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Alternative interpretations of the many-particle Lippmann-Schwinger equation

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Possible alternative interpretations of the Lippmann-Schwinger integral equation for multiparticle ($n>2$) systems are investigated and are shown to be equivalent if integrals which occur are uniformly convergent, as is reasonable. At real energies E, the derivation of the Lippmann-Schwinger equation from the Schrödinger equation involves various surface integrals at infinity in configuration space. It is shown that the values of these surface integrals are related to the values of certain volume integrals at complex energies ($E+i\epsilon$) in the limit $\epsilon \to 0$, originally examined by Lippmann. It is further proved that a number of these surface integrals vanish together, a result which—though plausible—previously had to be assumed. The results of this paper confirm previous studies showing that the solutions to the multiparticle Lippmann-Schwinger equation need not be unique. Because of certain convergence difficulties which can occur, the analysis of this paper is not wholly valid for "three-body" collisions (defined as collisions involving three independently incident aggregates of the fundamental particles comprising the multiparticle system), or for the even more complicated collisions involving $n>3$ incident aggregates.

I. INTRODUCTION

Recently[1,2] we have examined the problem of the nonuniqueness of solutions to the Lippmann-Schwinger (LS) integral equation[3,4] in an exactly solvable three particle model due to McGuire,[5] involving three equal mass spin zero particles moving on the same straight line and interacting via pairwise attractive δ-function potentials of equal strength. More specifically, in Ref. 2 (hereafter referred to as paper I) we were able to demonstrate explicitly and analytically that in the McGuire model there are many solutions Ψ to the LS equation at real energies E,

$$\Psi=\psi_i - G_i^{(+)}(E)V_i\Psi , \tag{1.1}$$

where $\psi_i(E)$ is the "incident" wave and $G_i^{(+)}$ is the outgoing Green's function; here, and elsewhere in this paper unless otherwise stated, we use the notation employed in I.

A "proof" that solutions to Eq. (1.1) need not be unique was given first by Foldy and Tobocman,[6] using operator algebra techniques. They recognized that the nonuniqueness was implied by the existence of nontrivial solutions Ψ to the homogeneous LS equation $\Psi=-G_i^{(+)}V_i\Psi$; they inferred the existence of such solutions from the vanishing of certain limits as $\epsilon\to 0$, where $\epsilon>0$ is the imaginary part of the complex energy $E+i\epsilon$. These limits, known in the literature as Lippmann's identities, were derived originally by Lippmann,[4] also using operator techniques. Gerjuoy[7] has derived the LS equation (1.1) from the Schrödinger equation via conventional mathematical operations in configuration space; in this derivation the existence of solutions to the homogeneous LS equation can be inferred from the vanishing of certain integrals at infinity in configuration space. Gerjuoy did not discuss the relationship, if any, between his surface integrals and Lippmann's identities; he concluded the surface integrals would vanish on the basis of very plausible assumptions about the asymptotic behaviors of $G_i^{(+)}$ and other relevant functions. Even today, the asymptotic behaviors of these quantities have not been established. Therefore Gerjuoy's analysis, although straightforward and not obscured by operator techniques, cannot be termed mathematically rigorous, in the sense, e.g., of Faddeev's[8,9] treatment of the three-particle LS equation.

The nonuniqueness of solutions to the multiparticle LS equation has been questioned.[10,11] Mukherjee,[10] in a series of publications during the last decade, has insisted that solutions to Eq. (1.1) are unique. In so insisting Mukherjee claims to have found errors in both Gerjuoy's[7] and Lippmann's[4] rather different approaches to multiparticle scattering theory; he also attributes errors to later analyses[12] which extend and confirm Gerjuoy's approach. In particular, Mukherjee claims that Lippmann's identities *and* Gerjuoy's predicted values of the aforementioned surface integrals are wrong.

Very recently, Benoist-Gueutal[11] has published renewed criticisms of Gerjuoy's approach and of the analysis by Levin and Sandhas[13] which confirmed that approach. Benoist-Gueutal does not insist that solutions to Eq. (1.1) are unique, but she does contend the arguments of Gerjuoy[7] and Levin and Sandhas[13] are sufficiently faulty that whether solutions to Eq. (1.1) really are nonunique remains a doubtful question.

The disputes about the nonuniqueness of solutions to the LS equation appear to be associated with ambiguities in the meaning of the implied limit $\epsilon \to 0$ in the usual operator form[3] of the LS equation,

$$\Psi = \psi_i - \frac{1}{H_i - E - i\epsilon} V_i \Psi \ . \tag{1.2}$$

For example, as discussed more fully in later sections of this paper, it is not clear whether Ψ in Eq. (1.2) should be regarded as ϵ dependent when the limit $\epsilon \to 0$ is taken. Such ambiguities pertain not only to the LS equation (1.2), but also to various formulations of connected kernel equations for multiparticle scattering, e.g., to the Faddeev equations.[8,9] In fact, using one of the possible ways of taking the limit $\epsilon \to 0$, Komarov, Shablov, Popova, and Osborn[14] claimed that Faddeev-type equations may have nonunique scattering solutions. Subsequently[15] it was pointed out that Komarov *et al.* reached such a conclusion because of a misapplication of the $\epsilon \to 0$ limit.

