PHYSICAL REVIEW C VOLUME 34, NUMBER 3 SEPTEMBER 1986

Pionic retardation effects in two-pion-exchange three-nucleon forces

S. A. Coon
Department of Physics, University of Arizona, Tucson, Arizona 85721

J. L. Friar
Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
(Received 12 May 1986)

Those two-pion-exchange three-nucleon forces which arise from nuclear processes that involve only pions and nucleons are calculated. Among the processes which contribute are pion seagulls (e.g., nucleon-antinucleon pair terms) and overlapping, retarded pion exchanges. The resulting potential is shown to be a $(v/c)^2$ relativistic correction, and satisfies nontrivial constraints from special relativity. The relativistic ambiguities found before in treatments of relativistic corrections to the one-pion-exchange nuclear charge operator and two-body potential are also present in the three-nucleon potential. The resulting three-nucleon force differs from the original Tucson-Melbourne potential only in the presence of several new nonlocal terms, and in the specification of the choice of ambiguity parameters in the latter potential.

I. INTRODUCTION

Three-body forces (3BF) in nuclear physics[1–4] constitute a topic which is nearly as old as nuclear physics itself. The earliest calculations[5] used models which are now known to be inadequate, and were supplanted by efforts to incorporate the rapidly developing phenomenology of hadronic reactions[6] during the 1950s. The latter work and much of the later work emphasized the importance of the pion, the lightest known hadron, and the internal structure of the nucleon (i.e., the Δ resonance). Paralleling the motivation of long-range three-body forces between atoms,[7] the pion-mediated three-nucleon forces have the longest range and might be expected to be the dominant component of such forces in the dilute trinucleon systems.

A recurring theoretical problem was the off-shell behavior of the pion-nucleon (π-N) interaction.[8,9] The familiar pseudoscalar (PS) and pseudovector (PV) interaction vertices have identical on-mass-shell forms, but very different off-mass-shell behavior. Without further guidance, vastly different three-nucleon forces can be constructed using these dissimilar forms. The pure PS form has a strong coupling to negative-energy states (z graphs) which leads to a large even-isospin pion-nucleon scattering length. Accurate experimental studies long ago showed that both of the isotopic scattering lengths were quite small. The discovery of (Goldstone mode) approximate chiral symmetry led to an understanding of this result[9,10] and of the small mass of the pion. While these discoveries did not completely specify the form of the pion-nucleon interaction, they provided sufficient constraints that formulation of the long-range three-nucleon forces was far less arbitrary.

The modern era of this topic began with the dictum of Brown, Green, and Gerace[11] that, no matter which theoretical π-N models are used, they should be consistent with (broken) chiral symmetry. That is, they should possess well-defined chiral limits. The first potential of this type was constructed by Yang.[12] Later, the Tucson-Melbourne group emphasized that one should also incorporate as much appropriate phenomenological input[13–15] (which includes chiral-symmetry-breaking information) as possible. The implementation of both rules led to the Tucson-Melbourne three-nucleon potential[16–18] via the use of PCAC and current algebra to determine the form of the (off-shell) pion-nucleon scattering amplitude for small pion and nucleon momenta. This amplitude mediates the three-nucleon force and is shown in Fig. 1(a), which depicts only one of the many possible pion-charge configurations that give this force its rich isotopic structure. Among the possible processes subsumed by the

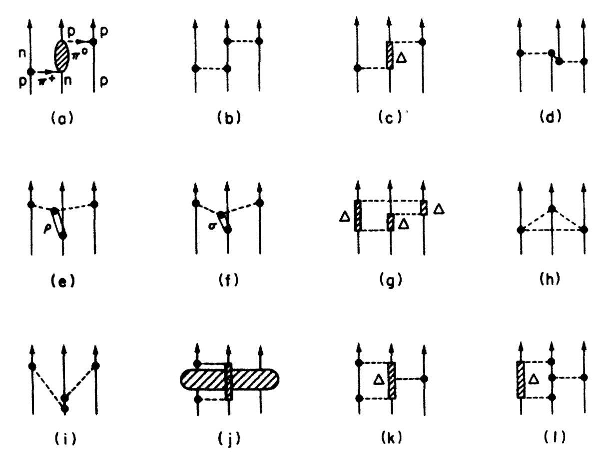

FIG. 1. Various physical processes contributing to the 2π3BF which are subsumed by the "blob" in panel (a) are depicted in (b)—(f), and (i). Shorter range processes are illustrated in (g)—(h) and (j)—(l).

"blob" are the disconnected graph [i.e., the pion-exchange Born term of Fig. 1(b)], the Fujita-Miyazawa[6] Δ-mediated interaction in Fig. 1(c), the z graph or nucleon-antinucleon "pair" process in Fig. 1(d), and the ρ- and σ-meson mediated forces in Figs. 1(e) and 1(f). Shorter-range processes such as those of Figs. 1(g), 1(h), and 1(j)—1(l) are also possible. For future reference we will denote those processes represented by Figs. 1(b), 1(d), and 1(i) as "purely pionic processes," because they involve only pions and nucleons.

Overlapping pion exchanges[19–23] depicted by the time-ordered graph of Fig. 1(i) present a special problem. The latter processes are clearly related to Fig. 1(b), but contain the modification due to retardation, the finite time required for a pion to propagate between two nucleons. Moreover, we expect intuitively that Fig. 1(b) is simply the iteration of the one-pion-exchange potential (OPEP) in the three-body system and should therefore be excluded. That is, it is not a "true" three-body force because it is merely an unavoidable consequence of solving the Schrödinger equation with two-body forces in a many-body system.

These disparate considerations lead to three theoretical problems which have continued to plague us. (1) The construction of a pion-nucleon amplitude with good properties for small momenta is adequate to describe the long-range behavior of the force, but is clearly inadequate to specify the short-range behavior. (2) Our intuitive analysis of Fig. 1 forces us to consider time-ordered graphs such as panel 1(i) at the same time that we "subtract out" the interaction of the OPEP, Fig. 1(b). The former processes are nontrivial and represent a kind of medium modification of the OPEP. (3) We will see subsequently that the purely pionic processes, when the Born terms are subtracted, have the schematic form V_π^2/Mc^2 or f^4/Mc^2, where V_π is the (two-body) OPEP, M is the nucleon mass, f is the (pseudovector) π-N coupling constant ($f=g/2M$), and the dimensionless renormalized π-N coupling constant has the value

$$f_R^2=\left[\frac{g\mu_\pi}{2M}\right]^2\Big/4\pi\simeq 0.079\ , \tag{1}$$

determined by the pion mass μ_π and the pseudoscalar π-N coupling constant g. Hence these contributions are relativistic corrections.[24]

It has been known for a decade[24] that relativistic corrections in nuclei are subject to a generic problem, which has prevented a proper study of their size and influence. The problem is precisely the one which arises in mapping the Bethe-Salpeter equation (a four-dimensional equation) into an "equivalent" quasipotential (three-dimensional) equation. A succinct statement of the problem is that there are many such equations.[25] In the more limited context of relativistic corrections, the statement is that an ambiguity in how to handle the *relative* times in a relativistic many-body system [or, equivalently, the fourth (time) components of the four-momenta transferred between nucleons] leads to a unitary ambiguity in *all* operators. The rules of quantum mechanics dictate a solution to the problem: unitary transformations cannot alter matrix elements, *provided* that the wave functions and the operators are consistently calculated. Our current difficulty is that the so-called "realistic" potential models, used to calculate the wave functions, do not have the requisite form to guarantee that the matrix elements can be unambiguously calculated. The two-body potentials to order $(v/c)^2$ must be momentum dependent, and this momentum dependence is not arbitrary. Moreover, the pion-exchange part of the nuclear charge and isoscalar current operators suffer from the same problem.[24] For completeness, we list in the Appendix the two-body relativistic corrections to the OPEP, and the complete OPE charge operator, both of which were calculated in Ref. 24. These operators depend on two arbitrary parameters, μ and ν, which label the ambiguity and which must cancel from any matrix element.

Because there is a confluence of problems associated with the purely pionic terms, we will restrict this work to those processes. We therefore take $m_\sigma=m_\rho=m_\Delta=\infty$, deferring until a later paper the explicit Δ-, σ-, and ρ-pole terms. That is, we will work with those processes in Fig. 1 which contain only nucleon pole terms, pion-exchange, and pion seagull terms. Much of the final result has been derived before[17] for a specific choice of μ, with the exception of the medium-modification, or retarded pion exchange, processes. The method we have chosen to perform this calculation was used in the first[24] complete [to order $(v/c)^2$] treatment of the pion-exchange two-body force and is particularly well suited for calculating the effect of medium modification of the one-pion-exchange potential, to be discussed later. The basic elements of this approach consist of a Foldy-Wouthuysen reduction procedure[26] (an expansion in powers of $1/M$) to define the pion-nucleon vertices and the use of time-dependent perturbation theory to tie together the nonrelativistic nucleons with pions. The Foldy-Wouthuysen transformation on the relativistic (four-component) equations of motion for a single nucleon interacting with a pion field results in a two-component equation. The latter is first order in time and no longer involves explicit antinucleon degrees of freedom. Those degrees of freedom have been "frozen out" and lead to seagull-like terms in the effective Hamiltonian which results. This approach contrasts with the calculation of Ref. 17, which constructs the entire three-body amplitude corresponding to a three-body force, subtracts the (so-called) forward-propagating Born terms, and then makes a small momentum expansion, while ignoring retardation in the pion lines. We will see in Sec. V that the results of these two very different approaches are nearly identical.

