Analytical treatment of high-energy nucleus-nucleus collisions including mean field effects: Boltzmann equation approach

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We study nucleus-nucleus collisions at high energy on the basis of Boltzmann's kinetic theory under the assumption that the incident energy per nucleon is high enough for nucleon-nucleon collisions to dominate the strength of the mean field so that the latter may be treated perturbatively. Then, by use of multiple collision expansion as well as linearization techniques and the eikonal approximation, we obtain the stationary distribution function from the Boltzmann equation with the nuclear mean field as external force. The distribution consists of an infinite sum over multiple collision components. Their momentum dependent parts obey a transport equation linear in the nuclear mean force field, and can be evaluated analytically. They differ from the solution of a Fokker-Planck-type equation without external forces, obtained previously, only by a shift in the momentum caused by the nuclear mean field. The present investigation confirms a recently developed phenomenological approach which has been successful in explaining the sideward kinetic energy flow of the particles emitted in individual nucleus-nucleus collision events.

I. INTRODUCTION

Boltzmann's kinetic theory has proven to be very successful in the description of dilute many particle systems in various branches of physics. Its application to high-energy nucleus-nucleus collisions is relatively new,[1,2] though the derivation of an equation of Boltzmann type from a Watson multiple scattering formulation for high-energy nuclear reactions has been discussed earlier (cf., e.g., Refs. 3–6). In fact, the applicability of Boltzmann's theory to high-energy heavy ion physics where strong compression of nuclear matter might possibly occur is not evident, since the former actually is a theory for rarefied gases. In the present work, we will, however, take the point of view that Boltzmann's theory is an appropriate framework for the description of high-energy nucleus-nucleus collisions. In other words, we assume these collisions to be transparent enough so that only moderate nuclear densities are reached during the collision. Among others, questions concerning the generalization of the theory to higher densities and to quantum mechanical systems (though in branches other than nuclear physics) are summarized in Ref. 7 and are discussed, e.g., in various subsequent contributions in the same proceedings. We will not address such questions here, except for a brief discussion of quantum mechanical corrections in Sec. III.

Once accepting the validity of Boltzmann's theory for our purposes we, however, still face serious difficulties. In spite of its beauty and physical simplicity, the Boltzmann equation, governing the time evolution of the one-particle distribution function, is of a high mathematical complexity. Mainly due to its nonlinearity, it cannot be solved in general; even the existence and the uniqueness of the solution are far from being proven for arbitrary initial and boundary conditions. Eluding mathematical rigor, we will assume that a unique solution exists for the case we are dealing with. In this uncomfortable situation, there are, in principle, three possible ways of tackling the problem: (i) One could try to solve the Boltzmann equation directly numerically. This, however, is, for the problem under consideration, tremendously difficult at the present state of computational art. (ii) One can simulate the physical situation itself on a computer rather than solving the Boltzmann equation as such, by so-called Monte Carlo or molecular dynamics procedures (for the methods in general, see, e.g., Refs. 8 and 9). This has been done, for the investigation of high-energy nuclear collisions, by various authors (see, e.g., Refs. 10–13). (iii) One can introduce approximations, suitable for the physical problem under consideration, such that the Boltzmann equation becomes accessible to an (at least partly) analytical treatment. We settle for this third possibility because the main shortcoming of all purely numerical investigations is their lack of transparency; in order to gain as much physical insight as possible, analytical investigations are indispensible. Unfortunately, classical techniques for obtaining an approximate solution of the Boltzmann equation, such as linearization about local thermal equilibrium, can hardly be applied to high-energy nucleus-nucleus collisions, since we are, at least in the initial phase of the reaction, concerned with a physical system far from equilibrium. To reach our goal, we make use of the multiple collision and linearization techniques developed especially for this particular nonequilibrium situation.[14] These methods have proven successful in the interpretation of a large variety of measured particle inclusive cross sections over a wide range of incident energies. In Ref. 14 the reaction was described exclusively in terms of sequences of binary collisions; collective effects due to the presence of a mean field have been neglected. Such effects may, however, be important to explain observables constructed from the momenta of particles measured in individual nucleus-nucleus collision events, like the sideward kinetic energy flow,[15] and they must therefore be included. This will be done in the present work.

After giving an outline of the underlying conception in Sec. II, we briefly describe the techniques of Ref. 14 in Sec. III, along with an extension to incorporate the presence of a mean field. The result is a set of linear kinetic equations for the one-particle distribution function, expanded in terms of collision numbers and powers of the force. In Sec. IV we show how these equations can be solved to first order in the force. This result is then used in Sec. V to derive an analytically solvable transport equation in momentum space, dependent on the collision number. The physical interpretation of the solution is discussed. Its connection to a recently developed phenomenological model[16] is pointed out. Our main results are briefly summarized in Sec. VI, and concluding remarks are given. Finally, technical details of the calculations can be found in various Appendices.

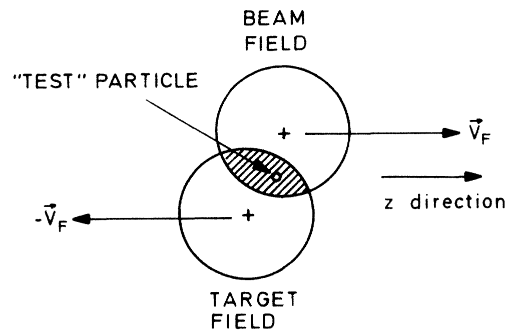

FIG. 1. A high-energy nucleus-nucleus collision in the center-of-mass frame according to our notion. The z direction is the direction of the beam.

II. THE UNDERLYING CONCEPTION

During the nucleus-nucleus collision, a particle under consideration ("test" particle) undergoes a series of binary interactions with other surrounding particles (shaded area in Fig. 1), by which its initial motion is changed. In addition to these binary collisions, the particle is subject to the influence of two nuclear average force fields, one of which is built up by the beam- (B-) like and the other by the target- (T-) like nucleons. As sketched in Fig. 1, the force B and T fields move with velocities $\mathbf{v}_F$ and $-\mathbf{v}_F$, respectively, relative to the center of mass of the two nuclei which are assumed to possess equal mass numbers A. Furthermore, these force fields are assumed not to disperse essentially during the interaction time and, therefore, not to be destroyed. In other words, the nuclei are supposed to show a large degree of transparency in the incident energy regime considered. Thus, the "test" particle is viewed as being influenced by external fields with time independent shapes.

We emphasize the essential difference of our treatment to those of Refs. 12 and 13 and others. In the latter, there is a common mean field acting which is built up by all the nucleons participating in the reaction, and which may change its shape appreciably during the course of the collision. No difference between B- and T-like nucleons is made, and consequently the mean field depends on the density arising from all nucleons in the interaction region. In contrast, our point of view is, based on the transparency of the reaction, that there exist two distinct classes of nucleons, namely B- and T-like ones which, on the average, move in opposite directions. Because of the wide separation in momentum space of the two classes of nucleons, these give rise to two corresponding mean fields which move in opposite directions with velocities $\mathbf{v}_F$ and $-\mathbf{v}_F$, respectively, and depend on the B and T densities separately. Hence the nuclear densities on which the B and T fields depend will be near normal nuclear matter density, and consequently the fields always remain attractive, whereas in Refs. 12 and 13 the nuclear mean field—in contrast—can become strongly repulsive if high compression is reached. In our treatment, compression effects are *a priori* absent.

Throughout Secs. III and IV we consider for simplicity the motion of a "test" nucleon in only one moving nuclear mean field. The generalization to two moving fields will be discussed in Sec. V.

III. REDUCTION OF THE BOLTZMANN EQUATION

Within Boltzmann's kinetic theory the time evolution of the one-particle distribution function $N(\mathbf{r},\mathbf{p},t)$ is governed by the equation

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla_r+\mathbf{F}(\mathbf{r})\cdot\nabla_p\right]N(\mathbf{r},\mathbf{p},t)=\int d^3p_1 d^3p' d^3p'_1 W(\mathbf{p},\mathbf{p}_1;\mathbf{p}',\mathbf{p}'_1)[N(\mathbf{r},\mathbf{p}',t)N(\mathbf{r},\mathbf{p}'_1,t)-N(\mathbf{r},\mathbf{p},t)N(\mathbf{r},\mathbf{p}_1,t)] \ . \quad (3.1)$$

Here, m denotes the mass of the particles under consideration, and $\mathbf{F}(\mathbf{r})$ is the local nuclear mean force field viewed as an external force. Since we henceforth choose the rest system of the field as our reference frame, and since furthermore in this system the field is assumed to be static (cf. Sec. II), we are allowed to omit the time dependence of the force. Some results, however, are given in the center-of-mass system of the colliding nuclei whenever it is convenient; in this case we will clearly specify this choice so that no confusion may arise. The transition probability $W(\mathbf{p},\mathbf{p}_1;\mathbf{p}',\mathbf{p}'_1)$ is a measure for finding two particles, undergoing an interaction with initial momenta $\mathbf{p}'$ and $\mathbf{p}'_1$, finally to possess momenta $\mathbf{p}$ and $\mathbf{p}_1$; it is expressed through the differential scattering cross section $d\sigma/d\Omega$ and energy and momentum conservation:

$$W(\mathbf{p},\mathbf{p}_1;\mathbf{p}',\mathbf{p}_1')=\frac{2}{m}\frac{d\sigma}{d\Omega}\delta(\mathbf{p}+\mathbf{p}_1-\mathbf{p}'-\mathbf{p}_1')\times\delta\left[\frac{p^2}{2}+\frac{p_1^2}{2}-\frac{p'^2}{2}-\frac{p_1'^2}{2}\right] . \quad (3.2)$$

Two remarks concerning extensions of the Boltzmann equation are appropriate at this stage:

(i) One of the quantum mechanical effects mentioned in the Introduction, namely the influence of the Pauli exclusion principle, has been investigated for the case of high-energy heavy ion collisions in an analytical fashion in Ref. 17, by using the Uehling-Uhlenbeck equation without mean field,[18] which is a transport equation of Boltzmann type with a semiclassical collision integral. It was shown in Ref. 17 that the Boltzmann solution is modified only within very limited intervals around the initial momenta of beam and target. We anticipate that this result will remain valid if the nuclear mean field is included. Therefore, we feel free, in order to work out the influence of the nuclear field as clearly as possible, to apply Boltzmann's original theory without modifications due to the Pauli principle.

(ii) It is well known how the Boltzmann equation (3.1) can be extended to the relativistic regime in a covariant manner (see, e.g., Refs. 1 and 19); we will, however, restrict ourselves to the nonrelativistic formulation throughout the investigation, since in the energy range we are concerned with (beam energies of several hundred MeV per nucleon), corrections due to relativistic kinematics are expected to be small.

A. Multiple collision expansion and linearization techniques

The first task now is to make the Boltzmann equation accessible to an analytical treatment in the context of the special conditions for high-energy nucleus-nucleus collisions. As mentioned in the Introduction, an appropriate method for the case of a vanishing force on the left hand side of Eq. (3.1) is given in Ref. 14, and we will use this method as a framework for our investigations. We will then subsequently show how it can be extended to take account of the presence of a force term in the Boltzmann equation. We do not repeat here all the considerations of Ref. 14, but only mention the essential steps required for an analytically tractable version of Eq. (3.1).