The study reported in I largely was undertaken to refute Mukherjee's criticisms. The explicit analytic results of that study show that Mukherjee's criticisms are not well taken. At the very least, we asserted in I, it is necessary to explain why the results of I cannot be extrapolated to actual three-dimensional multiparticle systems, as well as why purported proofs that the solutions of Eq. (1.1) are unique should fail in one dimension but not in three dimensions. Benoist-Gueutal[11] has offered an explanation of why the results of I cannot be extrapolated to actual three-dimensional multiparticle systems, but for reasons which have been stated elsewhere[16] we do not agree with Benoist-Gueutal's criticisms.

Nevertheless, these continued criticisms of different aspects of Gerjuoy's[7] and Lippmann's[4] results have impelled us to reexamine the LS equation nonuniqueness problem, in order to clarify some issues which apparently continue to be sources of confusion in the literature. As foreshadowed above, this reexamination is concerned mainly with the various ways of taking the $\epsilon \to 0$ limit in Eq. (1.2). In so doing, we establish a heretofore unrecognized connection between Gerjuoy's surface integrals and Lippmann's identities, without recourse to operator techniques. This connection confirms the conclusion that the multiparticle LS equation has nonunique solutions, in that Lippmann's identities now are seen to validate the reasoning of Ref. 7, and vice versa. Still, a rigorous proof of Lippmann's identities or a rigorous evaluation of Gerjuoy's surface integrals does not seem possible at this time. Therefore, in future papers[17] we will show by explicit calculation that in the three particle McGuire model Gerjuoy's surface integrals and Lippmann's ($\epsilon \to 0$) limits have precisely the values predicted by these authors, thereby confirming the connection established in the present paper as well as further confirming the aforesaid conclusion of nonuniqueness.

Section II below presents the derivation of the connection between Lippmann's identities and the values of certain surface integrals in configuration space, while at the same time demonstrating the equivalence (under reasonable assumptions) of the various ways of taking the $\epsilon \to 0$ limit in Eq. (1.2). This section is divided into subsections, wherein the implications of the possible ways of taking the $\epsilon \to 0$ limit in Eq. (1.2) are examined. At the end of Sec. II we explain, as the reader now is cautioned to take note, that because of certain convergence difficulties our analysis is not valid for arbitrarily complicated collisions (though we do not doubt our conclusion that solutions to the multiparticle LS equation generally are not unique); what we mean by "arbitrarily complicated collisions" is made precise at the end of Sec. II. A summary of our results is presented in Sec. III. We also have included an Appendix proving that a sufficient condition for inverting the order of the operations limit as $\epsilon \to 0$ and integration over an infinite interval is the uniform convergence of the sequence of ϵ-dependent infinite integrals as $\epsilon \to 0$; this condition plays an important role in our analysis, and we have been unable to locate a proof in the literature to which readers usefully could be directed.

II. THE LS EQUATION AT REAL AND AT COMPLEX ENERGIES

The LS equation originally was derived[3,4] in the form (1.2), but the operator techniques employed throughout this derivation do not clearly reveal how the implied limit $\epsilon \to 0$ in Eq. (1.2) is to be interpreted. Later (still basically operator) formulations[18] of the LS equation also do not make the meaning of this limit precise, nor do they answer various quite fundamental questions which stem from this imprecision. In particular, it is not even clear from the operator formulations whether Ψ in Eq. (1.2) is ϵ dependent. In other words, are we to suppose $\Psi \equiv \Psi(E+i\epsilon)$ in Eq. (1.2) or is Ψ to be regarded as a $\Psi(E)$, quite independent of ϵ? Of course, ψ_i in Eqs. (1.1) and (1.2) satisfies

$$(H_i - E)\psi_i = 0 \ , \tag{2.1}$$

so that $\psi_i \equiv \psi_i(\mathbf{r};E)$ does not depend on ϵ.

If $\Psi \equiv \Psi(E)$ in Eq. (1.2), the limit $\epsilon \to 0$ must be taken within Eq. (1.2); otherwise the right-hand side of Eq. (1.2) would be ϵ dependent while the left-hand side would be ϵ independent. In the coordinate representation, the operator $(H_i - E - i\epsilon)^{-1}$ must be identical with the Green's function $G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)$ satisfying

$$(H_i - E - i\epsilon)G_i(\mathbf{r},\mathbf{r}';E+i\epsilon) = \mathbf{I} \equiv \delta(\mathbf{r}-\mathbf{r}') \ . \tag{2.2}$$