The basic elements of the calculation, including the Foldy-Wouthuysen expansion of our basic vertices, is presented in Sec. II. The troublesome Born term subtraction is dealt with in Sec. III, while the seagull contributions are treated in Sec. IV. These diverse elements are combined in Sec. V, where we display explicit forms for the three-body potential operators, verify the constraints of special relativity, and make comparisons with previous results. In Sec. VI we present our conclusions.

II. VERTEX FUNCTIONS

We wish to determine the leading-order (in powers of $1/M$) contributions to the purely pionic parts of the two-pion-exchange three-body force (2π3BF). In order to do this we use the "rules of scale" developed previously.[27] We treat the nonrelativistic two-body potential energy V as the same order of magnitude as the kinetic energy, T ($\sim 1/M$). Thus, relativistic corrections to the Hamiltonian are schematically of order $1/M^3$ (i.e., $1/c^2$ corrections), and can be explicitly of order $1/M^3$ (the kinetic energy), V/M^2 (two-body potential), or V^2/M (two- and three-body potentials). Consequently, we will limit *explicit* powers of $1/M$ to first order. Moreover, we will calculate only three-body forces, which limits our requirements for pion-nucleon vertices to orders f, f/M, f^2, and f^2/M.

A detailed discussion of chiral (effective) Lagrangians is not needed for the purpose of this work, and we defer it to a later paper. All that is necessary is the observation that PV coupling manifests a "soft" pion-nucleon interaction, which is the simplest way to build in many of the consequences of chiral symmetry for the π-N interaction, particularly for our purposes here. This does not mean that models with PS coupling are wrong. Indeed, it is a relatively easy task[28] to show that the nonlinear σ model (based on PS coupling) can be transformed by means of a simple pion field transformation and a unitary nucleon field transformation into the (unrenormalized) Weinberg nonlinear model[29] (based on PV coupling). These models are therefore physically identical. The renormalized pion-nucleon coupling terms for the latter model have the approximate form

$$L_{N\pi} \cong \overline{N}\left[-f\gamma_5\gamma^\mu\partial_\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi} - \frac{f^2}{g_A^2}\gamma^\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}\right]N\,, \tag{2}$$

where the first term displays the $(\gamma^\mu\partial_\mu)$ PV form, $\boldsymbol{\pi}(\mathbf{x},t)$ is the (isovector) pion field, and N is the nucleon field. We use the conventions of Ref. 30. The second term, a seagull, is necessary for the pion-nucleon scattering amplitude to exhibit the correct behavior[10] at the (off-shell) Adler-Weisberger point.

In the absence of a sigma term, the πN amplitude at the Cheng-Dashen, Adler, and Weinberg symmetry points[10] are equal to the PV Born terms, calculated using the first term in (2). Alternatively, the reduced amplitude, defined as the difference of the full amplitude and the PV Born terms, will vanish at these points when there is no sigma term. Note that we have used a renormalized seagull coupling by introducing the axial vector coupling constant g_A. We also note that the latter process enters into some effective Lagrangians via a ρ-exchange process.[31] Freezing out the ρ leads to Eq. (2).

The essence of the transmogrification of the nonlinear σ model discussed above is the nucleon field transformation. It has the form

$$N_{PV} = e^{iS}N_{PS}\,, \tag{3a}$$

or

$$N_{PS} = e^{-iS}N_{PV}\,, \tag{3b}$$

where

$$S = \gamma_5 h(\xi)\boldsymbol{\tau}\cdot\boldsymbol{\xi}\,, \tag{3c}$$

and thus

$$e^{iS} = \frac{1 + i\gamma_5 h\boldsymbol{\tau}\cdot\boldsymbol{\xi}}{(1+h^2\xi^2)^{1/2}}\,, \tag{3d}$$

with $h(0)=1$ and $\boldsymbol{\xi}=\lambda f\boldsymbol{\pi}$. For our purposes we can take $h=1$, because we will work only to order f^2. Clearly, one is not required to work with pure PV or PS couplings, and a linear combination may suffice. Converting Eq. (2) to a Hamiltonian and performing the transformation (3b) with $\lambda = 1-\mu$ results in

$$\begin{aligned} H' \cong\ & (\boldsymbol{\alpha}\cdot\mathbf{p}+\beta M) + 2i\beta\gamma_5 fM\boldsymbol{\tau}\cdot\boldsymbol{\pi}(1-\mu) \\ & - f\mu\gamma_5\boldsymbol{\tau}\cdot[\dot{\boldsymbol{\pi}} + (\boldsymbol{\alpha}\cdot\nabla)\boldsymbol{\pi}] \\ & + f^2\left[\mu^2 + \frac{1}{g_A^2} - 1\right]\boldsymbol{\tau}\cdot\boldsymbol{\pi}\times[\dot{\boldsymbol{\pi}} + (\boldsymbol{\alpha}\cdot\nabla)\boldsymbol{\pi}] \\ & - 2\beta f^2 M\boldsymbol{\pi}^2(1-\mu)^2 + \cdots\,. \end{aligned} \tag{4}$$

In addition to the free Dirac Hamiltonian $[H_D]$, the next two terms are the usual PS (for $\mu=0$) and PV (for $\mu=1$) π-N couplings, followed by the Adler-Weisberger term (for $\mu=1$) [i.e., the seagull term in Eq. (2)], while the remaining term is necessary to cancel the large unphysical z-graph part of the πN amplitude in PS coupling. For $\mu=0$ and $g_A=1$, we reproduce the nonlinear σ model Hamiltonian to order f^2; the additional term which arises for $g_A \neq 1$ (i.e., renormalized) converts this to an effective Lagrangian which satisfies the chiral constraints at the symmetry points, as it must, because the unitary transformation cannot alter the physical content of the model.

The parameter μ in Eq. (4) is precisely the same parameter which labels one of the ambiguities in OPE relativistic corrections in a nucleus, and has its origin as a fractional (i.e., $1-\mu$) Weinberg (or Dyson)[32] transformation. We can construct nuclear operators from this Hamiltonian by performing a Foldy-Wouthuysen transformation.[26] The pionic terms to order f^2/M are given by

$$H^{\pi}_{\mu}=-f(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{\tau}\cdot\boldsymbol{\pi}-\frac{f}{4M}(1+\mu)\{\boldsymbol{\sigma}\cdot\mathbf{p},\boldsymbol{\tau}\cdot\dot{\boldsymbol{\pi}}\}+\frac{f^2}{g_A^2}\boldsymbol{\tau}\cdot\boldsymbol{\pi}\times\dot{\boldsymbol{\pi}}+\frac{f^2}{2M}(1-\mu)\boldsymbol{\pi}\cdot\nabla^2\boldsymbol{\pi}+\frac{f^2}{2M}(\mu-1+1/g_A^2)\{p^i,\boldsymbol{\tau}\times\boldsymbol{\pi}\nabla^i\cdot\boldsymbol{\pi}\}$$
$$+\frac{f^2}{2Mg_A^2}\sigma^k\epsilon^{kij}\tau^{\gamma}\epsilon^{\gamma\alpha\beta}(\nabla^i\pi^{\alpha}\nabla^j\pi^{\beta})-\frac{f^2(1-\mu)}{2M}\{\boldsymbol{\sigma}\times\mathbf{p},\pi^{\alpha}\cdot\nabla\pi^{\alpha}\}+\cdots \quad (5)$$

In addition, we have dropped several terms nominally of order f^2/M which involve two time derivatives. A time derivative of a pion field can be thought of as order $1/M$ and, consequently, these terms should not be kept. We note that the final result has the form

$$H^{\pi}_{\mu}=H^{\pi}_{\rm PV}-i(\mu-1)[U_B,H^{\pi}_{\rm PV}+H_D]-(\mu-1)\dot{U}_B\ , \quad (6a)$$

where

$$U_B=\frac{f}{4M}\{\boldsymbol{\sigma}\cdot\mathbf{p},\boldsymbol{\tau}\cdot\boldsymbol{\pi}\}\ . \quad (6b)$$

Note that four of the seven terms in Eq. (5), including three of the five seagull terms, are modified by the unitary transformation. The operators in Eq. (5) are the basic ingredients for our perturbation theory calculation. The first (μ-independent) term in Eq. (5) defines the basic π-N vertex, J^0_{π}, while the second term defines its relativistic correction, J'_{π}.