First, we split up the one-particle distribution function into two parts, one for B-like and the other for T-like nucleons:

$$N(\mathbf{r},\mathbf{p},t)=N^B(\mathbf{r},\mathbf{p},t)+N^T(\mathbf{r},\mathbf{p},t) . \quad (3.3)$$

This is justified because at high incident energies (i.e., energies considerably higher than the Fermi energy) the two kinds of nucleons are initially well distinguishable in momentum space; furthermore, this separation will last for a number of nucleon-nucleon (NN) collisions due to the forward-backward peaking of the differential NN scattering cross section at high energies. The second step consists of the simplification of taking only "violent" interactions between B- and T-like nucleons into account; the "gentle" interactions of nucleons of the same kind among each other are neglected in the early stage of the heavy ion collision and are responsible for composite particle formation in a later stage which we do not consider here. The third step is a multiple collision expansion of the one-particle distribution function

$$N(\mathbf{r},\mathbf{p},t)=\sum_{n=0}^{\infty}N_n(\mathbf{r},\mathbf{p},t)=\sum_{n=0}^{\infty}N_n^B(\mathbf{r},\mathbf{p},t)+\sum_{n=0}^{\infty}N_n^T(\mathbf{r},\mathbf{p},t) , \quad (3.4)$$

where the subscript n denotes the collision number of a nucleon. In the fourth and final step the distribution function of those nucleons on which the "test" particle scatters is approximated by a stationary distribution which is assumed to be of the form $\rho(\mathbf{r})f(\mathbf{p})$, where $\rho(\mathbf{r})$ is the nuclear density and $f(\mathbf{p})$ is a cold Fermi momentum distribution for $n=1$ and a Maxwell distribution for $n>1$.

Taking all this together, one finally arrives at the following set of coupled linear differential equations which relates each order of the multiple scattering expansion (3.4) for either B- or T-like particles to the next lower order [we henceforth drop the index B or T; Eq. (3.5) is valid for both kinds of particles]:

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla_r+\mathbf{F}(\mathbf{r})\cdot\nabla_p\right]N_n(\mathbf{r},\mathbf{p},t)=-\sigma\rho(\mathbf{r})\frac{p}{m}N_n(\mathbf{r},\mathbf{p},t)\int d^3p'K(\mathbf{p}'\,|\,\mathbf{p})+\sigma\rho(\mathbf{r})\int d^3p'\frac{p'}{m}K(\mathbf{p}\,|\,\mathbf{p}')N_{n-1}(\mathbf{r},\mathbf{p}',t) . \quad (3.5)$$

Here, σ denotes the total NN scattering cross section, assumed to be independent of the relative momentum between the colliding nucleons. The transition kernel $K(\mathbf{p}\,|\,\mathbf{p}')$ can be determined rigorously from the differential NN scattering cross section and the momentum distribution $f(\mathbf{p})$ of the partner nucleons.[2] In the present investigation, we will use for convenience a simplified kernel whose explicit form and properties will be given later on.

B. Expansion in powers of the force

The set of equations (3.5) can be solved recursively for the case of a vanishing force. We will give a brief review of the main results in the next subsection. The new feature now is the presence of a force term. To solve Eq. (3.5) with the force included, we choose a perturbative method, i.e., we expand the solution to Eq. (3.5) in terms of powers of the force. We are allowed to assume the convergence of this expansion, since in the region of high incident energy we are interested in, the reaction mechanism is dominated by sequences of NN collisions, and the collective field plays a role more subordinate than in reactions at lower energies.

In order to expand the distribution function in powers of the force, we formally provide the force with an ordering parameter λ (which may finally be taken equal to unity),

$$\mathbf{F}(\mathbf{r}) \rightarrow \lambda \mathbf{F}(\mathbf{r}) , \tag{3.6}$$

and expand the distribution function for any index n in powers of this ordering parameter:

$$N_n(\mathbf{r},\mathbf{p},t) = \sum_{i=0}^{\infty} \lambda^i N_n^{(i)}(\mathbf{r},\mathbf{p},t) . \tag{3.7}$$

Inserting both expressions (3.6) and (3.7) into the set of equations (3.5) and equating powers of the ordering parameter λ yields a new set of coupled equations for each order of the collision expansion and the field expansion of the distribution function:

$$\left[\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m}\cdot\nabla_r\right] N_n^{(i)}(\mathbf{r},\mathbf{p},t) = -\sigma\rho(\mathbf{r})\frac{p}{m}N_n^{(i)}(\mathbf{r},\mathbf{p},t)\int d^3p' K(\mathbf{p}'\,|\,\mathbf{p})$$
$$+\sigma\rho(\mathbf{r})\int d^3p' \frac{p'}{m} K(\mathbf{p}\,|\,\mathbf{p}') N_{n-1}^{(i)}(\mathbf{r},\mathbf{p}',t) - \mathbf{F}(\mathbf{r})\cdot\nabla_p N_n^{(i-1)}(\mathbf{r},\mathbf{p},t) , \tag{3.8}$$

where $N_n^{(-1)}$ vanishes by definition. Here, each order of the field expansion is coupled to the next lower order via the force [cf. the last term in Eq. (3.8)].

The set of equations (3.8) can be solved formally in a way which is known in the theory of partial differential equations as the method of characteristics. The result is an integral recurrence relation for the functions $N_n^{(i)}(\mathbf{r},\mathbf{p},t)$:

$$N_n^{(i)}(\mathbf{r},\mathbf{p},t) = \int_{-\infty}^{0} d\tau \left[\sigma\rho\left[\mathbf{r}+\frac{\mathbf{p}}{m}\tau\right]\int d^3p' K(\mathbf{p}\,|\,\mathbf{p}')\frac{p'}{m}N_{n-1}^{(i)}\left[\mathbf{r}+\frac{\mathbf{p}}{m}\tau, \mathbf{p}', t+\tau\right]\right.$$
$$\left. -\mathbf{F}\left[\mathbf{r}+\frac{\mathbf{p}}{m}\tau\right]\cdot\nabla_p N_n^{(i-1)}\left[\mathbf{r}+\frac{\mathbf{p}}{m}\tau, \mathbf{p}, t+\tau\right]\right]$$
$$\times\exp\left[-\int_{\tau}^{0} d\tau' \frac{p}{m}\sigma\rho\left[\mathbf{r}+\frac{\mathbf{p}}{m}\tau'\right]\int d^3p' K(\mathbf{p}'\,|\,\mathbf{p})\right] , \tag{3.9}$$

where in the expression

$$\nabla_p N_n^{(i-1)}(\mathbf{r}+\mathbf{p}\tau/m, \mathbf{p}, t+\tau)$$

the gradient acts on the second argument only. Here, we recognize in the first term on the right hand side the gain term of the linearized Boltzmann equation (3.8); the loss term manifests itself as an absorption factor [the exponential in Eq. (3.9)].

In order to obtain quantities which can be compared to experimental observations, we have to get rid of the time dependence in Eq. (3.9). This is achieved by introducing the stationary distribution function $P(\mathbf{r},\mathbf{p})$, which is essentially the time integral over the time dependent distribution function $N(\mathbf{r},\mathbf{p},t)$ and from which a cross section can easily be derived:[14]

$$P(\mathbf{r},\mathbf{p}) = \frac{p}{m}\int_{-\infty}^{\infty} dt\, N(\mathbf{r},\mathbf{p},t) \tag{3.10a}$$

and

$$P_n^{(i)}(\mathbf{r},\mathbf{p}) = \frac{p}{m}\int_{-\infty}^{\infty} dt\, N_n^{(i)}(\mathbf{r},\mathbf{p},t) . \tag{3.10b}$$

The justification for this integration procedure is discussed in Ref. 20. Integrating Eq. (3.9) over all times yields a hierarchy of equations for the stationary distribution functions (here and from now on we will only consider B-like particles):

$$P_n^{(i)}(\mathbf{r},\mathbf{p}) = \int_0^{\infty} d\zeta \left[\sigma\rho(\mathbf{r}-\boldsymbol{\zeta})\int d^3p' K(\mathbf{p}\,|\,\mathbf{p}') P_{n-1}^{(i)}(\mathbf{r}-\boldsymbol{\zeta}, \mathbf{p}') - \mathbf{F}(\mathbf{r}-\boldsymbol{\zeta})\cdot\nabla_p \frac{P_n^{(i-1)}(\mathbf{r}-\boldsymbol{\zeta}, \mathbf{p})}{p/m}\right]$$
$$\times\exp\left[-\int_0^{\zeta} d\zeta' \sigma\rho(\mathbf{r}-\boldsymbol{\zeta}')\int d^3p' K(\mathbf{p}'\,|\,\mathbf{p})\right] . \tag{3.11}$$

where $\boldsymbol{\zeta}$ is a vector along the momentum direction, $\boldsymbol{\zeta} = -(\mathbf{p}/m)\tau$. To treat Eq. (3.11) further analytically, we make use of the eikonal approximation, i.e., we replace the actual path of a particle through the surrounding medium by its projection on the beam (z) direction in the coordinate dependent part of Eq. (3.11). Fortunately, the eikonal approximation, which was originally introduced in the context of reactions at very high energy, has a range of validity down to energies as low as about 250 MeV per nucleon, as was demonstrated in Ref. 21. With the new integration variables $\eta=z-\zeta$ and $\eta'=z-\zeta'$, Eq. (3.11) then becomes

$$P_n^{(i)}(\mathbf{b},z;\mathbf{p})=\int_{-\infty}^{z} d\eta \left[\sigma\rho(\mathbf{b},\eta)\int d^3p' K(\mathbf{p}\,|\,\mathbf{p}')P_{n-1}^{(i)}(\mathbf{b},\eta;\mathbf{p}')-\mathbf{F}(\mathbf{b},\eta)\cdot\nabla_p \frac{P_n^{(i-1)}(\mathbf{b},\eta;\mathbf{p})}{p/m}\right]$$
$$\times\exp\left[-\int_{\eta}^{z} d\eta' \sigma\rho(\mathbf{b},\eta')\int d^3p' K(\mathbf{p}'\,|\,\mathbf{p})\right], \tag{3.12}$$

where we have decomposed the coordinate vector in a transverse and a longitudinal part, $\mathbf{r}=(\mathbf{b},z)$.

Our ultimate goal is to obtain from the set of coupled equations (3.12) a transport equation which can be solved directly analytically, and which we demand to be linear in the force $\mathbf{F}$. This is suggested by the linearity in $\mathbf{F}$ of the Boltzmann equation (3.1). To this end, we will determine the first order correction $P_n^{(1)}$ to the solution $P_n^{(0)}$ of Eq. (3.12) with a neglected force term (we will henceforth call these functions "free" solutions).