Then if Ψ is regarded as ϵ independent, we must interpret Eq. (1.2) either as

$$\text{(a)} \quad \Psi(\mathbf{r};E) = \psi_i(\mathbf{r};E) - \lim_{\epsilon \to 0} \int d\mathbf{r}' G_i(\mathbf{r};\mathbf{r}';E+i\epsilon) \times V_i(\mathbf{r}')\Psi(\mathbf{r}';E) \ , \tag{2.3}$$

or

(b) $$\Psi(\mathbf{r};E)=\psi_i(\mathbf{r};E)-\int d\mathbf{r}'\left[\lim_{\epsilon\to 0} G_i(\mathbf{r};\mathbf{r}';E+i\epsilon)\right]\times V_i(\mathbf{r}')\Psi(\mathbf{r}';E)\ . \tag{2.4}$$

No other—except Eqs. (2.3) and (2.4)—even remotely rational alternative interpretations of Eq. (1.2) appear to exist when $\Psi\equiv\Psi(E)$.

On the other hand, if $\Psi\equiv\Psi(E+i\epsilon)$, the limit $\epsilon\to 0$ cannot be confined to the right-hand side of Eq. (1.2) [as in Eqs. (2.3) or (2.4)] because then the left-hand side of Eq. (1.2) would be ϵ dependent while the right-hand side would be ϵ independent. Now Eq. (1.2) can be interpreted sensibly in the limit $\epsilon\to 0$ only by introducing

$$\hat{\Psi}(E)=\lim_{\epsilon\to 0}\Psi(E+i\epsilon)\ , \tag{2.5}$$

where $\Psi(E+i\epsilon)$ solves Eq. (1.2) written in the form

$$\Psi(E+i\epsilon)=\psi_i(E)-G_i(E+i\epsilon)V_i\Psi(E+i\epsilon)\ , \tag{2.6a}$$

i.e.,

$$\Psi(\mathbf{r};E+i\epsilon)=\psi_i(\mathbf{r};E)-\int d\mathbf{r}' G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\times V_i(\mathbf{r}')\Psi(\mathbf{r}';E+i\epsilon)\ . \tag{2.6b}$$

Hopefully, with the interpretation (2.6b) of Eq. (1.2), the limit $\hat{\Psi}(E)$ in Eq. (2.5) will exist and will be a solution of the real energy LS equation (1.1) having desired properties. We remark that the existence of the limit in Eq. (2.5) need not imply immediately that Eq. (2.6a) will yield

$$\lim_{\epsilon\to 0}\Psi(E+i\epsilon)=\psi_i(E)-\lim_{\epsilon\to 0}[G_i(E+i\epsilon)V_i\Psi(E+i\epsilon)]\ , \tag{2.6c}$$

i.e., need not imply that

$$\hat{\Psi}(\mathbf{r};E)=\psi_i(\mathbf{r};E)-\lim_{\epsilon\to 0}\int d\mathbf{r}' G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\times V_i(\mathbf{r}')\Psi(\mathbf{r}';E+i\epsilon)\ , \tag{2.6d}$$

because there is no immediate guarantee that the limit on the right-hand side of Eq. (2.6d) exists. We further remark that the foregoing alternative possible interpretations of the implied $\epsilon\to 0$ limit of Eq. (1.2) also pertain to the connected kernel equations.[15]

We proceed to explore the implications of the alternative possible interpretations of Eq. (1.2) which Eqs. (2.3), (2.4) and (2.6b) provide.

A. Implications of LS equation (2.4)

Equation (2.4) is identical to Eq. (1.1), because essentially by definition

$$G_i^{(+)}(E)=\lim_{\epsilon\to 0} G_i(E+i\epsilon),\ \ \epsilon>0\ . \tag{2.7}$$

Thus examining the implications of Eq. (2.4) inevitably involves some recapitulation of the results in Ref. 7, which concentrated on the connection between the real energy LS equation (1.1) and the real energy Schrödinger equation

$$(H-E)\Psi=0\ . \tag{2.8}$$

Let Ψ be a solution to Eq. (2.8) of form

$$\Psi=\psi_i+\Phi\ , \tag{2.9}$$

where the "scattered part" Φ may or may not be "everywhere outgoing," a phrase whose meaning requires discussion (see below); ψ_i is the "incident wave" satisfying Eq. (2.1). We seek to determine a sufficient condition for Ψ to satisfy Eq. (2.4). The real energy outgoing Green's function $G_i^{(+)}(E)$ satisfies

$$(H_i-E)G_i^{(+)}(\mathbf{r},\mathbf{r}';E)=\delta(\mathbf{r}-\mathbf{r}')\ . \tag{2.10}$$