III. PERTURBATION THEORY

Given the πN vertices developed in the preceding section, we can now connect them together using time-dependent perturbation theory to form physical amplitudes.[24] The amplitudes are represented graphically by Figs. 2 and 3. The individual vertices are depicted in Fig. 2, with panels (a) and (b) representing the π-nucleus vertices of order f (J^0_{π}) and f/M (J'_{π}), respectively, while panels (c) and (d) show the seagull vertices of order f^2 and f^2/M, respectively. These are tied together in the usual way with pion propagators to form *nuclear* amplitudes. We emphasize that the double lines depict nucleus wave functions and Green's functions. Explicit forms for the Green's functions, the π-nucleus vertices, and the appropriate Feynman rules are given in Ref. 24, where Figs. 3(a) and 3(b) are discussed in detail. We present here a succinct symmetry of that work with those minor modifications necessary to extend our procedures to order f^4, and work out a simple example.

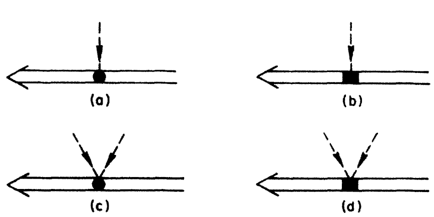

FIG. 2. Basic pion-nucleus vertices for one pion, panels (a) and (b), and two pions, panels (c) and (d). The solid circles represent the nonrelativistic forms, while squares represent relativistic corrections.

In the absence of pion exchange we suppose that the nuclear Hamiltonian is given by H_0. To H_0 we add the one-pion-exchange potential V_{π} to form the basis Hamiltonian H used to calculate the wave functions and Green's functions (double lines) in Fig. 3; consequently, we must also subtract V_{π} and treat it in perturbation theory together with the other pion-exchange processes shown in Fig. 3. Thus the pion-exchange potential V_{π} is a "counter term" of order f^2 (represented by the circles with overscored crosses) and is the analogue of mass counter terms in quantum field theory.[30] The problem with this procedure (and Rayleigh-Schrödinger perturbation theory in general) is that the meson exchanges generate energy-dependent potentials, and such potentials have undesirable properties: the orthonormality relationship for the wave functions explicitly involves the potentials. Alternatively, these potentials are not Hermitian in the usual sense, and are state dependent. Eliminating this energy dependence is accomplished in precise analogy to the mass and wave function renormalization procedures of quantum field theory,[30] leading to an energy (state) -independent pion-exchange potential.

The operator corresponding to the pion exchange in Fig. 2(a) can be constructed easily using the Feynman

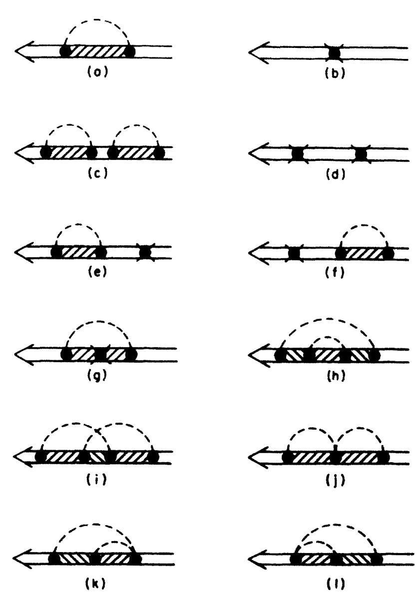

FIG. 3. Time-ordered pion-exchange graphs of order f^2 are shown in panels (a) and (b), with the $\times$ representing the OPEP counter term. The remaining contributions are of order f^4; they are the disconnected 2π-exchange graphs, (c)—(f), the overlapping graphs of (g)—(i), and the seagull graphs of (j)—(l), respectively. The cross hatching indicates virtual nuclear excitation driven by retarded pion exchanges.

rules, or taken directly from Eq. (35a) of Ref. 24. We find that the "pionic self-energy" Σ, or equivalently the retarded one-pion-exchange potential $V_\pi(E)$, is given by

$$\Sigma \equiv V_\pi(E)$$
$$= -\sum_m \int \frac{d^3q_1}{(2\pi)^3 2E_1} \frac{J_\pi^{0\alpha}(\mathbf{q}_1)\,|\,m\,\rangle\langle\, m\,|\,J_\pi^{0\alpha}(-\mathbf{q}_1)}{E_1+(E_m-E)-i\epsilon} \quad (7a)$$
$$\equiv -\sum_m \int_{-1} \frac{J_1\,|\,m\,\rangle\langle\, m\,|\,J_1}{E_1+(E_m-E)-i\epsilon}\,, \quad (7b)$$

where

$$\int_n \equiv \int \frac{d^3q_1}{(2\pi)^3 2E_1^{n+2}}\,, \quad (7c)$$

and from Eq. (52) of Ref. 24,

$$J_\pi^{0\alpha}(\mathbf{q}) = -if\sum_{i=1}^{A} \tau_i^\alpha \sigma_i\cdot\mathbf{q}\, e^{i\mathbf{q}\cdot\mathbf{x}_i'} \quad (7d)$$

is the usual nonrelativistic pion-nucleus (many-body) vertex. The energy E occurring in the propagator of Eq. (7) is the energy of the initial state and E_1 is the pion energy, $(\mathbf{q}_1^2+\mu_\pi^2)^{1/2}$. The schematic form written in (7b) will be useful later for the two-pion exchange case. Clearly, $V_\pi(E)$ is a complicated many-body operator.

We next expand the propagator as a power series in $(E_m-E)/E_1$ to second order, because $(E_m-E)\sim 1/M$. Using closure we find, to this order,

$$V_\pi(E) \cong V_\pi + \tfrac{1}{2}\{E-H,\hat{Z}\} + (E-H)V_\pi''(E-H) + \cdots\,, \quad (8)$$

where

$$V_\pi = V_\pi^0 + \tfrac{1}{2}\int_1 [J_1,[H,J_1]] + \int_2 ([J_1,H]^2 - \tfrac{1}{2}[H,[H,J_1J_1]])\,, \quad (9a)$$
$$\hat{Z} = V_\pi' - \int_2 [[H,J_1],J_1]\,, \quad (9b)$$
$$V_\pi^0 = -\int_0 J_1J_1 \equiv -\int \frac{d^3q_1 J_\pi^{0\alpha}(\mathbf{q}_1)J_\pi^{0\alpha}(-\mathbf{q}_1)}{(2\pi)^3 2E_1^2}\,, \quad (9c)$$
$$V_\pi' = -\int_1 J_1J_1\,, \quad (9d)$$
$$V_\pi'' = -\int_2 J_1J_1\,. \quad (9e)$$

It is a simple exercise to use the explicit form of the operator J_π^0 to demonstrate that the two-body parts of Eq. (9c) comprise the usual nonrelativistic OPEP. All one-body terms here and elsewhere will be ignored, because they contribute only to the nucleon mass.

The energy-dependent factors in Eq. (8) can be eliminated in two ways.[24] One method adds the energy-dependent $V_\pi(E)$ to the Hamiltonian in Schrödinger's equation,

$$[E-H_0-V_\pi(E)]\Psi_E = 0\,, \quad (10)$$

and then manipulates it to order f^4/M into the usual Schrödinger form:

$$\{E-(H-\tfrac{1}{8}[V_\pi',[H,V_\pi']])\}\Psi = 0\,. \quad (11)$$

Alternatively, the perturbation series for $V_\pi(E)$ defined by Figs. 3(a)–3(f) can be rearranged so that only a series for $\Delta V_\pi(E)\equiv V_\pi(E)-V_\pi$ remains. The renormalization procedure of Ref. 24 then leads again to Eq. (11), which completes the treatment of the disconnected diagrams, Figs. 3(c)–3(f), constructed using J_π^0.

We repeat the calculation replacing J_π^0 [solid circles in Fig. 2(a)] by J_π' [solid squares in Fig. 2(b)] in Eq. (7a). This produces a new contribution to V_π, denoted $\Delta V_\pi'$:

$$\Delta V_\pi' = i[H,U_E] + i\int_0 \{J_1,[H,J_1']\}\,, \quad (12)$$

where

$$U_E = -\tfrac{1}{2}\int_0 \{J_1',J_1\}\,. \quad (13)$$

The disconnected diagrams play no role to order f^4/M. This completes our treatment of Figs. 3(a)–3(f).

These results are virtually identical to the corresponding calculation culminating in Eqs. (57) and (65) of Ref. 24. The only difference is the double commutator term in Eq. (11). Moreover, if we examine the second term in (9a), we find that, after excluding one-body terms, only the potential energy in H contributes; it does not generate three-body contributions and we can ignore this term also. In Ref. 24 two-body contributions of order f^2/M^2 to $\Delta V_\pi'$ were calculated.