IV. SOLUTION TO FIRST ORDER IN THE FORCE

The first order solution to Eq. (3.12), which we denote by $P_n^{(0+1)}(\mathbf{b},z;\mathbf{p})$, is a sum of the "free" solution and the first order correction:

$$P_n^{(0+1)}(\mathbf{b},z;\mathbf{p})=P_n^{(0)}(\mathbf{b},z;\mathbf{p})+P_n^{(1)}(\mathbf{b},z;\mathbf{p}) . \tag{4.1}$$

A. Solution without mean field

For $i=0$, which is equivalent to the case without the force term in the Boltzmann equation, Eq. (3.12) reads explicitly

$$P_n^{(0)}(\mathbf{b},z;\mathbf{p})=\int_{-\infty}^{z} d\eta\, \sigma\rho(\mathbf{b},\eta)\int d^3p' K(\mathbf{p}\,|\,\mathbf{p}')P_{n-1}^{(0)}(\mathbf{b},\eta;\mathbf{p}')\exp\left[-\int_{\eta}^{z} d\eta' \sigma\rho(\mathbf{b},\eta')\int d^3p' K(\mathbf{p}'\,|\,\mathbf{p})\right] . \tag{4.2}$$

The method of solving this equation is described in detail in Ref. 14. We only give here the essential results. The functions $P_n^{(0)}(\mathbf{b},z;\mathbf{p})$ factorize into a coordinate- and a momentum-dependent part:

$$P_n^{(0)}(\mathbf{b},z;\mathbf{p})=G_n(\mathbf{b},z)M_n(\mathbf{p}) . \tag{4.3}$$

The coordinate dependent functions $G_n(\mathbf{b},z)$ turn out to be the well-known Glauber-Matthiae factors[22] for nucleon-nucleus scattering:

$$G_n(\mathbf{b},z)=\frac{1}{n!}[\sigma\tilde{\rho}(\mathbf{b},z)]^n\exp[-\sigma\tilde{\rho}(\mathbf{b},z)] , \tag{4.4}$$

where we made use of the "z-integrated density"

$$\tilde{\rho}(\mathbf{b},z)=\int_{-\infty}^{z} d\eta\, \rho(\mathbf{b},\eta) , \tag{4.5}$$

which, multiplied by the total NN cross section σ, is a measure for the (optical) path of a nucleon on its way through the nucleus. The geometrical weight functions (4.4) can easily be generalized for the nucleus-nucleus case. They just have to be folded with the nuclear density. This leads to[14]

$$\mathcal{G}_n(\mathbf{b},z)=\frac{1}{A}\int d^2s\, \tilde{\rho}(\mathbf{s}-\mathbf{b}, z\to\infty)G_n(\mathbf{s},z) , \tag{4.4'}$$

where $\mathbf{b}$ denotes the actual impact parameter in the nucleus-nucleus collision. The momentum dependent functions $M_n(\mathbf{p})$ obey the recurrence relation

$$M_n(\mathbf{p})=\int d^3p' K(\mathbf{p}\,|\,\mathbf{p}')M_{n-1}(\mathbf{p}') . \tag{4.6}$$

As mentioned, the transition kernel $K(\mathbf{p}\,|\,\mathbf{p}')$ depends upon the differential NN scattering cross section. Since the latter shows a smooth dependence on the momentum transfer $\mathbf{p}-\mathbf{p}'$, the right hand side (rhs) of Eq. (4.6) can be expanded to second order in the momentum transfer, with the result that the momentum distribution $M_n(\mathbf{p})$ obeys an equation of Fokker-Planck type, expressed here in the center-of-mass frame of the colliding nuclei:

$$\frac{\partial}{\partial n}M_n(\mathbf{p})=\beta(\nabla_p\cdot\mathbf{p}+m\tau\Delta_p)M_n(\mathbf{p}) . \tag{4.7}$$

The quantities β and τ, the analogues of which in the conventional theory of Brownian motion are the friction coefficient and the temperature, respectively, are completely determined by the slope of the differential NN cross section on one hand and by energy conservation on the other.[14] The solution of the transport equation (4.7) depends on the initial momentum distribution $M_1(\mathbf{p})$ according to

$$M_n(\mathbf{p})=\int d^3p' H_{n-1}(\mathbf{p},\mathbf{p}')M_1(\mathbf{p}') , \tag{4.8}$$

with the propagator

$$H_n(\mathbf{p},\mathbf{p}')=[2\pi m\tau(1-e^{-2\beta n})]^{-3/2}\times\exp\left[-\frac{(\mathbf{p}-\mathbf{p}'e^{-\beta n})^2}{2m\tau(1-e^{-2\beta n})}\right] . \tag{4.9}$$

An explicit expression for the initial distribution $M_1(\mathbf{p})$, i.e., the momentum distribution after the first collision, which is obtained by assuming the "test" particle to scatter on a Fermi distribution, can be found in Ref. 14.

B. Inclusion of the mean field

For the first order correction to the "free" solution (4.3) we need Eq. (3.12) with the index $i=1$:

$$P_n^{(1)}(\mathbf{b},z;\mathbf{p})=\int_{-\infty}^{z}d\eta\left[\sigma\rho(\mathbf{b},\eta)\int d^3p'K(\mathbf{p}\,|\,\mathbf{p}')P_{n-1}^{(1)}(\mathbf{b},\eta;\mathbf{p}')-G_n(\mathbf{b},\eta)\mathbf{F}(\mathbf{b},\eta)\cdot\nabla_p\frac{M_n(\mathbf{p})}{p/m}\right]\exp\left[-\int_\eta^z d\eta'\sigma\rho(\mathbf{b},\eta')\int d^3p'K(\mathbf{p}'\,|\,\mathbf{p})\right], \quad (4.10)$$

where we have used the factorization property (4.3) of the "free" solution.

At first, we have to specify the transition kernel $K(\mathbf{p}\,|\,\mathbf{p}')$. Instead of the one rigorously determined from the differential NN scattering cross section, which is a rather complicated function of its arguments, we use a simplified version:

$$K(\mathbf{p}\,|\,\mathbf{p}')=[2\pi m\tau(1-e^{-2\beta})]^{-3/2}\times\exp\left[-\frac{(\mathbf{p}-\mathbf{p}'e^{-\beta})^2}{2m\tau(1-e^{-2\beta})}\right]. \quad (4.11)$$

We obtain this transition kernel in an *a posteriori* fashion: On one hand, we have the recurrence relation (4.6) for the momentum distribution $M_n(\mathbf{p})$, which contains the kernel $K(\mathbf{p}\,|\,\mathbf{p}')$. On the other hand, we can express the functions $M_n(\mathbf{p})$ through the initial distribution $M_1(\mathbf{p})$ according to Eq. (4.8), which contains the propagator $H_{n-1}(\mathbf{p},\mathbf{p}')$ of the transport equation (4.7). By comparison of Eqs. (4.6) and (4.8) we are able to express the kernel $K(\mathbf{p}\,|\,\mathbf{p}')$ through the propagator $H_n(\mathbf{p},\mathbf{p}')$. This is demonstrated in Appendix A. The result is

$$K(\mathbf{p}\,|\,\mathbf{p}')=H_1(\mathbf{p},\mathbf{p}'), \quad (4.12)$$

which is Eq. (4.11). Therefore we may call the kernel (4.11) the "one collision propagator" of the Fokker-Planck-type equation (4.7), in analogy to the short time propagator of the conventional Fokker-Planck equation. Since the transport equation (4.7) has been obtained through a second order expansion of the recurrence relation (4.6) which contains the exact transition kernel, the simplified kernel (4.11) agrees with the latter at least in its first and second moments. In addition, the kernel (4.11) has the desired properties that (i) the average momentum of the distribution $M_n(\mathbf{p})$ decreases exponentially in the center-of-mass frame of the two colliding nuclei, and (ii) the width is broadened, reaching the thermal width $m\tau$ after a sufficiently large number of collisions. Finally, we remark that so far our arguments for using the simplified kernel (4.11) refer only to collision numbers $n>1$; as mentioned above, the first collision is usually treated in a different manner,[14] namely by assuming the "test" particle to scatter on a cold partner distribution of Fermi type, than on a thermalized heat bath. This turns out, however, to hardly influence the first and second moments in the energy range under consideration, so that we may use the simplified kernel (4.11) for each collision number.

We mention that the transition kernel $K(\mathbf{p}\,|\,\mathbf{p}')$ has the form of Eq. (4.11) in the *center-of-mass system* of the two colliding nuclei. Since we have, however, chosen the rest system of the force field as our reference frame, the transition kernel has to be modified. This is done in an obvious manner. Both arguments of the kernel (4.11) are shifted by a momentum $-\mathbf{p}_F=-m\mathbf{v}_F$ (where $\mathbf{v}_F$ is the velocity of the rest system of the field with respect to the nucleus-nucleus center-of-mass frame; cf. Sec. II):

$$K_F(\mathbf{p}\,|\,\mathbf{p}')=[2\pi m\tau(1-e^{-2\beta})]^{-3/2}\times\exp\left\{-\frac{[\mathbf{p}+\mathbf{p}_F-(\mathbf{p}'+\mathbf{p}_F)e^{-\beta}]^2}{2m\tau(1-e^{-2\beta})}\right\}. \quad (4.13)$$

We will drop the index F, since non confusion can arise by this.

It is immediately seen that the transition kernel $K(\mathbf{p}\,|\,\mathbf{p}')$ is normalized to unity with respect to its first argument,

$$\int d^3p\,K(\mathbf{p}\,|\,\mathbf{p}')=1, \quad (4.14)$$

for all values of the second argument. Therefore the exponential in Eq. (4.10) only contains the integral over the density. Furthermore, we replace the absolute value of the actual momentum $\mathbf{p}$ in the denominator of the expression $\nabla_p[M_n(\mathbf{p})/(p/m)]$ by the absolute value of the average momentum of the "free" distribution, which we denote by $\langle p\rangle_n$ (of course, with respect to the specific choice of our reference frame). This is justified since the momentum itself varies slowly compared to the pronouncedly peaked momentum distribution $M_n(\mathbf{p})$. For the collision number $n=0$ the solution of Eq. (4.10) then becomes

$$P_0^{(1)}(\mathbf{b},z;\mathbf{p})=-\int_{-\infty}^{z}d\eta\,G_0(\mathbf{b},\eta)\times\exp\left[-\int_\eta^z d\eta'\sigma\rho(\mathbf{b},\eta')\right]\times\mathbf{F}(\mathbf{b},\eta)\cdot\nabla_p\frac{M_0(\mathbf{p})}{\langle p\rangle_0/m}. \quad (4.15)$$

With the explicit expression for the Glauber-Matthiae factors, Eq. (4.4), we obtain, after a simple manipulation [cf. Eq. (C2) in Appendix C],

$$P_0^{(1)}(\mathbf{b},z;\mathbf{p})=-G_0(\mathbf{b},z)\frac{m}{\langle p\rangle_0}\times\int_{-\infty}^{z}d\eta\,\mathbf{F}(\mathbf{b},\eta)\cdot\nabla_pM_0(\mathbf{p}), \quad (4.16)$$

or

$$P_0^{(0+1)}(\mathbf{b},z;\mathbf{p})=G_0(\mathbf{b},z)\times[M_0(\mathbf{p})-\boldsymbol{\alpha}_0(\mathbf{b},z)\cdot\nabla_p M_0(\mathbf{p})]\ , \quad (4.17)$$

where the quantity $\boldsymbol{\alpha}_0(\mathbf{b},z)$, defined by

$$\boldsymbol{\alpha}_0(\mathbf{b},z)=\frac{m}{\langle p\rangle_0}\int_{-\infty}^{z} d\eta\,\sigma\mathbf{F}(\mathbf{b},\eta)\ , \quad (4.18)$$

has a very simple meaning: In the limit of large distances ($z\to\infty$), which we are interested in, it is the momentum transfer suffered by a particle traversing the force field $\mathbf{F}(\mathbf{r})$ in the z direction with an initial momentum $\langle p\rangle_0$ at a distance $\mathbf{b}$ from the center of the field. For a spherically symmetric potential this momentum transfer obviously points in the $\mathbf{b}$ direction.