Rewrite Eq. (2.8) in the form

$$(H_i-E)\Psi=-V_i\Psi\ . \tag{2.11}$$

Then multiply Eq. (2.10) on the left-hand side by $\Psi(\mathbf{r})$; multiply Eq. (2.11) on the left-hand side by $G_i^{(+)}(\mathbf{r},\mathbf{r}')$; subtract, and integrate over all $\mathbf{r}$. It follows, as in Ref. 7, after relabeling the variables so that the integrals are over $\mathbf{r}'$, that

$$\Psi(\mathbf{r})=\mathscr{I}[G_i^{(+)}(\mathbf{r},\mathbf{r}'),\Psi(\mathbf{r}')]-\int d\mathbf{r}' G_i^{(+)}(\mathbf{r},\mathbf{r}')V_i(\mathbf{r}')\Psi(\mathbf{r}')\ , \tag{2.12}$$

where $\mathscr{I}(X,Y)$ is the surface integral over the sphere at infinity in configuration space, defined by

$$\mathscr{I}(X,Y)=\int d\mathbf{r}[Y(\mathbf{r})TX(\mathbf{r})-X(\mathbf{r})TY(\mathbf{r})] \tag{2.13}$$

after applying Green's theorem in $\mathbf{r}$ space to the right-hand side of Eq. (2.13), wherein T is the kinetic energy operator in H (or H_i); of course, in Eq. (2.12) $\mathscr{I}$ is evaluated on the sphere at infinity in $\mathbf{r}'$ space, for arbitrary fixed $\mathbf{r}$.

Comparing Eqs. (2.4) and (2.12), it is apparent that a solution Ψ to the Schrödinger equation (2.8) also is a solution to the real energy LS equation in the form (2.4) when but only when

$$\mathscr{I}[G_i^{(+)}(\mathbf{r},\mathbf{r}'),\Psi(\mathbf{r}')]=\psi_i(\mathbf{r})\ . \tag{2.14}$$

Moreover, whenever Eq. (2.14) can be satisfied by more than one independent solution Ψ to (2.8), the LS equation (2.4) has nonunique solutions satisfying the Schrödinger equation.

Proceeding as in the preceding paragraphs, but starting with Eq. (2.1) instead of Eq. (2.11), we deduce

$$\mathscr{I}[G_i^{(+)}(\mathbf{r};\mathbf{r}'),\psi_i(\mathbf{r}')]=\psi_i(\mathbf{r})\ . \tag{2.15}$$

But any solution Ψ to Eq. (2.8) can be written in the form (2.9), if there is no requirement that Φ be everywhere outgoing. Comparing Eqs. (2.14) and (2.15), we conclude that a solution Ψ to the Schrödinger equation also satisfies the LS equation (2.4) when and only when the scattered part Φ of this solution satisfies

$$\mathscr{I}[G_i^{(+)}(\mathbf{r};\mathbf{r}'),\Phi(\mathbf{r}')]=0\ . \tag{2.16}$$

In other words, when Eq. (2.16) holds the corresponding Ψ defined by Eq. (2.9) necessarily satisfies Eq. (2.14) and—if also a solution of Eq. (2.8)—also satisfies Eq. (2.4).

In potential scattering the outgoing Green's function $G_i^{(+)}(\mathbf{r},\mathbf{r}')$ is $\sim e^{ikr'}/r'$ as $r'\to\infty$. Still in potential scattering, the scattered part Φ of a scattering solution to the Schrödinger equation is said to be "everywhere outgoing" if $\Phi(\mathbf{r}')$ has no component $\sim e^{-ikr'}/r'$ at infinite r', i.e., if $\Phi(\mathbf{r}')$ behaves like the outgoing Green's function as $\mathbf{r}'\to\infty$ along any direction in $\mathbf{r}'$ space. Furthermore, application of Green's theorem to the three-dimensional (in potential scattering) integral on the right-hand side of Eq. (2.13) makes $\mathscr{I}(X,Y)$ proportional to terms of the form $(Y\nabla X - X\nabla Y)$ integrated over the sphere at infinity. In potential scattering, therefore, whenever Φ is everywhere outgoing the leading terms in $(Y\nabla X - X\nabla Y)$ cancel and Eq. (2.16) is satisfied; correspondingly, in potential scattering the solution Ψ to Eq. (2.8) whose scattered part Φ is everywhere outgoing surely satisfies the real energy LS equation (2.4).[19]

Even for two-particle systems, in the laboratory coordinate system the "everywhere outgoing" scattered part $\Phi(\mathbf{r}')$ of a scattering solution $\Psi(\mathbf{r}')$ to Eq. (2.8) having incident wave $\psi_i(\mathbf{r}')$ generally does not behave like the outgoing Green's function $G_i^{(+)}(\mathbf{r},\mathbf{r}')$ for $\mathbf{r}'=r'\mathbf{n}'\to\infty$ along arbitrary directions $\mathbf{n}'$ in $\mathbf{r}'$ space, because[7] $\Phi(\mathbf{r}')$—along with $\Psi(\mathbf{r}')$—must have the factor $e^{i\mathbf{K}\cdot\mathbf{R}'}$ expressing conservation of momentum of the center of mass $\mathbf{R}'$. For systems of three or more particles, the asymptotic behavior at infinity of the scattered part $\Phi(\mathbf{r}')$ associated with a physically sensible scattering solution of Eq. (2.8) can be different from the asymptotic behavior of $G_i^{(+)}(\mathbf{r},\mathbf{r}')$ even in the center of mass coordinate system, especially when more than two independent bodies (which may be unbound particles or bound particle "aggregates") are initially incident.[20] Nevertheless, it is reasonable on physical grounds to assume—and Gerjuoy[7] did assume—that Eq. (2.16) will be satisfied whenever $\Phi(\mathbf{r}')$ is the scattered part of a physically sensible scattering solution of the Schrödinger equation, i.e., whenever it is physically sensible to term $\Phi(\mathbf{r}')$ "everywhere outgoing."