The last term in Eq. (9a) has a special significance; it gives the effect of retardation, or medium modification of the pion propagator, on the one-pion-exchange potential. Moreover, if one defines (with $\nu=0$ for the moment)

$$U_R = \frac{i(1-\nu)}{2}\int_2 [H,J_1J_1]\,, \quad (14)$$

the disconnected diagrams calculated using J_π^0 generate an effective three-body force contained in Eqs. (9a) and (11),

$$V_\pi^D = V_\pi^0 - \tfrac{1}{8}[V_\pi',[H,V_\pi']] + \int_2 [J_1,H]^2 + i[H,U_R]\,, \quad (15)$$

together with the contribution in Eq. (12) from J_π':

$$V_\pi'^D = i[H,U_E] + i\int_0 \{J_1,[H,J_1']\}\,. \quad (16)$$

The terms containing U_E and U_R are first-order unitary transformations, have vanishing expectation value, and could be eliminated by performing the inverse of that transformation. Can we simply ignore them for this reason? In Eq. (9a) the problem actually lies with the third term and the identity

$$[H,[H,J_1J_1]]/2 = [H,J_1]^2 + \{J_1,[H,[H,J_1]]\}/2\,. \quad (17)$$

We can eliminate the final two terms in (15) in terms of another with a different structure. Physically, the $[H,J_1]^2$ term corresponds to the pion retardation taking place equally at *both* π-N vertices in the OPEP, while performing the substitution of Eq. (17) into Eq. (15) leads to a single contribution which corresponds to retardation taking place asymmetrically. The former is equivalent to eliminating the commutator (i.e., $\nu=1$) and is labeled the "soft" representation, while the latter corresponds to the "standard" representation ($\nu=0$). Note that those parts of the two-body potential in H which commute with J_π^0 do not generate a three-body force in the soft representation. A similar problem exists for $\Delta V'_\pi$, which can be rearranged into various forms using commutator identities or by the unitary transformation U_E. The two-body parts of Eqs. (15) and (16), which are relativistic corrections to V_π^0, are derived by replacing H with T. These terms do not give us any guidance about which values of μ and ν to use (see the Appendix). No values of these parameters produce a potential which is momentum independent. Consequently, we are unable to make a choice which corresponds to a "realistic" potential model. Insisting on consistency requires the use of a two-body Hamiltonian which contains *essential* momentum dependence that follows from the requirements of special relativity.

Having dealt with the medium modification graphs, we must still contend with Figs. 1(b) and 1(d), as well as the overlapping time orderings (TO's) of Fig. 1(i). The derivation of these contributions is exceptionally tedious, and we will only quote the final result. The unretarded part of the graphs 3(g)—3(i) is a local two-body 2π-exchange potential and will be ignored, as will the $\hat{Z}$ terms which arise from the energy dependence of the propagators. Those terms can be removed by the procedure we discussed earlier and lead to high-order contributions (in $1/M$) than we have agreed to keep. We find

$$\Delta V_\pi^{\mathrm{TO}} = \tfrac{1}{8}[V'_\pi,[H,V'_\pi]] + i[H,V'_{2\pi}] \ , \qquad (18a)$$

where

$$V'_{2\pi} = \frac{i}{16}\int_1\int_1 \frac{E_2-E_1}{E_2+E_1}[J_1^2,J_2^2] \ . \qquad (18b)$$

The first of these terms exactly cancels the second term in Eq. (15), while the undesirable second term has the form of a unitary transformation. This term is unique in form and, unlike the others, can (and should) be removed by unitary transformation, since it leads to a momentum-dependent two-body potential of 2π range. Note that (18b) would vanish if the meson-nucleon vertex were spin and isospin independent. Repeating the same process for the π-N vertex J'_π leads to a vanishing result: $V_\pi^{\mathrm{TO}'}=0$.

IV. SEAGULL CONTRIBUTIONS

The seagull π-N vertices, Figs. 2(c) and 2(d), are all that remain from Eq. (5). Those contributions which involve seagulls of order f^2/M can be easily evaluated, because retardation plays no role. We define the seagull in Fig. 2(c) so that a term of the form

$$\int\int S^{\alpha\beta}(\mathbf{x},\mathbf{y})\pi^\alpha(\mathbf{x})\pi^\beta(\mathbf{y})$$

in the Hamiltonian generates a momentum space vertex with the properties

$$S^{\alpha\beta}(\mathbf{q}_1,\mathbf{q}_2) \equiv S_{12}^{\alpha\beta} = S_{21}^{\beta\alpha} = [S^{\alpha\beta}(-\mathbf{q}_1,-\mathbf{q}_2)]^\dagger \ . \qquad (19)$$

In the static limit, Figs. 3(j)—3(l) give the following contribution:

$$\Delta V_\pi^{\mathrm{SG}} = \int_0\int_0 \left[\frac{E_2 J_\pi^\alpha(-\mathbf{q}_1)J_\pi^\beta(-\mathbf{q}_2)S^{\alpha\beta}(\mathbf{q}_1,\mathbf{q}_2)}{E_1+E_2} + \frac{E_1 S^{\alpha\beta}(\mathbf{q}_1,\mathbf{q}_2)J_\pi^\alpha(-\mathbf{q}_1)J_\pi^\beta(-\mathbf{q}_2)}{E_1+E_2} + J_\pi^\beta(-\mathbf{q}_2)S^{\alpha\beta}(\mathbf{q}_1,\mathbf{q}_2)J_\pi^\alpha(-\mathbf{q}_1)\right]$$

$$\to \int_0\int_0 \{J_1^\alpha J_2^\beta, S_{12}^{\alpha\beta}\} \ , \qquad (20)$$

where the last form is developed by using the symmetries (19) and dropping one-body terms.

The Adler-Weisberger effective interaction has a time-derivative term which generates the seagull in Fig. 2(d) of the form

$$\int\int \bar{S}^{\alpha\beta}(\mathbf{x},\mathbf{y})\pi^\alpha(\mathbf{x},t)\dot{\pi}^\beta(\mathbf{y},t) \ ,$$

where

$$\bar{S}^{\alpha\beta}(\mathbf{q}_1,\mathbf{q}_2) = -\bar{S}^{\beta\alpha}(\mathbf{q}_2,\mathbf{q}_1) \ . \qquad (21)$$

A tedious repetition of the previous calculation leads to unretarded seagull terms which are two-body (only), and retarded three-body terms

$$\Delta V_\pi^{\mathrm{SG}'} = i[H,V''_{2\pi}] - i\int_0\int_0 \{\bar{S}_{12}^{\alpha\beta},\{[H,J_1^\alpha],J_2^\beta\}\} \ , \qquad (22a)$$

where

$$V''_{2\pi} = \tfrac{1}{8}\int_0\int_0 \{\bar{S}_{12}^{\alpha\beta},\{J_1^\alpha,J_2^\beta\}\}\left[\frac{E_2-E_1}{E_1+E_2}\right] \ . \qquad (22b)$$

As before, the $V''_{2\pi}$ term is unique and can be removed by a unitary transformation, $V''_{2\pi}$. Seagull diagrams involving two time derivatives are of higher order and can be ignored.

V. RESULTS

Our formal expressions in the preceding sections must be evaluated in order to generate explicit forms for the three-body force. Because these forms are lengthy and there are numerous separate terms, we define

$$\Delta V_\pi = \frac{f^4}{4M}\sum_{i\neq j\neq k} \tau_i\cdot\tau_j A_{ijk} + \tau_k\cdot\tau_i\times\tau_j B_{ijk} \ , \qquad (23a)$$

where

$$A_{ijk}=(\sigma_i\cdot\nabla_{ki})(\sigma_j\cdot\nabla_{kj})a_{ijk}+(\sigma_j\cdot\nabla_{kj})\{\sigma_i\cdot\mathbf{p}_i,c_{ijk}\}\ , \tag{23b}$$

and

$$B_{ijk}=(\sigma_i\cdot\nabla_{ki})(\sigma_j\cdot\nabla_{kj})b_{ijk}+(\sigma_j\cdot\nabla_{kj})\{\sigma_i\cdot\mathbf{p}_i,d_{ijk}\}\ . \tag{23c}$$

The unretarded seagull diagrams [Eq. (20] generate $c=d=0$ and

$$a_{ijk}^{\rm SG}=(1-\mu)[h_0(x_{kj})\nabla_{ki}^2h_0(x_{ki})+h_0(x_{ki})\nabla_{kj}^2h_0(x_{kj})]-(1-\mu)\{\sigma_k\times\mathbf{p}_k\cdot\nabla_k,h_0(x_{ki})h_0(x_{kj})\} \tag{24a}$$

and

$$b_{ijk}^{\rm SG}=-(1-\mu-1/g_A^2)\{\mathbf{p}_k\cdot(\nabla_{kj}-\nabla_{ki}),h_0(x_{ki})h_0(x_{kj})\}+(2/g_A^2)\sigma_k\cdot\nabla_{ki}h_0(x_{ki})\times\nabla_{kj}h_0(x_{kj})\ , \tag{24b}$$

where

$$h_0(z)=2\int_0 e^{i\mathbf{q}\cdot\mathbf{z}}=e^{-\mu_\pi z}/4\pi z\ . \tag{24c}$$

The retarded contributions from Eq. (22a) provide one additional seagull term. We ignore all contributions of shorter than two-pion range, which arise from $[V,J_\pi^0]$, and find $a'=c'=d'=0$ and

$$b_{ijk}^{\prime\rm SG}=(2/g_A^2)\{\mathbf{p}_i\cdot\nabla_{ki},h_0(x_{ki})h_0(x_{kj})\}\ , \tag{25}$$

which completes the seagull terms.