Next, we solve Eq. (4.10) recursively for each collision number n. For $n=1$, we get, using the $n=0$ correction of Eq. (4.10) along with the abbreviation $\mathbf{B}_0(\mathbf{b},z)=-(\langle p\rangle_0/m)G_0(\mathbf{b},z)\boldsymbol{\alpha}_0(\mathbf{b},z)$,

$$P_1^{(1)}(\mathbf{b},z;\mathbf{p})=\int_{-\infty}^{z} d\eta\left[\sigma\rho(\mathbf{b},\eta)\mathbf{B}_0(\mathbf{b},\eta)\cdot\frac{m}{\langle p\rangle_0}\int d^3p' K(\mathbf{p}\,|\,\mathbf{p}')\nabla_{p'}M_0(\mathbf{p}') - G_0(\mathbf{b},\eta)\mathbf{F}(\mathbf{b},\eta)\cdot\frac{m}{\langle p\rangle_0}\nabla_p M_1(\mathbf{p})\right]\exp\left[-\int_{\eta}^{z} d\eta'\sigma\rho(\mathbf{b},\eta')\right]\ . \quad (4.19)$$

The key to the solution of Eq. (4.19) [and, moreover, of Eq. (4.16) for all higher collision numbers in analogy] is to transform the momentum integral

$$\int d^3p' K(\mathbf{p}\,|\,\mathbf{p}')\nabla_{p'}M_0(\mathbf{p}')\ .$$

First, by a partial integration, we let the gradient act on the kernel; the surface term vanishes since the norm of the momentum distribution is finite. Using the important property

$$\nabla_{p'}K(\mathbf{p}\,|\,\mathbf{p}')=-e^{-\beta}\nabla_p K(\mathbf{p}\,|\,\mathbf{p}') \quad (4.20)$$

of the transition kernel $K(\mathbf{p}\,|\,\mathbf{p}')$, we obtain, with Eq. (4.6) for $n=1$,

$$\int d^3p' K(\mathbf{p}\,|\,\mathbf{p}')\nabla_{p'}M_0(\mathbf{p}')=e^{-\beta}\nabla_p M_1(\mathbf{p})\ . \quad (4.21)$$

As a consequence, the momentum dependent part $\nabla_p M_1(\mathbf{p})$ appears as a common factor in both terms on the rhs of (4.19). This property holds for all collision numbers. Therefore, the first order correction $P_n^{(1)}(\mathbf{b},z;\mathbf{p})$ separates for each collision number in a scalar product of a geometrical vector function and the gradient of the "free" momentum distribution:

$$P_n^{(1)}(\mathbf{b},z;\mathbf{p})=\frac{m}{\langle p\rangle_n}\mathbf{B}_n(\mathbf{b},z)\cdot\nabla_p M_n(\mathbf{p})\ , \quad (4.22)$$

where the coordinate dependent part obeys the equation

$$\mathbf{B}_n(\mathbf{b},z)=\int_{-\infty}^{z} d\eta\left[\sigma\rho(\mathbf{b},\eta)e^{-\beta}\frac{\langle p\rangle_n}{\langle p\rangle_{n-1}}\mathbf{B}_{n-1}(\mathbf{b},\eta)-\mathbf{F}(\mathbf{b},\eta)G_n(\mathbf{b},\eta)\right]\exp\left[-\int_{\eta}^{z} d\eta'\sigma\rho(\mathbf{b},\eta')\right] \quad (4.23)$$

derived in Appendix B.

Thus we have succeeded in obtaining a recurrence relation for the geometrical function $\mathbf{B}_n(\mathbf{b},z)$. However, this relation is rather inconvenient to handle since it connects any $\mathbf{B}_n$ with $\mathbf{B}_{n-1}$ by an integration. Therefore the numerical effort in evaluating the functions $\mathbf{B}_n$ becomes rather large for increasing collision numbers; furthermore, (4.23) is not quite adequate to gain a physical understanding of the effect of the mean field on the distribution function. It turns out, however, that the geometrical functions $\mathbf{B}_n(\mathbf{b},z)$ can be expressed in a form much more compact than Eq. (4.23); they can be written essentially in terms of powers of the z-integrated density $\tilde{\rho}(\mathbf{b},z)$ and of one-dimensional integrals over products of the force and powers of the z-integrated density:

$$\mathbf{B}_n(\mathbf{b},z)=\exp[-\sigma\tilde{\rho}(\mathbf{b},z)]\times\sum_{j=0}^{n} b_{jn}[\sigma\tilde{\rho}(\mathbf{b},z)]^{n-j}\times\int_{-\infty}^{z} d\eta\,\mathbf{F}(\mathbf{b},\eta)[\sigma\tilde{\rho}(\mathbf{b},\eta)]^{j}\ . \quad (4.24)$$

This is shown in Appendix C, where the expansion coefficients b_{jn} are also determined. They are numbers, independent of the coordinates, and can be computed with the following algebraic recurrence relation:

$$b_{00}=-1\ ,$$
$$b_{jn}=e^{-\beta}\frac{\langle p\rangle_n}{\langle p\rangle_{n-1}}\frac{b_{j,n-1}}{n-j}\quad \text{if } j<n\ , \quad (4.25)$$
$$b_{nn}=-\left[\frac{1}{n!}+\sum_{j=0}^{n-1} b_{jn}\right]\ .$$

We have now obtained an explicit expression for the first order correction to the nucleon distribution function. It is given by a scalar product of two vector functions, one depending only on the coordinates and the other on the momentum. The momentum dependent part is the gradient of the momentum distribution without the field; the coordinate dependent part is a functional of the density and the force.

C. Further simplifications

Though the coordinate dependent part of the first order correction (4.22) can be calculated with only moderate numerical effort from (4.24) and (4.25), its physical meaning is still somewhat obscure because of the various combinations of the force and the density in Eq. (4.24), except for the case $n=0$ [cf. (4.17) and (4.18)]. In order to remove this lack of transparency, we simplify Eq. (4.24) slightly. This is done by again making use of the fact that the elementary differential NN cross section prefers forward-backward scattering at high relative momenta. This implies a small "friction" constant β, since the latter is strongly related to the forward-backward enhancement of the NN cross section. Consequently, the average nucleon momenta belonging to successive collision numbers are not very different from each other, especially in the rest system of the mean field. Hence, we replace the quantity $e^{-\beta}\langle p\rangle_n/\langle p\rangle_{n-1}$ in Eq. (4.25) by unity. This replacement is less crucial than it looks at first sight since the small momentum transfer approximation is applied only to the, in general, not very sensitive geometrical part of the distribution function; the much more essential momentum dependent part remains untouched.

With the above mentioned replacement, it is easy to show (cf. Appendix D) that the expansion coefficients b_{jn} then reduce to

$$b_{jn}=-\frac{1}{n!}\delta_{j0}\ , \tag{4.26}$$

where δ denotes the Kronecker symbol. Therefore, only the first term ($j=0$) contributes to the sum in Eq. (4.24). The geometrical function $\mathbf{B}_n(\mathbf{b},z)$ then has a particularly simple form, since it becomes a product of the z-integrated force with a functional of the z-integrated density which we recognize as the Glauber-Matthiae factor for the collision number n:

$$\mathbf{B}_n(\mathbf{b},z)=-G_n(\mathbf{b},z)\int_{-\infty}^{z}d\eta\,\mathbf{F}(\mathbf{b},\eta)\ . \tag{4.27}$$

Now, the analogy to the case $n=0$ [Eqs. (4.16)–(4.18)] is complete. The first order correction to the "free" distribution function is in the limit $z\rightarrow\infty$ (from now on we suppress the argument z; this is to be understood as taking the limit $z\rightarrow\infty$),

$$P_n^{(1)}(\mathbf{b};\mathbf{p})=-G_n(\mathbf{b})\frac{m}{\langle p\rangle_n}\int_{-\infty}^{\infty}d\eta\,\mathbf{F}(\mathbf{b},\eta)\cdot\nabla_p M_n(\mathbf{p})\ , \tag{4.28}$$

or, with the "free" distribution function added,

$$P_n^{(0+1)}(\mathbf{b};\mathbf{p})=G_n(\mathbf{b})[M_n(\mathbf{p})-\boldsymbol{\alpha}_n(\mathbf{b})\cdot\nabla_p M_n(\mathbf{p})]\ . \tag{4.29}$$

The quantity $\boldsymbol{\alpha}_n(\mathbf{b})$, given by

$$\boldsymbol{\alpha}_n(\mathbf{b})=\frac{m}{\langle p\rangle_n}\int_{-\infty}^{\infty}d\eta\,\mathbf{F}(\mathbf{b},\eta)\ , \tag{4.30}$$

is the momentum transfer acquired by a particle passing the field in the z direction with impact parameter $\mathbf{b}$ and momentum $\langle p\rangle_n$.

We now have found the connection between the nuclear field and the distribution function to first order: the key quantity appearing in the expression for the distribution is the "z-integrated force"; in other words, it is the momentum transfer suffered by a particle on its way through the field. Next, we will derive a transport equation for the distribution function.

V. TRANSPORT EQUATION IN MOMENTUM SPACE

A. A "test" nucleon traversing a force field

The aim now is, with the help of the first order distribution function given by Eq. (4.29), to derive a transport equation for the momentum distribution in dependence on the collision number, i.e., a generalization of the "free" transport equation (4.7) including the nuclear field. At this point, we note that there will be an essential difference between the old Fokker-Planck-type equation and the new transport equation. In the case without the mean field, the momentum distribution factorizes in a coordinate- and a momentum-dependent part for any collision number; cf. Eq. (4.3). This has lead to a transport equation for the momentum distribution which is independent of the coordinates of the "test" particle. This will not be the case anymore when the nuclear field is included. Though in Eq. (4.29) for the first order distribution function we were still able to separate the geometrical weight functions $G_n(\mathbf{b})$ from the sum of the "free" distribution function and the first order correction, the remainder, i.e., the momentum distribution, now contains the momentum transfer, which is a function of the impact parameter [cf. Eq. (4.30)]. Therefore, the new transport equation for the momentum distribution regarding the mean field will contain the coordinates as a parameter. From now on, we will again choose the nucleus-nucleus center-of-mass system as our reference frame; Eqs. (4.28)–(4.30) hold in any reference frame, of course, with adequately transformed average momenta $\langle p\rangle_n$.

To derive the desired transport equation, we denote the new momentum distribution by $Q_n(\mathbf{p})$, where we have dropped for convenience the argument $\mathbf{b}$. Obviously, we have, to first order,

$$Q_n(\mathbf{p})=M_n(\mathbf{p})-\boldsymbol{\alpha}_n\cdot\nabla_p M_n(\mathbf{p})\ . \tag{5.1}$$

The old Fokker-Planck-type transport equation for the function $M_n(\mathbf{p})$ reads, after a trivial manipulation,

$$\frac{\partial}{\partial n}M_n(\mathbf{p})=\beta(3+\mathbf{p}\cdot\nabla_p+m\tau\Delta_p)M_n(\mathbf{p})\ . \tag{5.2}$$

Next, we act on this equation with the gradient and then multiply the resulting equation from the left with $\boldsymbol{\alpha}_n$; after some transformations with details given in Appendix E, we obtain

$$\frac{\partial}{\partial n}[\boldsymbol{\alpha}_n\cdot\nabla_p M_n(\mathbf{p})]-\nabla_p M_n(\mathbf{p})\cdot\frac{\partial\boldsymbol{\alpha}_n}{\partial n}=\beta\{4\boldsymbol{\alpha}_n\cdot\nabla_p M_n(\mathbf{p})+\mathbf{p}\cdot\nabla_p[\boldsymbol{\alpha}_n\cdot\nabla_p M_n(\mathbf{p})]+m\tau\Delta_p[\boldsymbol{\alpha}_n\cdot\nabla_p M_n(\mathbf{p})]\}\ . \tag{5.3}$$

Subtraction of Eq. (5.3) from Eq. (5.2) yields, with the help of Eq. (5.1),

$$\frac{\partial}{\partial n}Q_n(\mathbf{p})+\nabla_p M_n(\mathbf{p})\cdot\frac{\partial \boldsymbol{\alpha}_n}{\partial n}=\beta[3Q_n(\mathbf{p})+\mathbf{p}\cdot\nabla_p Q_n(\mathbf{p})+m\tau\Delta_p Q_n(\mathbf{p})-\boldsymbol{\alpha}_n\cdot\nabla_p M_n(\mathbf{p})] . \tag{5.4}$$

This equation has been obtained by use of the first order relation (5.1). Consequently, we may add in Eq. (5.4) terms which are quadratic in the mean field. In other words, we may replace the function $M_n(\mathbf{p})$ by the function $Q_n(\mathbf{p})$ whenever the former appears in combination with the mean field, i.e., in the last terms on the left and the right hand sides of Eq. (5.4). We finally obtain

$$\frac{\partial}{\partial n}Q_n(\mathbf{p})=\left[\beta\nabla_p\cdot\mathbf{p}-\left[\beta\boldsymbol{\alpha}_n+\frac{\partial\boldsymbol{\alpha}_n}{\partial n}\right]\cdot\nabla_p+m\beta\tau\Delta_p\right]\times Q_n(\mathbf{p}) . \tag{5.5}$$

This equation is our main result. It governs the evolution of the collision-number-dependent momentum distribution of a "test" particle in the presence of the nuclear field. The latter enters through the momentum transfer $\boldsymbol{\alpha}_n$ and through its derivative with respect to the collision number. If we neglect the nuclear field, i.e., if the momentum transfer vanishes, we recover the old Fokker-Planck-type transport equation (4.7).