In particular, if Φ_i now denotes the everywhere outgoing scattered part of a solution Ψ_i to Eq. (2.8) with incident wave ψ_i in the i channel, and if Φ_f correspondingly denotes the everywhere outgoing scattered part of the solution Ψ_f to Eq. (2.8) with incident wave ψ_f in the $f\neq i$ channel, then Gerjuoy assumed[7]

$$\mathscr{I}[G_i^{(+)}(\mathbf{r},\mathbf{r}'),\Phi_i(\mathbf{r}')]=0 \tag{2.17a}$$

and

$$\mathscr{I}[G_i^{(+)}(\mathbf{r},\mathbf{r}'),\Phi_f(\mathbf{r}')]=0\ . \tag{2.17b}$$

As will be seen below, however, Eqs. (2.17a) *and* (2.17b) can be *deduced* from a precise formulation of the everywhere outgoing condition (involving the Green's function associated with the full Hamiltonian H), along with very reasonable assumptions about the convergence of integrals like those on the right-hand sides of Eqs. (2.3) and (2.4); it is not necessary to rely solely on the physically reasonable interpretation of the phrase "everywhere outgoing." Furthermore, if the channel f is a true rearrangement of i, the outgoing Green's function $G_i^{(+)}(\mathbf{r},\mathbf{r}')$ does not propagate in the channel f, i.e., $G_i^{(+)}(\mathbf{r},\mathbf{r}')$ has a negligibly small projection on the bound state eigenfunctions characterizing the f channel as $\mathbf{r}'=r'\mathbf{n}'$ approaches infinity along directions $\mathbf{n}'$ corresponding physically to the particle groupings (aggregates) found in the f channel. Thus, recalling the definition (2.13) of $\mathscr{I}(X,Y)$, Gerjuoy[7] also assumed

$$\mathscr{I}[G_i^{(+)}(\mathbf{r},\mathbf{r}'),\psi_f(\mathbf{r}')]=0 \tag{2.17c}$$

whenever ψ_f is an incident wave in a channel f which is a true rearrangement of i. Equations (2.17b) and (2.17c) imply

$$\mathscr{I}[G_i^{(+)}(\mathbf{r},\mathbf{r}'),\Psi_f(\mathbf{r}')]=0\ . \tag{2.17d}$$

We know from Eq. (2.17a) [recall the implications of Eq. (2.16)] that

$$\mathscr{I}[G_i^{(+)}(\mathbf{r},\mathbf{r}'),\Psi_i(\mathbf{r}')]=\psi_i(\mathbf{r})\ . \tag{2.18a}$$

But Eq. (2.17d) then also implies

$$\mathscr{I}[G_i^{(+)}(\mathbf{r},\mathbf{r}'),\Psi_\lambda(\mathbf{r}')]=\psi_i(\mathbf{r})\ , \tag{2.18b}$$

where

$$\Psi_\lambda=\Psi_i+\lambda\Psi_f\ , \tag{2.19}$$

with λ any constant. Since Ψ_i and Ψ_f have been postulated to be solutions of the Schrödinger equation (2.8), Ψ_λ also solves the Schrödinger equation. Equation (2.18b) therefore implies [recalling Eq. (2.14)] that Ψ_λ satisfies the real energy LS equation (2.4).

In summary, if the assumed relations (2.17a)—(2.17c) really hold, then the real energy LS equation (2.4) has nonunique solutions. In a future paper[17] we will demonstrate by explicit calculation that Eqs. (2.17a)—(2.17c) indeed do hold in the McGuire three-particle model [which, of course, implies Eq. (2.17d) also holds]. We also have verified Eq. (2.15) in the McGuire model, although there really is no need to do so because Eq. (2.15) was *derived* above, not assumed, and the derivations made no assumptions about the asymptotic properties of $G_i^{(+)}(\mathbf{r},\mathbf{r}')$.