Additional forces are generated by the vertex J'_π in Eq. (16), which are dependent on the parameter μ, as are the first three terms in Eq. (24). The second term in Eq. (16) produces $c'^R=d'^R=0$ and

$$\bar a_{ijk}^{\prime R}=(\mu+1)([h_0(x_{kj})\nabla_{ki}^2h_0(x_{ki})+h_0(x_{ki})\nabla_{kj}^2h_0(x_{kj})]-\{\sigma_k\times\mathbf{p}_k\cdot\nabla_k,h_0(x_{kj})h_0(x_{ki})\})\ , \tag{26a}$$

and

$$\bar b_{ijk}^{\prime R}=-(\mu+1)\{\mathbf{p}_k\cdot(\nabla_{kj}-\nabla_{ki}),h_0(x_{ki})h_0(x_{kj})\}\ . \tag{26b}$$

Adding Eqs. (24) and (26) together produces a result independent of μ. The entire dependence on that parameter resides in the first (unitary transformation) term in Eq. (16). This result is clearly necessary in order that the transformation on the nucleon lines performed in Eq. (4) have a sensible interpretation. The first term in Eq. (16) produces a three-body force ΔV_π^μ when H is replaced by V_π, with the form

$$a_{ijk}^\mu=-(\mu+1)[h_0(x_{ki})\nabla_{kj}^2h_0(x_{kj})-\{\sigma_k\times\mathbf{p}_k\cdot\nabla_{kj},h_0(x_{kj})h_0(x_{ki})\}]\ , \tag{27a}$$

$$c_{ijk}^\mu=(\mu+1)\sigma_k\cdot\nabla_{kj}h_0(x_{kj})\times\nabla_{ki}h_0(x_{ki})\ , \tag{27b}$$

$$b_{ijk}^\mu=(\mu+1)\{\mathbf{p}_k\cdot\nabla_{kj},h_0(x_{kj})h_0(x_{ki})\}\ , \tag{27c}$$

$$d_{ijk}^\mu=-(\mu+1)\nabla_{kj}h_0(x_{kj})\cdot\nabla_{ki}h_0(x_{ki})\ . \tag{27d}$$

This completes the treatment of μ-dependent terms.

The remaining components of the three-body force are determined by the parameter ν and explicitly involve retardation of the OPEP. They are obtained from Eqs. (15) by replacing H with $T+V_\pi$; we find

$$\Delta V_\pi^R=\int_2\{[T,J_1],[V_\pi,J_1]\}-\frac{1-\nu}{2}\int_2([T,[V_\pi,J_1J_1]]+[V_\pi,[T,J_1J_1]])\ . \tag{28}$$

The first part of (28) (for $\nu=1$) comes from $[J_1,H]^2$, while the second term has its origin in the unitary transformation alone. The various commutators are straightforward to evaluate, but the final result is not simply related to the OPEP because of the index 2 on the Fourier transform integral, which requires a digression.

In Ref. 24 we noted that this form arises from expanding the relativistic OPEP propagator in momentum space about the point $q_0=0$:

$$V_\pi^0\sim f^2/[(\mathbf{q}^2+\mu_\pi^2)-q_0^2]\cong f^2/(\mathbf{q}^2+\mu_\pi^2)+q_0^2f^2/(\mathbf{q}^2+\mu_\pi^2)^2+\cdots$$
$$=f^2/(\mathbf{q}^2+\mu_\pi^2)-q_0^2\frac{d}{d\mathbf{q}^2}[f^2/(\mathbf{q}^2+\mu_\pi^2)]\ , \tag{29a}$$

for some "value" of the parameter q_0. The entire, lengthy renormalization procedure developed in this work has been devoted to defining what we mean by q_0 in a many-body system. If we introduce a π-N form factor, $F_{\pi\rm N}(\mathbf{q}^2-q_0^2)$, in the usual way[14,15] we generate[24]

$$V_\pi^0\cong[f^2F_{\pi\rm N}^2(\mathbf{q}^2)/(\mathbf{q}^2+\mu_\pi^2)]-q_0^2\frac{d}{d\mathbf{q}^2}[f^2F_{\pi\rm N}^2(\mathbf{q}^2)/(\mathbf{q}^2+\mu_\pi^2)]+\cdots\ , \tag{29b}$$

which tells us how to treat the retardation consistently, even when a πN form factor is included. Introducing this form factor leads to the following replacement for any function, $f(\mathbf{q})$:

$$\int_2\mathbf{q}f(\mathbf{q})\equiv\int\frac{d^3q\,\mathbf{q}f(\mathbf{q})}{(2\pi)^32E^4}\to-\tfrac{1}{2}\int\frac{d^3q\,f(\mathbf{q})}{(2\pi)^32}\nabla_q\left[\frac{F_{\pi\rm N}^2(\mathbf{q}^2)}{\mathbf{q}^2+\mu_\pi^2}\right]$$
$$=\tfrac{1}{2}\int\frac{d^3q}{2(2\pi)^3}[\nabla_qf(\mathbf{q})]\left[\frac{F_{\pi\rm N}^2(\mathbf{q}^2)}{\mathbf{q}^2+\mu_\pi^2}\right]\to\tfrac{1}{2}\int_0\nabla_qf(\mathbf{q})\ . \tag{30}$$

We emphasize that this procedure is the only one which correctly incorporates the π-N form factor in the retarded OPEP, where instead of Eq. (24c) we have henceforth

$$h_0(z)=2\int_0 e^{i\mathbf{q}\cdot\mathbf{z}}=\int\frac{d^3q\,e^{i\mathbf{q}\cdot\mathbf{z}}}{(2\pi)^3}\left[\frac{F_{\pi N}^2(\mathbf{q}^2)}{\mathbf{q}^2+\mu_\pi^2}\right] . \tag{31}$$

Using this prescription for treating retardation, we obtain the final form of ΔV_π^R, with $c^R=d^R=0$ and

$$a_{ijk}^R=-(1-\nu)(\nabla_{kj}\cdot\nabla_{ki})\nabla_{kj}h_0(x_{kj})\cdot\mathbf{x}_{ki}h_0(x_{ki})$$
$$-\{2(1-\nu)\mathbf{p}_{ki}\cdot\nabla_{ki}+(1-\nu)\mathbf{p}_{kj}\cdot\nabla_{kj}+2\mathbf{p}_i\cdot\nabla_{ki},\,\sigma_k\cdot\nabla_{kj}h_0(x_{kj})\times\mathbf{x}_{ki}h_0(x_{ki})\} , \tag{32a}$$

$$b_{ijk}^R=-(1-\nu)(\nabla_{kj}\cdot\nabla_{ki})\sigma_k\cdot\nabla_{kj}h_0(x_{kj})\times\mathbf{x}_{ki}h_0(x_{ki})$$
$$+\{2(1-\nu)\mathbf{p}_{ki}\cdot\nabla_{ki}+(1-\nu)\mathbf{p}_{kj}\cdot\nabla_{ki}+2\mathbf{p}_i\cdot\nabla_{ki},\,\nabla_{kj}h_0(x_{kj})\cdot\mathbf{x}_{ki}h_0(x_{ki})\} , \tag{32b}$$

where $\mathbf{p}_{ki}\equiv\mathbf{p}_k-\mathbf{p}_i$, etc.