Equation (5.5) is a second order partial differential equation with varying coefficients, whose solution is not *a priori* obvious. In the present case, however, we are able to obtain the solution in closed form. The momentum transfer $\boldsymbol{\alpha}_n$, given by Eq. (4.30), may, in principle, be an arbitrarily complicated function of the coordinate and the collision number. The coordinate dependence is unimportant for obtaining the solution since the quantity $\mathbf{b}$ appears in Eq. (5.5) only in parametric form. The explicit dependence on the collision number will not enter the form of the solution, as we will see in a moment.

To solve Eq. (5.5), we first perform the trivial reordering

$$\left[\frac{\partial}{\partial n}+\frac{\partial\boldsymbol{\alpha}_n}{\partial n}\cdot\nabla_p\right]Q_n(\mathbf{p})=\beta[\nabla_p\cdot(\mathbf{p}-\boldsymbol{\alpha}_n)+m\tau\Delta_p]Q_n(\mathbf{p}) . \tag{5.6}$$

This suggests the introduction of new collision number and momentum variables ν and $\mathbf{q}$, respectively, according to

$$\nu(\mathbf{p},n)=n,\quad \mathbf{q}(\mathbf{p},n)=\mathbf{p}-\boldsymbol{\alpha}_n ; \tag{5.7}$$

the new momentum distribution as a function of these variables is denoted by $\tilde{Q}_\nu(\mathbf{q})$. From Eq. (5.7) we obtain for the differential operators

$$\nabla_q=\nabla_p,\quad \Delta_q=\Delta_p,\quad \frac{\partial}{\partial\nu}=\frac{\partial}{\partial n}+\frac{\partial\boldsymbol{\alpha}_n}{\partial n}\cdot\nabla_p . \tag{5.8}$$

With the help of the relations (5.7) and (5.8), Eq. (5.6) becomes

$$\frac{\partial}{\partial\nu}\tilde{Q}_\nu(\mathbf{q})=\beta(\nabla_q\cdot\mathbf{q}+m\tau\Delta_q)\tilde{Q}_\nu(\mathbf{q}) . \tag{5.9}$$

This is identical to the old Fokker-Planck-type transport equation (4.7), expressed through the new variables ν and $\mathbf{q}$. Thus, the solution of Eq. (5.9) is identical to that of Eq. (4.7) for the variables ν and $\mathbf{q}$:

$$\tilde{Q}_\nu(\mathbf{q})=M_n(\mathbf{q}) . \tag{5.10}$$

Going back to the original momentum distribution $Q_n(\mathbf{p})$ for the original variables n and $\mathbf{p}$, we obtain, by use of the relations (5.7), the final result

$$Q_n(\mathbf{p})=M_n(\mathbf{p}-\boldsymbol{\alpha}_n) . \tag{5.11}$$

The stationary distribution function then reads

$$P_n(\mathbf{b};\mathbf{p})=G_n(\mathbf{b})M_n[\mathbf{p}-\boldsymbol{\alpha}_n(\mathbf{b})] . \tag{5.12}$$

Now the physical interpretation of the result becomes clear. The distribution function under the influence of the nuclear field agrees with the one without the field, taken at a different momentum value. The original momentum is shifted by an amount which is given by the momentum transfer suffered by the "test" particle traversing the force field with an average momentum $\langle\mathbf{p}\rangle_n$. Thus, the change of the momentum distribution compared to the "free" one depends on the collision number. Furthermore, it depends on the transverse coordinate $\mathbf{b}$ of the particle, since the momentum shift varies with the distance of the particle trajectory from the center of the force. Though the factorization of the distribution function in a geometrical weight function (the Glauber-Matthiae factor G_n) and a momentum distribution is maintained for any collision number, the momentum distribution now depends on the coordinate in parametric form through its dependence on the momentum shift. The *form* of the momentum distribution is preserved; the only difference from the "free" distribution consists of a shift of the argument.

Some remarks are appropriate at this stage. Though we have used the first order momentum distribution only to derive the transport equation (5.5), its solution (5.11) now contains the mean field to all orders. Hence it appears that the transport equation (5.5) has a wider range of validity than its derivation at first sight suggests. This is supported by the fact that this solution as well as the accompanying transport equation (5.5) have been obtained previously in Ref. 23 in a quite different manner based on phenomenological grounds. In the latter work, the starting point is the momentum distribution (5.11), whose form is demanded; then it is shown that this distribution obeys the transport equation (5.5). In contrast, in the present investigation the transport equation is derived from a microscopic point of view, starting with the Boltzmann equation. Furthermore, an explicit expression for the momentum shift entering the transport equation is obtained.

Finally, we express the momentum distribution $Q_n(\mathbf{p})$ in terms of the "free" momentum distribution $M_1(\mathbf{p})$

after the first collision, since this representation is sometimes convenient for practical applications. After a simple calculation, the momentum distribution reads

$$Q_n(\mathbf{p})=\int d^3p' R_{n-1}(\mathbf{p},\mathbf{p}')M_1(\mathbf{p}')\ , \tag{5.13}$$

with the propagator

$$R_n(\mathbf{p},\mathbf{p}')=[2\pi m\tau(1-e^{-2\beta n})]^{-3/2} \times\exp\left[-\frac{(\mathbf{p}-\boldsymbol{\alpha}_n-\mathbf{p}'e^{-\beta n})^2}{2m\tau(1-e^{-2\beta n})}\right]. \tag{5.14}$$

B. Extension to nucleus-nucleus collisions

Until now, we have restricted our investigation to the consideration of a "test" particle traversing a force field with a distance b from its center. For the case of an actual nucleus-nucleus collision, the distribution $P_n(\mathbf{b},\mathbf{p})$ of Eq. (5.12) has to be slightly modified with respect to the coordinate dependent quantities. As in Sec. II, we restrict ourselves to a collision between nuclei of equal nucleon number A.

First, we note that now we are concerned with two force fields rather than one, built up by B and T nucleons. Their velocities relative to the center-of-mass system of the colliding nuclei are $\mathbf{v}_F$ and $-\mathbf{v}_F$, respectively. The total field is a sum of the two fields,

$$\mathbf{F}(\mathbf{r},t)=\mathbf{F}_B(\mathbf{r},t)+\mathbf{F}_T(\mathbf{r},t)\ , \tag{5.15}$$

and the linearized Boltzmann equation (3.5) reads

$$\left\{\frac{\partial}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla_r+[\mathbf{F}_B(\mathbf{r},t)+\mathbf{F}_T(\mathbf{r},t)]\cdot\nabla_p\right\}N_n(\mathbf{r},\mathbf{p},t) =-\sigma\rho(\mathbf{r})\frac{p}{m}N_n(\mathbf{r},\mathbf{p},t)\int d^3p' K(\mathbf{p}'\,|\,\mathbf{p})+\sigma\rho(\mathbf{r})\int d^3p'\frac{p'}{m}K(\mathbf{p}\,|\,\mathbf{p}')N_{n-1}(\mathbf{r},\mathbf{p}',t)\ . \tag{5.16}$$

At this stage we are not able to suppress the time dependence of the force term, since there does not exist a reference frame in which both fields simultaneously are static. To expand the distribution function again in powers of the force, we provide both parts of the force, which do not depend upon each other, with two different ordering parameters λ_B and λ_T, respectively,

$$\mathbf{F}(\mathbf{r},t)\rightarrow\lambda_B\mathbf{F}_B(\mathbf{r},t)+\lambda_T\mathbf{F}_T(\mathbf{r},t)\ , \tag{5.17}$$

and expand the distribution function in powers of both ordering parameters:

$$N_n(\mathbf{r},\mathbf{p},t)=N_n^{(0)}(\mathbf{r},\mathbf{p},t) +\lambda_B N_{B,n}^{(1)}(\mathbf{r},\mathbf{p},t)+\lambda_T N_{T,n}^{(1)}(\mathbf{r},\mathbf{p},t) +\text{higher order terms}\ . \tag{5.18}$$

Here, the subscripts B and T of the distribution function denote the different *fields* and not the different *kinds of "test" particles*. Insertion of expressions (5.17) and (5.18) in Eq. (5.16) yields, to zeroth order, the old result, and, to first order,

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla_r\right]N_{B,n}^{(1)}(\mathbf{r},\mathbf{p},t)=-\sigma\rho(\mathbf{r})\frac{p}{m}N_{B,n}^{(1)}(\mathbf{r},\mathbf{p},t)\int d^3p' K(\mathbf{p}'\,|\,\mathbf{p}) +\sigma\rho(\mathbf{r})\int d^3p'\frac{p'}{m}K(\mathbf{p}\,|\,\mathbf{p}')N_{B,n-1}^{(1)}(\mathbf{r},\mathbf{p}',t)-\mathbf{F}_B(\mathbf{r},t)\cdot\nabla_p N_{B,n}^{(0)}(\mathbf{r},\mathbf{p},t) \tag{5.19}$$

and

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla_r\right]N_{T,n}^{(1)}(\mathbf{r},\mathbf{p},t)=-\sigma\rho(\mathbf{r})\frac{p}{m}N_{T,n}^{(1)}(\mathbf{r},\mathbf{p},t)\int d^3p' K(\mathbf{p}'\,|\,\mathbf{p}) +\sigma\rho(\mathbf{r})\int d^3p'\frac{p'}{m}K(\mathbf{p}\,|\,\mathbf{p}')N_{T,n-1}^{(1)}(\mathbf{r},\mathbf{p}',t)-\mathbf{F}_T(\mathbf{r},t)\cdot\nabla_p N_{T,n}^{(0)}(\mathbf{r},\mathbf{p},t)\ . \tag{5.20}$$

To treat Eqs. (5.19) and (5.20) further, we choose for the former the B-field rest system as our reference frame and for the latter the T-field rest system; this choice makes the time dependence of both force terms vanish, and all further considerations remain unchanged. Finally, we deal with two different momentum shifts:

$$\boldsymbol{\alpha}_n^{(-)}(\mathbf{s})=\frac{m}{p_F-\langle p\rangle_n}\int_{-\infty}^{\infty}d\eta\,\mathbf{F}(\mathbf{s},\eta)\ , \qquad \boldsymbol{\alpha}_n^{(+)}(\mathbf{s})=\frac{m}{p_F+\langle p\rangle_n}\int_{-\infty}^{\infty}d\eta\,\mathbf{F}(\mathbf{s},\eta)\ . \tag{5.21}$$