B. Implications of LS equation (2.3)

In general Eq. (2.3) is not identical with Eq. (2.4); such identity requires

$$\lim_{\epsilon\to 0}\int d\mathbf{r}' G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)V_i(\mathbf{r}')\Psi(\mathbf{r}',E) = \int d\mathbf{r}'\left[\lim_{\epsilon\to 0} G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\right]V_i(\mathbf{r}')\Psi(\mathbf{r}';E)\ , \tag{2.20a}$$

or, in the more condensed notation of Eqs. (1.1) and (2.6a),

$$\lim_{\epsilon\to 0}[G_i(E+i\epsilon)V_i\Psi]=\left[\lim_{\epsilon\to 0} G_i(E+i\epsilon)\right]V_i\Psi\ . \tag{2.20b}$$

Equations (2.20) have been proved for potential scattering, with potentials vanishing sufficiently rapidly at infinity.[21] Thus in potential scattering with reasonably behaved potentials, Eqs. (2.3) and (2.4) assuredly are identical. Accordingly, in potential scattering the interpretation that $\Psi\equiv\Psi(E)$ in Eq. (1.2) inevitably reduces Eq. (1.2) to the real energy LS equation (1.1). On the other hand, for systems of three (or more) particles interacting via two-body forces, the quantity $V_i(\mathbf{r})$ need not decrease rapidly as

$\mathbf{r}\equiv(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$ approaches infinity along all possible directions in configuration space;[22] correspondingly, the rapid decrease of $|V_i(\mathbf{r}')\Psi(\mathbf{r}';E)|$ as $\mathbf{r}'\to\infty$, which makes possible an easy proof of Eqs. (2.20) in potential scattering,[21] can no longer be relied on.

A sufficient condition for Eq. (2.20a) to hold is the uniform convergence[23] as $\epsilon\to0$ of the sequence of infinite integrals on the left-hand side of Eq. (2.20a), as is proved in the Appendix.[24] This uniform convergence property is not readily demonstrable for many-particle systems, even with quite restrictive assumptions about the particle interactions. Nevertheless, even though we cannot furnish a proof, there are reasons to believe that the sequence of integrals on the left-hand side of Eq. (2.20a) does converge uniformly as $\epsilon\to0$ whenever the right-hand side of Eq. (2.20a) is a convergent integral, as has been argued elsewhere in a somewhat different connection.[20] Moreover, for three-particle systems at least, it can be shown[24] convincingly, though perhaps not wholly rigorously, that the right-hand side of Eq. (2.20a) is a convergent integral *and* the left-hand side of Eq. (2.20a) is uniformly convergent as $\epsilon\to0$, if (for short range interactions) one makes the following reasonable assumption: Assume that at large distances the projections of the Green's function G_i on the bound state eigenfunctions characterizing the i channel behave like the free-space Green's function in the subspace (of the full configuration space) defined by the coordinates of the unbound particles and of the mass centers locating the various bound particle aggregates in channel i; these unbound particles and mass centers can go out to infinity along all directions in this subspace. The convergence of the right-hand side of Eq. (2.20a) in three-particle systems also follows from arguments given in the response by one of us[16] (E.G.) to Benoist-Gueutal's criticisms[11] of the real energy LS equation (1.1). Actually, Benoist-Gueutal[11] contends that integrals like the right-hand side of Eq. (2.20a) are not well defined rather than necessarily nonconvergent; the aforementioned response[16] rejects this contention.

We conclude therefore, for three-particle systems interacting via short range two-body potentials at any rate, that Eq. (2.20a) is true. Granting this conclusion (whose nonrigorous foundation we have described in the preceding paragraph), Eqs. (2.3) and (2.4) are equivalent for such three-particle systems, i.e., for such systems the interpretation $\Psi\equiv\Psi(E)$ in Eq. (1.2) reduces Eq. (1.2) to the real energy LS equation (1.1) whether Eq. (1.2) initially is interpreted as in Eq. (2.3) or as in Eq. (2.4). However, the equality of (2.20b) has additional interesting implications, as follows.

The (henceforth presumed convergent) infinite integrals in Eq. (2.20a) are the limits as $R\to\infty$ of integrals over spheres of radius R in configuration space. In other words, the equality of (2.20a) means

$$\lim_{\epsilon\to0}\left[\lim_{R\to\infty}\int^R d\mathbf{r}'G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)V_i(\mathbf{r}')\Psi(\mathbf{r}';E)\right] = \lim_{R\to\infty}\int^R d\mathbf{r}'\left[\lim_{\epsilon\to0}G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\right]V_i(\mathbf{r}')\Psi(\mathbf{r}';E)\ . \tag{2.21a}$$

But for any finite R the fact that the limit (2.7) exists makes it possible to pull the limit $\epsilon\to0$ outside the integral sign in Eq. (2.21a);[25] for this purpose it is sufficient, and we may assume, that the integrand on the left-hand side of Eq. (2.21a) is bounded for all values of $\mathbf{r}'$ in the sphere of radius R, except in the vicinity of $\mathbf{r}'=\mathbf{r}$ where $G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)$ has an integrable singularity and the limit (2.7) as $\epsilon\to0$ does not strictly exist.[26] Thus the equality (2.20a) becomes

$$\lim_{\epsilon\to0}\lim_{R\to\infty}\int^R d\mathbf{r}'G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)V_i(\mathbf{r}')\Psi(\mathbf{r}';E) = \lim_{R\to\infty}\lim_{\epsilon\to0}\int^R d\mathbf{r}'G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)V_i(\mathbf{r}')\Psi(\mathbf{r}';E)\ . \tag{2.21b}$$