All but two of these terms are proportional to $1-\nu$ and arise from the double commutators in Eq. (28). The first term in (28) generates the two terms in (32) proportional to $\mathbf{p}_i$, which will not vanish when $\nu=1$. Adding the results of Eqs. (24)–(27) and (32) produces our final result for

$$\Delta V_\pi=\Delta V_\pi^{SG}+\Delta V_\pi^{SG'}+\Delta V_\pi'^{R}+\Delta V_\pi^{\mu}+\Delta V_\pi^{R} ,$$

which is

$$a_{ijk}=-(\mu-3)h_0(x_{ki})\nabla_{kj}^2h_0(x_{kj})-(1-\nu)(\nabla_{kj}\cdot\nabla_{ki})\nabla_{kj}h_0(x_{kj})\cdot\mathbf{x}_{ki}h_0(x_{ki})+(\mu-3)\{\sigma_k\times\mathbf{p}_k;\nabla_{kj}h_0(x_{kj})h_0(x_{ki})\}$$
$$-\{2(1-\nu)\mathbf{p}_{ki}\cdot\nabla_{ki}+(1-\nu)\mathbf{p}_{kj}\cdot\nabla_{kj}+2\mathbf{p}_i\cdot\nabla_{ki},\,\sigma_k\cdot\nabla_{kj}h_0(x_{kj})\times\mathbf{x}_{ki}h_0(x_{ki})\} , \tag{33a}$$

$$c_{ijk}=(\mu+1)\sigma_k\times\nabla_{ki}h_0(x_{ki})\cdot\nabla_{kj}h_0(x_{kj}) , \tag{33b}$$

$$b_{ijk}=\frac{2}{g_A^2}\sigma_k\cdot\nabla_{ki}h_0(x_{ki})\times\nabla_{kj}h_0(x_{kj})-(1-\nu)(\nabla_{kj}\cdot\nabla_{ki})\sigma_k\cdot\nabla_{kj}h_0(x_{kj})\times\mathbf{x}_{ki}h_0(x_{ki})$$
$$-\frac{2}{g_A^2}\{\mathbf{p}_{ki}\cdot\nabla_{ki},\,h_0(x_{ki})h_0(x_{kj})\}+(\mu-3)\{\mathbf{p}_k\cdot\nabla_{kj},\,h_0(x_{kj})h_0(x_{ki})\}$$
$$+\{2(1-\nu)\mathbf{p}_{ki}\cdot\nabla_{ki}+(1-\nu)\mathbf{p}_{kj}\cdot\nabla_{kj}+2\mathbf{p}_i\cdot\nabla_{ki},\,\nabla_{kj}h_0(x_{kj})\cdot\mathbf{x}_{ki}h_0(x_{ki})\} , \tag{33c}$$

$$d_{ijk}=-(\mu+1)\nabla_{kj}h_0(x_{kj})\cdot\nabla_{ki}h_0(x_{ki}) . \tag{33d}$$

The final result is quite complex: 15 terms, of which five depend on μ and six on ν, 11 are momentum dependent, and only four are local. The terms proportional to the unitary parameters μ and ν can be easily checked, because they are generated by commutators. Moreover, six terms, including the c and d terms, can be completely checked by verifying the requirements of special relativity, which we do below.

If we change momentum variables to relative ($\boldsymbol{\pi}_i$) and (nuclear) center of mass ($\mathbf{P}$) for the ith nucleon, and define $m_t=AM$, we find

$$\mathbf{p}_i=\boldsymbol{\pi}_i+M\mathbf{P}/m_t , \tag{34}$$

the terms proportional to $\mathbf{P}$ in the potential, $V(\mathbf{P})$, play a special role. It was shown in Ref. 24 that the wave function of a many-body system must have the form

$$\Psi_{\mathbf{P}}\cong[1-i\chi(\mathbf{P})]\Psi_0 , \tag{35}$$

to order $(v/c)^2$. The requirement that the energy eigenvalue have the obvious form $[\mathbf{P}^2+(m_t+h)^2]^{1/2}$ leads to

$$V(\mathbf{P})\cong-\frac{\mathbf{P}^2V_0}{2m_t^2}-i[\chi_v(\mathbf{P}),h]-i[\chi_0(\mathbf{P}),V_0] , \tag{36a}$$

where V_0 is the nonrelativistic part of V, h is the nonrelativistic Hamiltonian, and χ_0 and χ_v are the potential-independent and potential-dependent parts of χ, respectively. The latter consists of two terms, given by Eq. (A21) in the Appendix. It was verified in Ref. 24 that the two-body relativistic corrections to the OPEP satisfy (36a). The three-body parts must satisfy

$$V_{3N}(\mathbf{P})+i[\chi_v,V_\pi^0]=0 . \tag{36b}$$

Inserting (34) into (33) generates $V_{3N}(\mathbf{P})$ and performing the commutator verifies (36b). Because momentum difference terms (e.g., $\mathbf{p}_{ki}$) do not contribute to $V_{3N}(\mathbf{P})$, only six of the 11 momentum-dependent terms are checked in this way. We emphasize that these six terms are verified completely by the requirements of special relativity.

The final check is a comparison with previous Tucson-Melbourne results. We have arranged our expressions so that nucleon "k" rescatters the pion. Thus our "k" is equivalent to the label "1" of Ref. 17, while "i" and "j" correspond to "2" and "3." A detailed comparison with the results of Ref. 17 determines that $\mu=-1$. This is no surprise,[25] since that choice corresponds to working with on-shell (i.e., "free") spinor amplitudes. Moreover, all of the $1-\nu$ terms as well as the $\mathbf{p}_i\cdot\nabla_{ki}$ terms in a_{ijk} and b_{ijk} are not present, because retardation in the pion propagator was not considered in that work. In addition, the $\sigma_k\times\mathbf{p}_k\cdot\nabla_k$ term in a_{ijk} is also missing. With those exceptions, our "purely pionic" results are in agreement with Ref. 17 for $\mu=-1$. The $\mathbf{p}_1\cdot\mathbf{p}_1'$ term in the potential of Ref. 18(a) is incompatible with the requirements of relativity and should be dropped; we have no such term. See also the comments of Ref. 18(b).

Finally, we note that there have been a number of incomplete or inconsistent calculations[19–22] of the overlapping and disconnected diagrams. All of these calculations founder on their treatment of the energy-dependent terms in Eq. (8), determined by V_π'. If the formalism which is used replaces H by T (this is the usual case), there is no obvious way that these terms can cancel, as they must. Note that the function V_π' is proportional to derivatives of the Bessel function, K_0, while all of our results depend on $K_{1/2}$, which is simply expressible in terms of exponentials. All previous calculations of the various time-ordered second-order (in f^2) processes obtained erroneous results determined by V_π'; the errors are signaled by the appearance of the functions K_0 and K_1 in their potentials. The iterated OPEP has the form $V_\pi G V_\pi$, and since the Green's function G behaves as $(p^2/M)^{-1}\sim M$, these processes are of order MV_π^2. As we emphasized earlier, our three-body force is of order V_π^2/M, or a $1/M^2$ [i.e., $(v/c)^2$] correction. Unless one introduces an energy-dependent potential, there are no operators of order V_π^2 (i.e., a v/c correction). This has been noted recently by Glöckle and Yang,[23] who state that the time-ordered processes should not be kept unless a correct relativistic treatment is made; we have performed such a calculation here.

VI. DISCUSSION AND CONCLUSIONS

The results of the preceding section complete the treatment of the purely pionic 2π3BF term; in addition, the Appendix contains the two-body relativistic corrections in the OPEP of order V_π/M^2. Both sets of potentials depend on the arbitrary parameters μ and ν. The (unitary) form of the result guarantees that matrix elements of the complete potential in a consistent calculation are independent of those parameters. The parameter μ labels the (incomplete) transformation from the Weinberg nonlinear model to the nonlinear σ model, with $\mu=1$ corresponding to the former (PV) and $\mu=0$ to the latter (PS). We have adopted a (renormalized) form of these equivalent (effective) Lagrangians which guarantees the proper form at the symmetry points. The Tucson-Melbourne potential in its original form corresponds to $\mu=-1$, and is missing one nonlocal term which depends on μ.

The parameter ν labels the manner in which the effects of retardation of the pion propagator are treated, and corresponds to the folded diagram ambiguity of Johnson.[33] As previously noted, the soft representation ($\nu=1$) eliminates many 3BF contributions, and that is true in this work as well. Most of the terms we have calculated that are not found in the original Tucson-Melbourne work are eliminated by choosing $\nu=1$. The latter work made no attempt to calculate the effect of pion-propagator modification.

The procedure we have advocated here differs in principle from that of Ref. 17. In that work the (off-shell) π-N scattering amplitude was expanded in powers of pion and nucleon momenta. In contradistinction, in this work we have expanded in powers of $1/M$. The results are unchanged by the difference in expansion philosophy. In future work, when we reincorporate the additional degrees of freedom (σ, ρ, etc.), this will not be the case.

Having gone to great lengths to define the (properly subtracted) 2π3BF corresponding to purely pionic terms, we should note that to the best of our knowledge none of the momentum-dependent terms in ΔV_π have ever been numerically estimated. It is not clear that they have a significant size compared to the Δ or σ contributions, for example. It is important that these estimates be made.

Finally, it is obvious that a consistent treatment of operators and wave functions needs to be performed in order that nuclear matrix elements be well defined. That is, wave functions should be calculated using two- and three-body potentials with the *same* values of μ and ν as the operators whose matrix elements we wish to calculate. It is not clear how important the ambiguous terms are in our three-nucleon force. No numerical estimate of any of the previously calculated pieces has been made.

The work of one of us (S.A.C.) was supported in part by the U.S. National Science Foundation, while that of J.L.F. was performed under the auspices of the U.S. Department of Energy.