Here, $\boldsymbol{\alpha}_n^{(-)}$ denotes the momentum transfer given to a B

nucleon by the B field, and $\alpha_n^{(+)}$ by the T field. The quantity $\langle p\rangle_n$ has to be understood now as the absolute value of the average momentum after n collisions with respect to the nucleus-nucleus center-of-mass frame. The momentum shifts $\boldsymbol{\delta}_n^{(-)}(\mathbf{b})$ and $\boldsymbol{\delta}_n^{(+)}(\mathbf{b})$ in a nucleus-nucleus collision are obtained by a folding procedure:

$$\boldsymbol{\delta}_n^{(-)}(\mathbf{b})=\frac{1}{A}\int d^2s\,\tilde{\rho}(\mathbf{s}+\mathbf{b})\boldsymbol{\alpha}_n^{(-)}(\mathbf{s})\;,$$
$$\boldsymbol{\delta}_n^{(+)}(\mathbf{b})=\frac{1}{A}\int d^2s\,\tilde{\rho}(\mathbf{s}-\mathbf{b})\boldsymbol{\alpha}_n^{(+)}(\mathbf{s})\;, \tag{5.22}$$

where $\mathbf{b}$ denotes the impact parameter in the nucleus-nucleus collision. The net momentum shift $\boldsymbol{\delta}_n(\mathbf{b})$ is

$$\boldsymbol{\delta}_n(\mathbf{b})=\boldsymbol{\delta}_n^{(-)}(\mathbf{b})+\boldsymbol{\delta}_n^{(+)}(\mathbf{b})\;. \tag{5.23}$$

After the change of the integration variable in the expression for $\boldsymbol{\delta}_n^{(-)}(\mathbf{b})$, this becomes

$$\boldsymbol{\delta}_n(\mathbf{b})=\frac{1}{A}\int d^2s\,\tilde{\rho}(\mathbf{b}-\mathbf{s})[\boldsymbol{\alpha}_n^{(+)}(\mathbf{s})-\boldsymbol{\alpha}_n^{(-)}(\mathbf{s})]\;, \tag{5.24}$$

or, with the help of Eq. (5.21),

$$\boldsymbol{\delta}_n(\mathbf{b})=-\frac{1}{A}\frac{2m\langle p\rangle_n}{p_F^2-\langle p\rangle_n^2}\int d^2s\,\tilde{\rho}(\mathbf{b}-\mathbf{s})\int_{-\infty}^{\infty}d\eta\,\mathbf{F}(\mathbf{s},\eta)\;. \tag{5.25}$$

The average momentum $\langle \mathbf{p}\rangle_n$ of the "free" momentum distribution after n collisions, which enters Eq. (5.25), is readily determined as the first moment of the old transport equation (4.7). The explicit expression reads

$$\langle \mathbf{p}\rangle_n=\langle \mathbf{p}\rangle_1\exp[-\beta(n-1)]\;. \tag{5.26}$$

Obviously, the two contributions to the momentum shift do not cancel each other as long as the average momentum $\langle \mathbf{p}\rangle_n$ does not vanish, i.e., as long as thermal equilibrium is not yet reached. In the limit $n\to\infty$, however, the quantity $\boldsymbol{\delta}_n(\mathbf{b})$ vanishes and we obtain a thermal distribution as the stationary solution of the transport equation. For completeness, we mention that for T-like "test" nucleons the momentum shift $\boldsymbol{\delta}_n(\mathbf{b})$ of Eq. (5.25) merely changes its sign. Thus, the time integrated solution of the Boltzmann equation finally can be written as, in the limit $z\to\infty$,

$$P(\mathbf{b};\mathbf{p})=\sum_n \mathscr{G}_n(\mathbf{b})\{M_n^B[\mathbf{p}-\boldsymbol{\delta}_n(\mathbf{b})]+M_n^T[\mathbf{p}+\boldsymbol{\delta}_n(\mathbf{b})]\}\;. \tag{5.27}$$

At this point we note that Eq. (5.25) for the momentum shift was already given in Ref. 16. It was determined by integrating the forces of the two moving sources acting on the moving "test" particles distributed over the overlap region (roughly speaking, the shaded area in Fig. 1) over the interaction time. Both expressions, our Eq. (5.25) and the corresponding one in Ref. 16, are, in fact, identical; this is shown in Appendix F. It has to be emphasized, however, that in the latter investigation it is *assumed* from a phenomenological point of view that the momentum shift in the distribution function has to be identified with the momentum transfer; this assumption is thereby confirmed by a microscopic derivation.

VI. SUMMARY AND CONCLUDING REMARKS

Within the framework of Boltzmann's kinetic theory, we have presented a microscopic approach to the description of high-energy nucleus-nucleus collisions, taking account of nuclear mean field effects. This has been achieved by making partly use of the multiple collision expansion and linearization methods developed in Ref. 14. In addition to these techniques, we have expanded the thereby obtained set of linear Boltzmann type equations in powers of the mean field. The result is a rather natural extension of the original theory. To be specific, in the eikonal limit we have obtained a linear transport equation for the nucleon momentum distribution in dependence on the collision number. The solution of this transport equation differs from that without mean field only by a momentum shift which is a functional of the density and the nuclear mean field and which depends on the collision number and the impact parameter. This shift can be interpreted as the momentum transferred to the nucleons in the overlap region by the moving nuclear mean fields. The resulting expressions are identical to the corresponding ones given in recent phenomenological investigations.[16,23] We may refrain here from the evaluation of observable quantities like the kinetic energy flow angle which is determined by the second moments of the distribution function (5.27) and triple differential cross sections which are essentially given by Eq. (5.27) itself; this has already been done in the latter investigations, where potentials derived from phenomenological Skyrme-type interactions have been used. Let us merely mention that a sideward flow angle is obtained which is not inconsistent in size with the experimentally observed one. Thus, as long as our underlying physical picture (large degree of transparency of the system) is valid, all results and conclusions reached in Refs. 16 and 23 are confirmed by their microscopic foundation now at hand.

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APPENDIX A: THE SIMPLIFIED COLLISION KERNEL

We briefly demonstrate how the simplified collision kernel $K(\mathbf{p}\,|\,\mathbf{p}')$ of Eq. (4.11) can be obtained from the propagator $H_n(\mathbf{p},\mathbf{p}')$ of the Fokker-Planck-type equation (4.7). This is conveniently performed by application of Fourier transformations. If a function in momentum space is denoted by a capital letter, its Fourier transform is denoted by the corresponding lower case letter. We define a new propagator $\tilde{H}_n(\mathbf{p}-\mathbf{q})$ according to

$$\tilde{H}_n(\mathbf{p}-\mathbf{q})=H_n(\mathbf{p},e^{\beta n}\mathbf{q})\;. \tag{A1}$$

Changing integration variables from $\mathbf{p}'$ to $\mathbf{q}$, Eq. (4.8) becomes

$$M_n(\mathbf{p})=e^{3\beta(n-1)}\int d^3q\,\tilde{H}_{n-1}(\mathbf{p}-\mathbf{q})M_1[e^{\beta(n-1)}\mathbf{q}]\ , \quad \text{(A2)}$$

By use of the Faltung theorem, this relation is readily Fourier transformed. We obtain

$$m_n(\mathbf{x})=(2\pi)^{3/2}\tilde{h}_{n-1}(\mathbf{x})m_1(e^{-\beta(n-1)}\mathbf{x})\ , \quad \text{(A3)}$$

and, using this equation for n and $n+1$ and eliminating the function m_1,

$$m_{n+1}(\mathbf{x})=\tilde{k}(\mathbf{x})m_n(e^{-\beta}\mathbf{x})\ , \quad \text{(A4)}$$

where we have introduced the function $\tilde{k}(\mathbf{x})$ through

$$\tilde{k}(\mathbf{x})=\tilde{h}_n(\mathbf{x})/\tilde{h}_{n-1}(e^{-\beta}\mathbf{x})\ . \quad \text{(A5)}$$

In momentum space, Eq. (A4) reads, again with the integration variable $\mathbf{p}'=e^{\beta}\mathbf{q}$,

$$M_{n+1}(\mathbf{p})=(2\pi)^{-3/2}\int d^3p'\,\tilde{K}(\mathbf{p}-e^{-\beta}\mathbf{p}')M_1(\mathbf{p}')\ . \quad \text{(A6)}$$

Therefore, the desired transition kernel is

$$K(\mathbf{p}\,|\,\mathbf{p}')=(2\pi)^{-3/2}\tilde{K}(\mathbf{p}-e^{-\beta}\mathbf{p}')\ . \quad \text{(A7)}$$

This equation is easily evaluated. The final result is Eq. (4.12).

APPENDIX B: INTEGRAL RECURRENCE RELATION FOR THE GEOMETRICAL FUNCTIONS $\mathbf{B}_n$

Inserting the first order correction in the form of Eq. (4.22) into Eq. (4.10) yields [with $\int d^3p'K(\mathbf{p}'\,|\,\mathbf{p})=1$ and p in the denominator replaced by $\langle p\rangle_n$]:

$$\frac{m}{\langle p\rangle_n}\mathbf{B}_n(\mathbf{b},z)\cdot\nabla_p M_n(\mathbf{p})=\int_{-\infty}^{z}d\eta\left[\sigma\rho(\mathbf{b},\eta)\frac{m}{\langle p\rangle_{n-1}}\mathbf{B}_{n-1}(\mathbf{b},\eta)\cdot\int d^3p'K(\mathbf{p}\,|\,\mathbf{p}')\nabla_{p'}M_{n-1}(\mathbf{p}')\right.$$
$$\left.-\frac{m}{\langle p\rangle_n}G_n(\mathbf{b},\eta)\mathbf{F}(\mathbf{b},\eta)\cdot\nabla_p M_n(\mathbf{p})\right]\exp\left[-\int_{\eta}^{z}d\eta'\sigma\rho(\mathbf{b},\eta')\right]\ . \quad \text{(B1)}$$

Again, we can evaluate the momentum integral by a partial integration and utilizing the property (4.20) of the transition kernel; this gives

$$\int d^3p'K(\mathbf{p}\,|\,\mathbf{p}')\nabla_{p'}M_{n-1}(\mathbf{p}')=e^{-\beta}\nabla_p K(\mathbf{p}\,|\,\mathbf{p}')M_{n-1}(\mathbf{p}') \quad \text{(B2)}$$

and, with the recurrence relation (4.6),

$$\int d^3p'K(\mathbf{p}\,|\,\mathbf{p}')\nabla_{p'}M_{n-1}(\mathbf{p}')=e^{-\beta}\nabla_p M_n(\mathbf{p})\ , \quad \text{(B3)}$$

so that we can factorize out the quantity $\nabla_p M_n(\mathbf{p})$ on the right hand side of Eq. (B1):

$$\mathbf{B}_n(\mathbf{b},z)\cdot\frac{m}{\langle p\rangle_n}\nabla_p M_n(\mathbf{p})=\int_{-\infty}^{z}d\eta\left[\sigma\rho(\mathbf{b},\eta)e^{-\beta}\frac{\langle p\rangle_n}{\langle p\rangle_{n-1}}\mathbf{B}_{n-1}(\mathbf{b},\eta)-\mathbf{F}(\mathbf{b},\eta)G_n(\mathbf{b},\eta)\right]$$
$$\times\exp\left[-\int_{\eta}^{z}d\eta'\sigma\rho(\mathbf{b},\eta')\right]\cdot\frac{m}{\langle p\rangle_n}\nabla_p M_n(\mathbf{p})\ . \quad \text{(B4)}$$

The momentum dependent parts on both sides of this equation agree. Equating the geometrical parts yields the integral recurrence relation (4.23).