Rewrite Eq. (2.8) as

$$(H_i-E-i\epsilon)\Psi=-i\epsilon\Psi-V_i\Psi\ . \tag{2.22}$$

Now manipulate Eqs. (2.2) and (2.22) much as we manipulated their respective counterparts (2.11) and (2.10) to derive Eq. (2.12). To be specific, multiply Eq. (2.2) on the left-hand side by $\Psi(\mathbf{r})$, multiply Eq. (2.22) on the left-hand side by $G_i(\mathbf{r};\mathbf{r}';E+i\epsilon)$, and integrate over $\mathbf{r}'$, but only over $|\mathbf{r}'|\le R$, not over all $\mathbf{r}'$ as previously. Denote the integral in Eq. (2.21b) by $F(\mathbf{r};R;\epsilon)$, and choose R sufficiently large that for any specified fixed $\mathbf{r}$ in $F(\mathbf{r};R;\epsilon)$ the inequality $R>|\mathbf{r}|$ holds, i.e., so that the integration volume $|\mathbf{r}'|\le R$ surely includes the point $\mathbf{r}'=\mathbf{r}$. Then the foregoing manipulations yield

$$F(\mathbf{r};R;\epsilon)=\mathscr{I}[G_i(\mathbf{r},\mathbf{r}';E+i\epsilon),\Psi(\mathbf{r}')]_R -\Psi(\mathbf{r})-i\epsilon\int^R d\mathbf{r}'G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\Psi(\mathbf{r}')\ . \tag{2.23}$$

where the notation $\mathscr{I}[G_i,\Psi]_R$ indicates that the surface integral obtained by applying Green's theorem to the right-hand side of Eq. (2.13) is evaluated at $|\mathbf{r}'|=R$, not (as previously) at infinity in $\mathbf{r}'$ space.

If in Eq. (2.23) we first take the limit $\epsilon\to0$ and then let $R\to\infty$, as on the right-hand side of Eq. (2.21b), then Eq. (2.23) reduces merely to a rearrangement of the terms in Eq. (2.12), because the last term in Eq. (2.23) vanishes as $\epsilon\to0$ for fixed R, and because the right-hand side of Eq. (2.21b) [which equals the right-hand side of Eq. (2.21a)] then becomes the infinite integral $G_i^{(+)}(E)V_i\Psi$ on the right-hand side of Eq. (2.12). On the other hand, taking in Eq. (2.23) first the limit $R\to\infty$ and then the limit $\epsilon\to0$, as on the left-hand side of Eq. (2.21b), should make the right-hand side of Eq. (2.23) simply

$$-\Psi(\mathbf{r})-\lim_{\epsilon\to0}i\epsilon\int^\infty d\mathbf{r}'G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\Psi(\mathbf{r}')\ , \tag{2.24}$$

because for every finite ϵ the quadratically integrable Green's function $G_i(E+i\epsilon)$ is expected to be exponentially decreasing at infinity [as it surely will be if, as assumed earlier, $G_i(E+i\epsilon)$ behaves at infinity like the free-space Green's function for the set of unbound particles and mass centers which go out to infinity in the i channel].

We now conclude, therefore, that whenever $\Psi(E)$ is a solution of the Schrödinger equation for which Eqs. (2.20) hold,

$$\mathscr{I}[G_i^{(+)}(\mathbf{r},\mathbf{r}';E),\Psi(\mathbf{r}')] = -\lim_{\epsilon\to 0} i\epsilon \int d\mathbf{r}' G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\Psi(\mathbf{r}') , \quad (2.25)$$

where the left-hand side of Eq. (2.25) again is evaluated on the sphere at infinity, and the right-hand side of Eq. (2.25) is integrated over all $\mathbf{r}'$. Thus from Eqs. (2.18a) and (2.17d) we expect that

$$-\lim_{\epsilon\to 0} i\epsilon \int d\mathbf{r}' G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\Psi_i(\mathbf{r}';E)=\psi_i(\mathbf{r};E) , \quad (2.26a)$$

$$-\lim_{\epsilon\to 0} i\epsilon \int d\mathbf{r}' G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\Psi_f(\mathbf{r}';E)=0 , \quad (2.26b)$$

where Ψ_i and Ψ_f are defined as previously. Moreover, for $\Psi \equiv \Psi_i$, Eq. (2.22) can be rewritten as

$$(H_i - E - i\epsilon)\Phi_i = -i\epsilon\Phi_i - V_i\Psi_i . \quad (2.27)$$

Then, manipulating Eqs. (2.27) and (2.2) as in the derivation of Eq. (2.23), we readily find

$$F_i(\mathbf{r};R;\epsilon)=\mathscr{I}[G_i(\mathbf{r},\mathbf{r}';E+i\epsilon),\Phi_i(\mathbf{r}')]_R - \Phi_i(\mathbf{r}) - i\epsilon\int^R d\mathbf{r}' G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\Phi_i(\mathbf{r}') , \quad (2.28)$$

where F_i denotes the integral in Eq. (2.21b) for $\Psi\equiv\Psi_i$. Equation (2.28) yields, in place of Eq. (2.25),

$$\mathscr{I}[G_i^{(+)}(\mathbf{r},\mathbf{r}';E),\Phi_i(\mathbf{r}')] = -\lim_{\epsilon\to 0} i\epsilon \int d\mathbf{r}' G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\Phi_i(\mathbf{r}') . \quad (2.29)$$