APPENDIX

We present here the results of previous calculation of the (two-body) relativistic corrections to the OPEP of order V_π/M^2 and to the charge operator. We present the former in two forms: for a general many-body system and for the center of mass of a two-body system. We follow Ref. 24, Eqs. (61)–(65), and (57)–(60) for $V_\pi^{\rm ret}$, and write

$$V_\pi=V_\pi^0+V_\pi^2+V_\pi^{\sigma'}+V_\pi^{\nu'}+V_\pi^{\mu'}+V_\pi^{\rm ret}\ , \qquad (A1)$$

where the leading-order (nonrelativistic) OPEP is given by

$$V_\pi^0=\frac{f^2}{2}\sum_{i\neq j}\tau_i\cdot\tau_j\sigma_i\cdot\nabla_{ij}\sigma_j\cdot\nabla_{ij}h_0(x_{ij})$$

$$\rightarrow f^2 t_{12}\sigma_1\cdot\nabla\,\sigma_2\cdot\nabla h_0(r)\ , \qquad (A2)$$

while

$$V_\pi^2 = -\frac{f^2}{4M^2}\sum_{i\neq j}\tau_i\cdot\tau_j\sigma_i\cdot\nabla_{ij}\sigma_j\cdot\nabla_{ij}\{\mathbf{p}_i^2, h_0(x_{ij})\}$$
$$\to -\{\mathbf{p}^2, V_\pi^0\}/2M^2 . \tag{A3}$$

The remaining contributions depend on the parameters μ and ν:

$$V_\pi^{\sigma'} = \frac{\mu f^2}{8M^2}\sum_{i\neq j}\tau_i\cdot\tau_j\sigma_j\cdot\nabla_{ij}\{\mathbf{p}_i\cdot\nabla_{ij}, \{\sigma_i\cdot\mathbf{p}_i, h_0(x_{ij})\}\}$$
$$\to\mu\left\{\frac{\mathbf{p}\cdot\nabla}{M}, U_E\right\} = i\mu[T, U_E] , \tag{A4}$$

where

$$U_E = \frac{f^2}{8M}\sum_{i\neq j}\tau_i\cdot\tau_j\{\sigma_i\cdot\mathbf{p}_i, \sigma_i\cdot\nabla_{ij}h_0(x_{ij})\}$$
$$\to\frac{f^2}{8M}t_{12}(\{\sigma_1\cdot\mathbf{p}, \sigma_2\cdot\nabla h_0\} + \{\sigma_2\cdot\mathbf{p}, \sigma_1\cdot\nabla h_0\}) . \tag{A5}$$

Moreover,

$$V_\pi^{v'} = -\frac{f^2(\mu-1)}{2M}\sum_{i\neq j}\tau_i\cdot\tau_j\sigma_j\cdot\nabla_{ij}h_0(x_{ij})\sigma_i\cdot\nabla_i V^0$$
$$= 2i(\mu-1)[V^0, U_E] , \tag{A6}$$

where V^0 is the spin- and isospin-independent nonrelativistic potential, which, together with the kinetic energy, forms h. In addition,

$$V_\pi^{\mu'} = -\frac{f^2(\mu+1)}{8M^2}\sum_{i\neq j}\tau_i\cdot\tau_j\sigma_j\cdot\nabla_{ij}\{\mathbf{p}_i\cdot\nabla_{ij}, \{\sigma_j\cdot\mathbf{p}_i, h_0(x_{ij})\}\} + i(\mu+1)[h-2V^0, U_E] \to -i(\mu+1)[V^0, U_E] . \tag{A7}$$

The retardation potential has the form

$$V_\pi^{\text{ret}} = i[h, U_R](1-\nu) + \frac{f^2}{16M^2}\sum_{i\neq j}\tau_i\cdot\tau_j\sigma_j\cdot\nabla_{ij}\sigma_j\cdot\nabla_{ij}\{\mathbf{p}_i\cdot\nabla_{ij}, \{\mathbf{p}_j;\mathbf{x}_{ij}h_0(x_{ij})\}\} \to i[h, U_R](1-\nu) - \frac{i}{2}[T, U_R] , \tag{A8}$$

where

$$U_R = -\frac{i}{2}[h, V_\pi''] = \frac{if^2}{8}\left[h, \sum_{i\neq j}\tau_i\cdot\tau_j\sigma_j\cdot\nabla_{ij}\sigma_i\cdot\mathbf{x}_{ij}h_0(x_{ij})\right]$$
$$= \frac{f^2}{8M}\sum_{i\neq j}\tau_i\cdot\tau_j\sigma_i\cdot\nabla_{ij}\sigma_j\cdot\nabla_{ij}\{\mathbf{p}_i;\mathbf{x}_{ij}h_0\} + \frac{f^4}{2}\sum_{i\neq j\neq k}(\tau_k\cdot\tau_i\times\tau_j)\sigma_i\cdot\nabla_{ki}\sigma_j\cdot\nabla_{kj}\nabla_{kj}h_0(x_{kj})\cdot\mathbf{x}_{ki}h_0(x_{ki})$$
$$-(\tau_i\cdot\tau_j)\sigma_i\cdot\nabla_{ki}\sigma_j\cdot\nabla_{kj}\sigma_k\times\nabla_{kj}h_0(x_{kj})\cdot\mathbf{x}_{ki}h_0(x_{ki})$$
$$\to\frac{f^2}{4M}t_{12}\sigma_1\cdot\nabla\,\sigma_2\cdot\nabla\{\mathbf{p};\mathbf{r}h_0\} . \tag{A9}$$

In the two-body (limiting) forms we have defined $\mathbf{r}=\mathbf{x}_{12}$, $\mathbf{p}=\mathbf{p}_1=-\mathbf{p}_2$, $t_{12}=\tau_1\cdot\tau_2$, and $\nabla=\nabla_{12}=-\nabla_{21}$. The terms proportional to μ and ν sum to the commutators $i[h, U_E]$ and $i[h, U_R]$, as they must. Moreover, the two-body form of U_R satisfies the additional relationship

$$U_R = 2U_E + U_G , \tag{A10}$$

where the Gross transformation is given by[24]

$$U_G = \frac{1}{8M}\sum_{i\neq j}\{\mathbf{p}_i, \cdot\mathbf{x}_{ij}V_\pi^0(ij)\} \to \frac{1}{4M}\{\mathbf{p}, \cdot\mathbf{r}V_\pi^0\} . \tag{A11}$$

Adding the (two-body case) terms together, we obtain, for the OPEP,

$$V_\pi = V_\pi^0 - \{\mathbf{p}^2, V_\pi^0\}/2M^2 + i[(1-\nu)V^0 + (\tfrac{1}{2}-\nu)T, U_G]$$
$$+ i[(\mu-2\nu-1)V^0 + (\mu-2\nu+1)T, U_E] . \tag{A12}$$

For completeness we include the corresponding momentum-dependent and ambiguous terms for a scalar- and a vector-exchange potential:

$$V = V_S^0 + V_V^0 + \{\mathbf{p}^2, V_V^0 - V_S^0\}/2M^2$$
$$+ i(\tfrac{1}{2}-\nu)[h, U_G^0] + \frac{i}{2}[V_V^0 + V_S^0, U_G^0] , \tag{A13}$$

where

$$U_G^0 = \frac{1}{4M}\{\mathbf{p}, \cdot\mathbf{r}(V_S^0 + V_V^0)\} \tag{A14}$$

is expressed in terms of the nonrelativistic components, V_S^0 and V_V^0, of the scalar- and vector-exchange potential. No analogue of the parameter μ is appropriate for $(v/c)^2$ corrections to these potentials. Equation (A12) was derived to order f^2 only; consequently, the potential V should not include the OPEP. This equation nevertheless illustrates all of the ambiguity problems we discussed earlier.

It is possible in Eqs. (A12) and (A13) to eliminate by means of a choice of representation certain of the momentum-dependent terms from $[T,U]$, at the cost of introducing shorter range local terms from $[V,U]$. The latter terms for the many-body problem generate three-body forces, however. Any representation for which $\beta=\mu-2\nu+1$ vanishes will lack the $[T,U_E]$ terms, while choosing $\nu=\frac{1}{2}$ [no-retardation representation] eliminates $[T,U_G]$. The choices $\mu=0, \nu=\frac{1}{2}$ eliminate both. Perhaps by way of coincidence, the $E1$ pion-exchange charge operator corrections to deuteron forward photodisintegration (to be discussed below) were shown to be very small for $\beta=0$. The various representations we have discussed here and elsewhere[24,25] are $\nu=[0, \frac{1}{2}, 1]$, [standard, no retardation, soft]; and $\mu=[-1,0,1,3]$, [free, PS, PV, soft].