APPENDIX C: EXPLICIT EXPRESSIONS FOR THE GEOMETRICAL FUNCTIONS $\mathbf{B}_n$

To prove Eq. (4.24) for the geometrical functions $\mathbf{B}_n(\mathbf{b},z)$, we go into the integral recurrence relation (4.23) with unspecified expansion coefficients b_{jn}:

$$\exp[-\sigma\tilde{\rho}(\mathbf{b},z)]\sum_{j=0}^{n}b_{jn}[\sigma\tilde{\rho}(\mathbf{b},z)]^{n-j}\int_{-\infty}^{z}d\eta\,\mathbf{F}(\mathbf{b},\eta)[\sigma\tilde{\rho}(\mathbf{b},\eta)]^{j}$$
$$=\int_{-\infty}^{z}d\eta\left\{\sigma\rho(\mathbf{b},\eta)e^{-\beta}\frac{\langle p\rangle_n}{\langle p\rangle_{n-1}}\sum_{j=0}^{n-1}b_{j,n-1}[\sigma\tilde{\rho}(\mathbf{b},\eta)]^{n-j-1}\int_{-\infty}^{\eta}d\eta'\mathbf{F}(\mathbf{b},\eta')[\sigma\tilde{\rho}(\mathbf{b},\eta')]^{j}\right.$$
$$\left.-\frac{1}{n!}\mathbf{F}(\mathbf{b},\eta)[\sigma\tilde{\rho}(\mathbf{b},\eta)]^{n}\right\}\exp[-\sigma\tilde{\rho}(\mathbf{b},\eta)]\exp\left[-\sigma\int_{\eta}^{z}d\eta''\rho(\mathbf{b},\eta'')\right]\ , \quad \text{(C1)}$$

where we have used Eq. (4.4) for the Glauber-Matthiae factors $G_n(\mathbf{b},z)$. With the definition (4.5) for the "z-integrated density" $\tilde{\rho}(\mathbf{b},z)$, both exponentials on the rhs of Eq. (C1) can be combined:

$$\exp[-\sigma\tilde{\rho}(\mathbf{b},\eta)]\exp\left[-\sigma\int_{\eta}^{z}d\eta''\rho(\mathbf{b},\eta'')\right]=\exp[-\sigma\tilde{\rho}(\mathbf{b},z)]\ . \quad \text{(C2)}$$

This expression does not depend upon the variable η and can therefore be cast in front of the integral; Eq. (C1) then reads

$$\sum_{j=0}^{n} b_{jn}[\sigma\tilde{\rho}(\mathbf{b},z)]^{n-j}\int_{-\infty}^{z} d\eta\,\mathbf{F}(\mathbf{b},\eta)[\sigma\tilde{\rho}(\mathbf{b},\eta)]^{j}$$

$$=\int_{-\infty}^{z} d\eta\left\{\sigma\rho(\mathbf{b},\eta)e^{-\beta}\frac{\langle p\rangle_n}{\langle p\rangle_{n-1}}\sum_{j=0}^{n-1} b_{j,n-1}[\sigma\tilde{\rho}(\mathbf{b},\eta)]^{n-j-1}\int_{-\infty}^{\eta} d\eta'\mathbf{F}(\mathbf{b},\eta')[\sigma\tilde{\rho}(\mathbf{b},\eta')]^{j}-\frac{1}{n!}\mathbf{F}(\mathbf{b},\eta)[\sigma\tilde{\rho}(\mathbf{b},\eta)]^{n}\right\}. \tag{C3}$$

Next, we have to evaluate the rhs of Eq. (C3). To this end, we have to make use of the following lemma:

$$\int_{a}^{z} dy\,f(y)\int_{a}^{y} dx\,g(x)=\int_{a}^{z} dy\,f(y)\int_{a}^{z} dx\,g(x)-\int_{a}^{z} dy\,g(y)\int_{a}^{y} dx\,f(x)\,, \tag{C4}$$

where f and g are arbitrary, sufficiently well-behaved functions, and a is an arbitrary number. Lemma (C4) can easily be proved by partial integration. Obviously, Eq. (C4) remains true if one or both of the integrands are vector functions. Applying lemma (C4) in the limit $a\to-\infty$ to the first part of the rhs of Eq. (C3), the latter becomes

$$\begin{aligned}\text{rhs}=e^{-\beta}\frac{\langle p\rangle_n}{\langle p\rangle_{n-1}}\sum_{j=0}^{n-1} b_{j,n-1}\Bigg\{&\int_{-\infty}^{z} d\eta'\sigma\rho(\mathbf{b},\eta')[\sigma\tilde{\rho}(\mathbf{b},\eta')]^{n-j-1}\int_{-\infty}^{z} d\eta\,\mathbf{F}(\mathbf{b},\eta)[\sigma\tilde{\rho}(\mathbf{b},\eta)]^{j}\\ &-\int_{-\infty}^{z} d\eta\,\mathbf{F}(\mathbf{b},\eta)[\sigma\tilde{\rho}(\mathbf{b},\eta)]^{j}\int_{-\infty}^{\eta} d\eta'\sigma\rho(\mathbf{b},\eta')[\sigma\tilde{\rho}(\mathbf{b},\eta')]^{n-j-1}\Bigg\}\\ &-\frac{1}{n!}\int_{-\infty}^{z} d\eta\,\mathbf{F}(\mathbf{b},\eta)[\sigma\tilde{\rho}(\mathbf{b},\eta)]^{n}\,.\end{aligned} \tag{C5}$$

To work out Eq. (C5) further, we make use of the identity

$$\int_{-\infty}^{z} d\eta\,\rho(\mathbf{b},\eta)[\tilde{\rho}(\mathbf{b},\eta)]^{i}=\frac{1}{i+1}[\tilde{\rho}(\mathbf{b},\eta)]^{i+1}\,, \tag{C6}$$

where i is an arbitrary positive integer or zero. We will prove this identity at the end of this Appendix. Application of relation (C6) to the first and fourth integral of Eq. (C5) yields

$$\begin{aligned}\text{rhs}=e^{-\beta}\frac{\langle p\rangle_n}{\langle p\rangle_{n-1}}\sum_{j=0}^{n-1}\frac{b_{j,n-1}}{n-j}&\left\{[\sigma\tilde{\rho}(\mathbf{b},z)]^{n-j}\int_{-\infty}^{z} d\eta\,\mathbf{F}(\mathbf{b},\eta)[\sigma\tilde{\rho}(\mathbf{b},\eta)]^{j}-\int_{-\infty}^{z} d\eta\,\mathbf{F}(\mathbf{b},\eta)[\sigma\tilde{\rho}(\mathbf{b},\eta)]^{n}\right\}\\ &-\frac{1}{n!}\int_{-\infty}^{z} d\eta\,\mathbf{F}(\mathbf{b},\eta)[\sigma\tilde{\rho}(\mathbf{b},\eta)]^{n}\,,\end{aligned} \tag{C7}$$

or, by reordering the terms,

$$\begin{aligned}\text{rhs}=&\,e^{-\beta}\frac{\langle p\rangle_n}{\langle p\rangle_{n-1}}\sum_{j=0}^{n-1}\frac{b_{j,n-1}}{n-j}[\sigma\tilde{\rho}(\mathbf{b},z)]^{n-j}\int_{-\infty}^{z} d\eta\,\mathbf{F}(\mathbf{b},\eta)[\sigma\tilde{\rho}(\mathbf{b},\eta)]^{j}\\ &-\left[e^{-\beta}\frac{\langle p\rangle_n}{\langle p\rangle_{n-1}}\sum_{j=0}^{n-1}\frac{b_{j,n-1}}{n-j}+\frac{1}{n!}\right]\int_{-\infty}^{z} d\eta\,\mathbf{F}(\mathbf{b},\eta)[\sigma\tilde{\rho}(\mathbf{b},\eta)]^{n}\,.\end{aligned} \tag{C8}$$

This expression is equal to the left hand side (lhs) of Eq. (C3) concerning the coordinate dependent terms; comparison of the individual powers yields a recurrence relation for the expansion coefficients b_{jn}. For $j<n$ we obtain

$$b_{jn}=e^{-\beta}\frac{\langle p\rangle_n}{\langle p\rangle_{n-1}}\frac{b_{j,n-1}}{n-j}\,, \tag{C9}$$

and, for $j=n$,

$$b_{nn}=-\left[\frac{1}{n!}+e^{-\beta}\frac{\langle p\rangle_n}{\langle p\rangle_{n-1}}\sum_{i=0}^{n-1}\frac{b_{i,n-1}}{n-i}\right], \tag{C10}$$

which is identical to the last of the relations (4.25) by use of Eq. (C9). The coefficient b_{00} is fixed by Eq. (4.16).

Finally, we have to furnish proof of the identity (C6). We will do this by mathematical induction. For $i=0$ there is nothing to prove because of the definition (4.5) of the "z-integrated density" $\tilde{\rho}(\mathbf{b},z)$. Suppose that Eq. (C6) is true for $i=j-1$:

$$\int_{-\infty}^{z} d\eta\,\rho(\mathbf{b},\eta)[\tilde{\rho}(\mathbf{b},\eta)]^{j-1}=\frac{1}{j}[\tilde{\rho}(\mathbf{b},z)]^{j}\,. \tag{C11}$$

For $i=j$, we write the lhs of Eq. (C6) as

$$\int_{-\infty}^{z} d\eta\, \rho(\mathbf{b},\eta)[\tilde{\rho}(\mathbf{b},\eta)]^{j} = \int_{-\infty}^{z} d\eta\, \rho(\mathbf{b},\eta)[\tilde{\rho}(\mathbf{b},\eta)]^{j-1} \int_{-\infty}^{\eta} d\eta' \rho(\mathbf{b},\eta') \; ; \tag{C12}$$

with the help of lemma (C4), this becomes

$$\int_{-\infty}^{z} d\eta\, \rho(\mathbf{b},\eta)[\tilde{\rho}(\mathbf{b},\eta)]^{j} = \int_{-\infty}^{z} d\eta\, \rho(\mathbf{b},\eta)[\tilde{\rho}(\mathbf{b},\eta)]^{j-1} \int_{-\infty}^{z} d\eta' \rho(\mathbf{b},\eta') - \int_{-\infty}^{z} d\eta\, \rho(\mathbf{b},\eta) \int_{-\infty}^{\eta} d\eta' \rho(\mathbf{b},\eta')[\tilde{\rho}(\mathbf{b},\eta')]^{j-1} \; . \tag{C13}$$

Applying, now, Eq. (C11) to both terms on the rhs, we obtain

$$\int_{-\infty}^{z} d\eta\, \rho(\mathbf{b},\eta)[\tilde{\rho}(\mathbf{b},\eta)]^{j} = \frac{1}{j}[\tilde{\rho}(\mathbf{b},z)]^{j+1} - \frac{1}{j}\int_{-\infty}^{z} d\eta\, \rho(\mathbf{b},\eta)[\tilde{\rho}(\mathbf{b},\eta)]^{j} \; , \tag{C14}$$

which is the same as

$$\int_{-\infty}^{z} d\eta\, \rho(\mathbf{b},\eta)[\tilde{\rho}(\mathbf{b},\eta)]^{j} = \frac{1}{j+1}[\tilde{\rho}(\mathbf{b},\eta)]^{j+1} \; . \tag{C15}$$

This proves the assertion.

