three orders in perturbation theory and, furthermore, that out of the N^2 channel to channel contributions only a subset of $2(3)\times N$ contributions are dominating.

ACKNOWLEDGMENT

The author wishes to express his thanks to Professor Dr. W. Glöckle for stimulating discussions and steady encouragement.

APPENDIX A

Equation (25) can be rewritten reordering finite sums:

$$\sum_{m=0}^{N}\psi^{(m)}=\sum_{m=0}^{N}\sum_{n=0}^{m}\epsilon^n G_0 t_0 P\psi^{(m-n)}+\sum_{m=0}^{N-1}\sum_{l=0}^{m}\sum_{n=0}^{l}\epsilon^n G_0 t_0 P\omega^{l-n}W(1+P)\psi^{m-l}$$

$$=\sum_{n=0}^{N}\epsilon^n\sum_{m=n}^{N}G_0 t_0 P\psi^{(m-n)}+\sum_{n=0}^{N-1}\epsilon^n G_0 t_0 G_0\sum_{m=n}^{N-1}\sum_{l=n}^{m}\omega^{l-n}W(1+P)\psi^{(m-l)}$$

$$=G_0 t_0 P\sum_{m=0}^{N}\psi^{(m)}+G_0 t_0 G_0\sum_{m=0}^{N-1}\sum_{l=0}^{m}\omega^l W(1+P)\psi^{(m-l)}$$

$$+\sum_{n=1}^{N}\epsilon^n\sum_{m=n}^{N}G_0 t_0 P\psi^{(m-n)}+\sum_{n=1}^{N-1}\epsilon^n G_0 t_0 G_0\sum_{m=n}^{N-1}\sum_{l=n}^{n}\omega^{l-n}W(1+P)\psi^{(m-l)}\,. \tag{A1}$$

Using

$$\sum_{n=1}^{N-1}\epsilon^n G_0 t_0 G_0\sum_{m=n}^{N-1}\sum_{l=n}^{m}\omega^{l-n}W(1+P)\psi^{(m-l)}=\sum_{n=0}^{N-2}\epsilon^{n+1}\left[G_0 t_0 G_0\sum_{m=0}^{N-2-n}\sum_{l=0}^{m}\omega^l W(1+P)\psi^{(m-l)}\right]$$

$$=\sum_{n=0}^{N-2}\epsilon^{n+1}\left[\sum_{n=0}^{N-1-n}\psi^{(m)}-G_0 t_0 P\sum_{m=0}^{N-1-n}\psi^{(m)}\right.$$

$$\left.-\epsilon^{(N-1-n)}\sum_{m=0}^{N-2-n}\psi^{(m)}+O(N-n)\right], \tag{A2}$$

Eq. (A1) gives

$$\sum_{m=0}^{N}\psi^{(m)}=G_0t_0P\sum_{m=0}^{N}\psi^{(m)}+G_0t_0G_0\sum_{m=0}^{N-1}\sum_{l=0}^{m}\omega^l W(1+P)\psi^{(m-l)}$$
$$+\epsilon^N\psi^{(0)}+\sum_{n=0}^{N-2}\epsilon^{n+1}\sum_{m=0}^{N-1-n}\psi^{(m)}-\sum_{n=0}^{N-2}\epsilon^{n+1}\epsilon^{(N-1-n)}\sum_{m=0}^{N-2-n}\psi^{(m)}+O(N+1)$$
$$=G_0t_0P\sum_{m=0}^{N}\psi^{(m)}+G_0t_0G_0\sum_{m=0}^{N-1}\sum_{l=0}^{m}\omega^l W(1+P)\psi^{(m-l)}+\epsilon\sum_{m=0}^{N-1}\psi^{(m)}+\sum_{n=0}^{N-3}\epsilon^{n+1}O(N-n)\sum_{m=0}^{N-2-n}\psi^{(m)}+O(N+1)$$
$$=G_0t_0P\sum_{m=0}^{N}\psi^{(m)}+\epsilon\sum_{m=0}^{N-1}\psi^{(m)}+G_0t_0G_0\sum_{m=0}^{N-1}\sum_{l=0}^{m}\omega^l W(1+P)\psi^{(m-l)}\,, \tag{A3}$$

which coincides with (26a).

It remains to be seen that the expression for the energetic shift, Eq. (26c), follows from Eq. (26a). To do this, we project Eq. (26c) with the left handed eigenfunction of the homogeneous equation

$$\langle\tilde{\psi}^{(0)}|=\langle\tilde{\psi}^{(0)}|G_0t_0P=\langle\psi^{(0)}|Pt_0P\,, \tag{A4}$$

and make use of the following auxiliary relations:

$$\langle\tilde{\psi}^{(0)}|G_0t_0G_0=\langle\psi^{(0)}|\,, \tag{A5a}$$
$$\langle\tilde{\psi}^{(0)}|G_0t_0P=\langle\psi^{(0)}|Pt_0P\,, \tag{A5b}$$
$$\langle\tilde{\psi}^{(0)}|(G_0+G_0t_0G_0)=\langle\psi^{(0)}|(1+P)\,. \tag{A5c}$$