Pion exchange also influences transition operators, such as the nuclear charge density $\rho_{\rm ch}(\mathbf{r})$ or, equivalently, the charge form factor $F_{\rm ch}(\mathbf{q})$. The impulse approximation form factor operator is

$$F_0(\mathbf{q})=\sum_k \tfrac{1}{2}(G_E^s+G_E^v\tau_k^3)e^{i\mathbf{q}\cdot\mathbf{x}_k}\ , \tag{A15}$$

where the nucleon intrinsic charge form factors are $G_E^s=G_E^{\rm p}+G_E^{\rm n}$, $G_E^v=G_E^{\rm p}-G_E^{\rm n}$, $G_E=F_1-q^2F_2/4M^2$, and $G_M=F_1+F_2$. Here, p, n, s, and v refer to proton, neutron, isoscalar, and isovector, respectively. The true (pion) exchange charge form factor is

$$F_{\rm ex}^{\pi}=-\frac{f^2F_\pi}{2M}\sum_{i\neq j}(\boldsymbol{\tau}_i\times\boldsymbol{\tau}_j)_3\boldsymbol{\sigma}_i\cdot\boldsymbol{\nabla}_i\boldsymbol{\sigma}_j\cdot\boldsymbol{\nabla}_j\left\{\mathbf{p}_i\cdot\boldsymbol{\nabla}_i,\ \int d^3x\, e^{i\mathbf{q}\cdot\mathbf{x}}\bar{h}_0(|\mathbf{x}-\mathbf{x}_i|)\bar{h}_0(|\mathbf{x}-\mathbf{x}_j|)\right\}\ , \tag{A16}$$

where F_π is the pion charge form factor and $\bar{h}_0$ contains only a single $F_{\pi\rm N}$. In addition, there are a seagull term and two recoil graph (vertex) contributions:

$$F_\pi^{\rm SG}=\frac{f^2}{4M}\sum_{i\neq j}F_1^V(\mu+1)(\boldsymbol{\tau}_i\times\boldsymbol{\tau}_j)_3\{\boldsymbol{\sigma}_i\cdot\mathbf{p}_i, e^{i\mathbf{q}\cdot\mathbf{x}_i}\boldsymbol{\sigma}_j\cdot\boldsymbol{\nabla}_{ij}h_0(x_{ij})\}+i(1-\mu)\boldsymbol{\sigma}_i\cdot\mathbf{q}\,\boldsymbol{\sigma}_j\cdot\boldsymbol{\nabla}_{ij}h_0(x_{ij})e^{i\mathbf{q}\cdot\mathbf{x}_i}(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_jF_1^s+\tau_j^3F_1^v)\ , \tag{A17}$$

$$F_\pi'^{v}=\frac{f^2}{8M}(\mu+1)\sum_{i\neq j}\boldsymbol{\sigma}_j\cdot\boldsymbol{\nabla}_{ij}\{i\boldsymbol{\sigma}_i\cdot\mathbf{q}\,e^{i\mathbf{q}\cdot\mathbf{x}_i}h_0(x_{ij})(F_1^s(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)+F_1^v\tau_j^3)-F_1^v(\boldsymbol{\tau}_1\times\boldsymbol{\tau}_j)_3\{\boldsymbol{\sigma}_i\cdot\mathbf{p}_i, h_0(x_{ij})(e^{i\mathbf{q}\cdot\mathbf{x}_i}+e^{i\mathbf{q}\cdot\mathbf{x}_j})\}\}\ , \tag{A18}$$

$$\widetilde{F}_\pi^{v}=i(1-\nu)[F_0,U_R]+\frac{f^2F_1^v}{4M}\sum_{i\neq j}(\boldsymbol{\tau}_i\times\boldsymbol{\tau}_j)_3\boldsymbol{\sigma}_i\cdot\boldsymbol{\nabla}_{ij}\boldsymbol{\sigma}_j\cdot\boldsymbol{\nabla}_{ij}\{\mathbf{p}_j;\mathbf{x}_{ij}e^{i\mathbf{q}\cdot\mathbf{x}_i}h_0(x_{ij})\}\ , \tag{A19}$$

where the former recoil graph contribution (A18) arises from J_π' and the latter (A19) from J_π^0. Note that the latter three terms depend on μ and ν. It is easy to show that all of the μ dependence in (A17) plus (A18) has the form $i\mu[F_0,U_E]$, as one might surmise. In addition, there is a contribution from nuclear motion,

$$F_m(\mathbf{q})=\frac{i}{2}\{\chi_v(\mathbf{q}),F_0(\mathbf{q})\}\ , \tag{A20}$$

where

$$\chi_v(\mathbf{P})=-\frac{1}{2m_t}\sum_{i\neq j}\mathbf{x}_i'\cdot\mathbf{P}\,V_{ij}^0+\frac{f^2(\mu-1)}{4m_t}\sum_{i\neq j}\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j\boldsymbol{\sigma}_i\cdot\mathbf{P}\,\boldsymbol{\sigma}_j\cdot\boldsymbol{\nabla}_{ij}h_0(x_{ij})\ . \tag{A21}$$

These results for $F_{\rm ch}$ assume a chiral coupling model (PV); *pure* PS coupling leads to an unphysical result which involves the nucleon anamolous magnetic moments[34] F_2.

Specializing to the deuteron ground state, one finds

$$F_{\rm ch}(q^2)=F_0+i(1-\nu)[F_0,U_G]+i(\mu-2\nu-1)[F_0,U_E]\ . \tag{A22}$$

The previously found ambiguities affect every term but the impulse approximation. The choices $\mu=3$, $\nu=1$ [soft, soft] completely eliminate the exchange terms. These are, however, not the choices which eliminate the undesirable $\mathbf{p}$ terms in the potential of Eq. (A12).

[1]B. H. J. McKellar and R. Rajaraman, in *Mesons in Nuclei*, edited by M. Rho and D. Wilkinson (North-Holland, Amsterdam, 1979), p. 358.

[2]B. H. J. McKellar and W. Glöckle, Nucl. Phys. **A416**, 435c (1984).

[3]W. Glöckle and P. U. Sauer, Europhys. News **15-2**, 5 (1984).

[4]J. L. Friar, B. F. Gibson, and G. L. Payne, Annu. Rev. Nucl. Part. Sci. **34**, 403 (1984). This reference and the preceding three review three-nucleon forces extensively.

[5]H. Primakoff and T. Holstein, Phys. Rev. **55**, 1218 (1939).

[6]J.-I. Fujita and H. Miyazawa, Prog. Theor. Phys. **17**, 360 (1957).
[7]B. M. Axilrod and E. Teller, J. Chem. Phys. **11**, 299 (1943); Y. Muto, Proc. Phys. Math. Soc. (Jpn.) **17**, 629 (1943).
[8]M. Scadron, *Advanced Quantum Mechanics* (Springer, Heidelberg, 1979).
[9]D. K. Campbell, in *Nuclear Physics with Heavy Ions and Mesons,* edited by R. Balian *et al.* (North-Holland, Amsterdam, 1978), p. 551.
[10]M. Scadron, in *Few-Body Dynamics,* edited by A. N. Mitra *et al.* (North-Holland, Amsterdam, 1976), p. 325; Rep. Prog. Phys. **44**, 213 (1981).
[11]G. E. Brown, A. M. Green, and W. J. Gerace, Nucl. Phys. **A115**, 435 (1968).
[12]S. N. Yang, Phys. Rev. C **10**, 2067 (1974).
[13]G. Höhler, in *Landolt-Börnstein (New Series),* edited by H. Schopper (Springer, Berlin, 1983), Vol. I/9b, Pt. 2.
[14]S. A. Coon and M. D. Scadron, Phys. Rev. C **23**, 1150 (1981).
[15]C. A. Dominguez, Riv. Nuovo Cimento **8**, No. 6 (1985).
[16]S. A. Coon, B. R. Barrett, M. D. Scadron, D. W. E. Blatt, and B. H. J. McKellar, in *Few-Body Dynamics* (Ref. 10), p. 739.
[17]S. A. Coon, M. D. Scadron, P. C. McNamee, B. R. Barrett, D. W. E. Blatt, and B. H. J. McKellar, Nucl. Phys. **A317**, 242 (1979).
[18](a) S. A. Coon and W. Glöckle, Phys. Rev. C **23**, 1790 (1981); (b) W. Glöckle, *ibid.* **31**, 1045 (1985).
[19]K. A. Brueckner, C. A. Levinson, and H. M. Mahmoud, Phys. Rev. **95**, 217 (1954).
[20]K. Hasegawa, Prog. Theor. Phys. **30**, 827 (1963).
[21]C. Pask, Phys. Lett. **25B**, 78 (1967).
[22]S. N. Yang (unpublished).
[23]S. N. Yang and W. Glöckle, Phys. Rev. C **33**, 1774 (1986).
[24]J. L. Friar, Ann. Phys. (N.Y.) **104**, 380 (1977).
[25]J. L. Friar, Phys. Rev. C **22**, 796 (1980). This work contains an extensive discussion of these problems.
[26]L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).
[27]J. L. Friar, Phys. Rev. C **27**, 2078 (1983).
[28]J. L. Friar (unpublished). In view of the results of Ref. 29, this slight modification of previously published work is not unexpected.
[29]S. Weinberg, Phys. Rev. Lett. **18**, 188 (1967).
[30]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).
[31]S. Weinberg, Phys. Rev. **166**, 1568 (1968).
[32]F. J. Dyson, Phys. Rev. **47**, 929 (1948).
[33]M. B. Johnson, Ann. Phys. (N.Y.) **97**, 400 (1976).
[34]J. L. Friar and B. F. Gibson, Phys. Rev. C **15**, 1779 (1977).