APPENDIX D: THE GEOMETRICAL FUNCTIONS $\mathbf{B}_n$ FOR SMALL MOMENTUM TRANSFER

With the replacement

$$e^{-\beta} \langle p \rangle_n / \langle p \rangle_{n-1} = 1 \; , \tag{D1}$$

the expansion coefficients b_{jn} of Eq. (4.25) become

$$b_{00} = -1 \; ,$$
$$b_{jn} = \frac{b_{j,\,n-1}}{n-j} \quad \text{if } j<n \; , \tag{D2}$$
$$b_{nn} = -\left[\frac{1}{n!} + \sum_{j=0}^{n-1} b_{jn} \right] .$$

We will prove relation (4.26) by mathematical induction with respect to the index n. For $n=0$, Eq. (4.26) is trivial. Suppose now that Eq. (4.26) holds for $n=m-1$, for each j between zero and $m-1$:

$$b_{j,\,m-1} = -\frac{1}{(m-1)!}\delta_{j0} \; . \tag{D3}$$

For $n=m$ and $j<m$, we have, with the second relation (D2),

$$b_{jm} = \frac{b_{j,\,m-1}}{m-j} \; , \tag{D4}$$

which becomes, with Eq. (D3),

$$b_{jm} = -\frac{1}{m!}\delta_{j0} \; . \tag{D5}$$

In the case $j=m$, we have, with the third relation (D2),

$$b_{mm} = -\left[\frac{1}{m!} + \sum_{j=0}^{m-1} b_{jm} \right] , \tag{D6}$$

which becomes, with Eq. (D5) for $j<m$,

$$b_{mm} = -\left[\frac{1}{m!} + \frac{1}{m!}\sum_{j=0}^{m-1} \delta_{j0} \right] = 0 \; . \tag{D7}$$

Hence, Eq. (4.26) is true; Eq. (4.27) follows immediately.

APPENDIX E: DERIVATION OF EQ. (5.3)

Applying the gradient to the transport equation (5.2) yields (we drop for the moment the argument $\mathbf{p}$ of the momentum distribution M_n)

$$\frac{\partial}{\partial n}\nabla_p M_n = \beta[3\nabla_p M_n + \nabla_p(\mathbf{p}\cdot\nabla_p M_n) + m\tau\Delta_p \nabla_p M_n] \; . \tag{E1}$$

The second term on the rhs becomes

$$\nabla_p(\mathbf{p}\cdot\nabla_p M_n) = [(\nabla_p M_n)\cdot\nabla_p]\mathbf{p} + (\nabla_p M_n)\times(\nabla_p\times\mathbf{p}) + (\mathbf{p}\cdot\nabla_p)\nabla_p M_n + \mathbf{p}\times(\nabla_p\times\nabla_p M_n) \; . \tag{E2}$$

Both curls vanish. The first term on the rhs of this equation is simply

$$[(\nabla_p M_n)\cdot\nabla_p]\mathbf{p} = \nabla_p M_n \; . \tag{E3}$$

Thus, Eq. (E1) becomes

$$\frac{\partial}{\partial n}\nabla_p M_n = \beta[4\nabla_p M_n + (\mathbf{p}\cdot\nabla_p)\nabla_p M_n + m\tau\Delta_p \nabla_p M_n] \; . \tag{E4}$$

Multiplying this equation by $\boldsymbol{\alpha}_n$, we obtain

$$\boldsymbol{\alpha}_n\cdot\frac{\partial}{\partial n}\nabla_p M_n = \beta[4\boldsymbol{\alpha}_n\cdot\nabla_p M_n + \mathbf{p}\cdot\nabla_p(\boldsymbol{\alpha}_n\cdot\nabla_p M_n) + m\tau\Delta_p(\boldsymbol{\alpha}_n\cdot\nabla_p M_n)] \; , \tag{E5}$$

since $\boldsymbol{\alpha}_n$ commutes with the operators $\mathbf{p}\cdot\nabla_p$ and Δ_p on the rhs. On the lhs, $\boldsymbol{\alpha}_n$ does not commute with the operator $\partial/\partial n$; instead, we have

$$\boldsymbol{\alpha}_n\cdot\frac{\partial}{\partial n}\nabla_p M_n = \frac{\partial}{\partial n}(\boldsymbol{\alpha}_n\cdot\nabla_p M_n) - \nabla_p M_n\cdot\frac{\partial\boldsymbol{\alpha}_n}{\partial n} \; ; \tag{E6}$$

inserted into Eq. (E5), this yields Eq. (5.3).

APPENDIX F: ANOTHER REPRESENTATION FOR THE MOMENTUM TRANSFER

In Ref. 16, the momentum transfer suffered by the beamlike particles present in the overlap region of the two nuclei under the influence of the nuclear fields was determined by integrating the force acting on the particles over the reaction time. The explicit expression was given by (for momentum independent potentials)

$$\boldsymbol{\delta}_n(\mathbf{b})=\boldsymbol{\delta}_n^{(-)}(\mathbf{b})+\boldsymbol{\delta}_n^{(+)}(\mathbf{b})=\frac{1}{A}\left[\int_{-\infty}^{\infty}dt\int d^3\eta\,\mathbf{F}\left[\boldsymbol{\eta}_\perp-\frac{\mathbf{b}}{2},\eta_\parallel-\frac{p_F}{m}t\right]\rho\left[\boldsymbol{\eta}_\perp+\frac{\mathbf{b}}{2},\eta_\parallel-\frac{\langle p\rangle_n}{m}t\right]\right.$$
$$\left.+\int_{-\infty}^{\infty}dt\int d^3\eta\,\mathbf{F}\left[\boldsymbol{\eta}_\perp+\frac{\mathbf{b}}{2},\eta_\parallel+\frac{p_F}{m}t\right]\rho\left[\boldsymbol{\eta}_\perp-\frac{\mathbf{b}}{2},\eta_\parallel-\frac{\langle p\rangle_n}{m}t\right]\right]\;; \tag{F1}$$

here, we have used the notation of the present paper. We will now show, by simple changes of the integration variables, that Eq. (F1) agrees with our expressions (5.22) and (5.23) for the momentum transfer. Introducing a new variable $\boldsymbol{\zeta}$, defined by

$$\boldsymbol{\zeta}_\perp=\boldsymbol{\eta}_\perp\mp\frac{\mathbf{b}}{2},\quad \zeta_\parallel=\eta_\parallel-\frac{\langle p\rangle_n}{m}t\;, \tag{F2}$$

where the upper (lower) sign holds for the first (second) term of Eq. (F1), the latter becomes

$$\boldsymbol{\delta}_n(\mathbf{b})=\frac{1}{A}\left[\int_{-\infty}^{\infty}dt\int d^3\zeta\,\mathbf{F}\left[\boldsymbol{\zeta}_\perp,\zeta_\parallel-\frac{p_F-\langle p\rangle_n}{m}t\right]\rho(\boldsymbol{\zeta}_\perp+\mathbf{b},\zeta_\parallel)+\int_{-\infty}^{\infty}dt\int d^3\zeta\,\mathbf{F}\left[\boldsymbol{\zeta}_\perp,\zeta_\parallel+\frac{p_F+\langle p\rangle_n}{m}t\right]\rho(\boldsymbol{\zeta}_\perp-\mathbf{b},\zeta_\parallel)\right]\,. \tag{F3}$$

Interchanging the order of time and coordinate integration and replacing the time integration by a spatial one via

$$z=\zeta_\parallel\mp\frac{p_F\mp\langle p\rangle_n}{m}t\;, \tag{F4}$$

we obtain

$$\boldsymbol{\delta}_n(\mathbf{b})=\frac{1}{A}\left[\int d^3\zeta\,\rho(\boldsymbol{\zeta}_\perp+\mathbf{b},\zeta_\parallel)\frac{m}{p_F-\langle p\rangle_n}\int_{-\infty}^{\infty}dz\,\mathbf{F}(\boldsymbol{\zeta}_\perp,z)+\int d^3\zeta\,\rho(\boldsymbol{\zeta}_\perp-\mathbf{b},\zeta_\parallel)\frac{m}{p_F+\langle p\rangle_n}\int_{-\infty}^{\infty}dz\,\mathbf{F}(\boldsymbol{\zeta}_\perp,z)\right]\,. \tag{F5}$$

By use of the "z-integrated density" $\tilde{\rho}$ and the quantities $\boldsymbol{\alpha}_n^{(-)}$ and $\boldsymbol{\alpha}_n^{(+)}$ defined by Eq. (5.21), this becomes

$$\boldsymbol{\delta}_n(\mathbf{b})=\frac{1}{A}\left[\int d^2\zeta_\perp\tilde{\rho}(\boldsymbol{\zeta}_\perp+\mathbf{b})\boldsymbol{\alpha}_n^{(-)}(\boldsymbol{\zeta}_\perp)+\int d^2\zeta_\perp\tilde{\rho}(\boldsymbol{\zeta}_\perp-\mathbf{b})\boldsymbol{\alpha}_n^{(+)}(\boldsymbol{\zeta}_\perp)\right]\,, \tag{F6}$$

which is identical to Eqs. (5.22) and (5.23).

[1]R. Malfliet, Nucl. Phys. **A363**, 429 (1981).
[2]B. Schürmann and D. P. Min, Nucl. Phys. **A370**, 496 (1981).
[3]K. M. Watson, Phys. Rev. **118**, 886 (1960).
[4]M. K. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964), p. 826.
[5]J. Hüfner, Ann. Phys. (N.Y.) **115**, 43 (1978).
[6]M. Thies, Ann. Phys. (N.Y.) **123**, 411 (1979).
[7]G. E. Uhlenbeck, Acta Phys. Austr. Suppl. ***X***, 107 (1973).
[8]W. W. Wood, Acta Phys. Austr. Suppl. ***X***, 451 (1973).
[9]F. H. Ree, in *Physical Chemistry,* edited by H. Eyring, D. Henderson, and W. Jost (Academic, New York, 1971), Vol. VIII A, p. 157.
[10]J. Cugnon, Nucl. Phys. **A387**, 191c (1982).
[11]Z. Fraenkel, Nucl. Phys. **A428**, 373c (1984).
[12]G. F. Bertsch, H. Kruse, and S. Das Gupta, Phys. Rev. C **29**, 673 (1984).
[13]R. Malfliet, Phys. Rev. Lett. **53**, 2386 (1984).
[14]R. Malfliet and B. Schürmann, Phys. Rev. C **31**, 1275 (1985).
[15]H. A. Gustafsson, H. H. Gutbrod, H. Löhner, M. R. Maier, A. M. Poskanzer, T. Renner, H. Riedesel, H. G. Ritter, H. Spieler, A. Warwick, F. Weik, and H. Wieman, Phys. Rev. Lett. **52**, 1590 (1984).
[16]B. Schürmann and W. Zwermann, Phys. Lett. **158B**, 366 (1985).
[17]S. Mies and B. Schürmann, Ann. Phys. (N.Y.) **166**, 189 (1986).
[18]E. A. Uehling and G. E. Uhlenbeck Phys. Rev. **43**, 552 (1933).
[19]S. R. de Groot, W. A. van Leeuwen, and Ch.G. van Weert, *Relativistic Kinetic Theory* (North-Holland, Amsterdam, 1980).
[20]J. Hüfner and M. Thies, Phys. Rev. C **20**, 273 (1979).
[21]B. Schürmann, Phys. Rev. C **20**, 1607 (1979).
[22]R. J. Glauber and G. Matthiae, Nucl. Phys. **B21**, 135 (1970).
[23]B. Schürmann and W. Zwermann, Phys. Rev. C **33**, 1668 (1986).