This results in

$$0=-\Delta E\sum_{m=0}^{N-1}\langle\psi^{(0)}|1+P|\psi^{(m)}\rangle$$
$$+\sum_{m=0}^{N-1}\sum_{n=0}^{m}\langle\psi^{(0)}|\omega^n W(1+P)|\psi^{(m-n)}\rangle\,. \tag{A6}$$

The last term can be rewritten as

$$\sum_{m=0}^{N-1}\sum_{n=0}^{m}\omega^n W(1+P)\psi^{(m-n)}$$
$$=\sum_{m=0}^{N=1}W(1+P)\psi^{(m)}+\sum_{m=1}^{N-1}\sum_{n=1}^{m}\omega^n W(1+P)\psi^{(m-n)}$$
$$=\sum_{m=0}^{N-1}W(1+P)\psi^{(m)}$$
$$+WG_0\sum_{m=0}^{N-2}\sum_{n=0}^{m}\omega^n W(1+P)\psi^{(m-n)}$$
$$-\Delta EG_0\sum_{m=0}^{N-2}\sum_{n=0}^{m}\omega^n W(1+P)\psi^{(m-n)}\,, \tag{A7}$$

which completes our proof by induction for Eqs. (26b) and (26c).

APPENDIX B

To be more elucidatory, we present explicit representations for lower orders of our formulas (26). We use $K=G_0t_0P$ throughout in this appendix. For zeroth order,

$$\psi^{(0)}=K\psi^{(0)}\,, \tag{B1a}$$
$$\Psi^{(0)}=(1+P)\psi^{(0)}\,. \tag{B1b}$$

For first order,

$$\Delta E^{(1)}=\frac{\langle\Psi^{(0)}|W_1|\Psi^{(0)}\rangle}{\langle\psi^{(0)}|\Psi^{(0)}\rangle}\,, \tag{B2a}$$
$$\phi^{(1)}=\phi_1^{(1)}+\phi_2^{(1)}\,,$$
$$\phi_1^{(1)}=-\Delta E^{(1)}(G_0+G_0t_0G_0)\psi^{(0)}\,, \tag{B2b}$$
$$\phi_2^{(1)}\equiv G_0t_0\tilde{\phi}_2^{(1)}=G_0t_0[G_0(1+P)W_1\Psi^{(0)}]\,,$$
$$\psi^{(1)}=\phi^{(1)}+K\psi^{(1)}\,, \tag{B2c}$$
$$\Psi^{(1)}=\Psi^{(0)}+(1+P)\psi^{(1)}+\tilde{\phi}_2^{(1)}\,. \tag{B2d}$$

For second order,

$$\Delta E^{(2)}=\frac{\langle\Psi^{(0)}|W_1|\Psi^{(1)}\rangle}{\langle\psi^{(0)}|\Psi^{(1)}\rangle}\,, \tag{B3a}$$
$$\phi^{(2)}=\phi_1^{(2)}+\phi_2^{(2)}\,,$$
$$\phi_1^{(2)}=-\Delta E^{(2)}(G_0+G_0t_0G_0)(\psi^{(0)}+\psi^{(1)})\,, \tag{B3b}$$
$$\phi_2^{(2)}\equiv G_0t_0\tilde{\phi}_2^{(2)}=G_0t_0[G_0(1+P)W_1\Psi^{(1)}-\Delta E^{(2)}G_0\tilde{\phi}_2^{(1)}]\,,$$
$$(\psi^{(1)}+\psi^{(2)})=\phi^{(2)}+K(\psi^{(1)}+\psi^{(2)})\,, \tag{B3c}$$
$$\Psi^{(2)}=\Psi^{(0)}+(1+P)(\psi^{(1)}+\psi^{(2)})+\tilde{\phi}_2^{(2)}\,. \tag{B3d}$$

For nth order,

$$\Delta E^{(n)}=\frac{\langle\Psi^{(0)}|W_1|\Psi^{(n-1)}\rangle}{\langle\psi^{(0)}|\Psi^{(n-1)}\rangle}\,, \tag{B4a}$$
$$\phi^{(n)}=\phi_1^{(n)}+\phi_2^{(n)}\,,$$
$$\phi_1^{(n)}=-\Delta E^{(n)}(G_0+G_0t_0G_0)(\psi^{(0)}+\cdots+\psi^{(n-1)})\,, \tag{B4b}$$
$$\phi_2^{(n)}\equiv G_0t_0\tilde{\phi}_2^{(n)}=G_0t_0[G_0(1+P)W_1\Psi^{(n-1)}-\Delta E^{(n)}G_0\tilde{\phi}_2^{(n-1)}]\,,$$
$$(\psi^{(1)}+\cdots+\psi^{(n)})=\phi^{(n)}+K(\psi^{(1)}+\cdots+\psi^{(n)})\,, \tag{B4c}$$
$$\Psi^{(n)}=\Psi^{(0)}+(1+P)(\psi^{(n)}+\cdots+\psi^{(n)})+\tilde{\phi}_2^{(n)}\,. \tag{B4d}$$

Note the common structure for the driving terms $\phi^{(i)}$ and the energy shifts $\Delta E^{(i)}$ which are ideal for a recursive numerical evaluation.

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