Thus from Eq. (2.17a) we also expect

$$-\lim_{\epsilon\to 0} i\epsilon \int d\mathbf{r}' G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\Phi_i(\mathbf{r}')=0 . \quad (2.30)$$

Equation (2.1) can be rewritten as

$$(H_i - E - i\epsilon)\psi_i = -i\epsilon\psi_i . \quad (2.31)$$

Now the previous manipulations of Eqs. (2.31) and (2.2) yield

$$\mathscr{I}[G_i(\mathbf{r},\mathbf{r}';E+i\epsilon),\psi_i(\mathbf{r}')]_R = \psi_i(\mathbf{r}) + i\epsilon\int^R d\mathbf{r}' G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\psi_i(\mathbf{r}') . \quad (2.32)$$

Equation (2.32) is no more than the result of subtracting Eq. (2.28) from Eq. (2.23). Letting $R\to\infty$ for finite ϵ causes the left-hand side of Eq. (2.32) to vanish. Hence we should have, for any $\epsilon>0$ (perhaps only for ϵ not too far from $\epsilon=0$, however),

$$-i\epsilon\int d\mathbf{r}' G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\psi_i(\mathbf{r}')=\psi_i(\mathbf{r}) , \quad (2.33a)$$

which, of course, implies

$$-\lim_{\epsilon\to 0} i\epsilon \int d\mathbf{r}' G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\psi_i(r')=\psi_i(\mathbf{r}) . \quad (2.33b)$$

The incident wave ψ_f of Eq. (2.17c) satisfies

$$(H_f - E)\psi_f = 0 , \quad (2.34)$$

where the total Hamiltonian H of Eq. (2.8) is

$$H = H_i + V_i = H_f + V_f . \quad (2.35)$$

Equation (2.35) means Eq. (2.34) can be rewritten as

$$(H_i - E - i\epsilon)\psi_f = (V_f - V_i)\psi_f - i\epsilon\psi_f . \quad (2.36)$$

The by now customary manipulations of Eqs. (2.36) and (2.2) yield

$$D_{if}(\mathbf{r};R;\epsilon)=\mathscr{I}[G_i(\mathbf{r},\mathbf{r}';E+i\epsilon),\psi_f(\mathbf{r}')]_R - \psi_f(\mathbf{r}) - i\epsilon\int^R d\mathbf{r}' G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\psi_f(\mathbf{r}') , \quad (2.37a)$$

where

$$D_{if}(\mathbf{r};R;\epsilon)=\int^R d\mathbf{r}' G_i(\mathbf{r},\mathbf{r}';E+i\epsilon) \times [V_i(\mathbf{r}')-V_f(\mathbf{r}')]\psi_f(\mathbf{r}') . \quad (2.37b)$$

It can be seen that as $R\to\infty$ the integral on the right-hand side of Eq. (2.37b) will converge in the limit $\epsilon\to 0$ if $G_i^{(+)}(\mathbf{r},\mathbf{r}')$ does not propagate in the channel f, as was assumed previously in connection with Eq. (2.17c). Moreover, arguments that implied the uniform convergence of the sequence of integrals on the left-hand side of Eq. (2.20a) as $\epsilon\to 0$ also imply that the sequence of integrals on the right-hand side of Eq. (2.37b) is uniformly convergent as $R\to\infty$. In other words, we expect

$$\lim_{\epsilon\to 0}\lim_{R\to\infty} D_{if}(\mathbf{r};R;\epsilon)=\lim_{R\to\infty}\lim_{\epsilon\to 0} D_{if}(\mathbf{r};R;\epsilon) , \quad (2.38)$$

which is the analogue of Eq. (2.21b). Thus, as in Eqs. (2.25) and (2.29), the pair of Eqs. (2.37a) and (2.38) yields

$$\mathscr{I}[G_i^{(+)}(\mathbf{r},\mathbf{r}';E),\psi_f(\mathbf{r}')] = -\lim_{\epsilon\to 0} i\epsilon \int d\mathbf{r}' G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\psi_f(\mathbf{r}') . \quad (2.39)$$

Consequently, if Eq. (2.17c) holds we also expect

$$-\lim_{\epsilon\to 0} i\epsilon \int d\mathbf{r}' G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\psi_f(\mathbf{r}')=0 . \quad (2.40a)$$

Equations (2.17c), (2.37), and (2.40a) imply

$$\psi_f(\mathbf{r})=\int d\mathbf{r}' G_i^{(+)}(\mathbf{r},\mathbf{r}';E)[V_f(\mathbf{r}')-V_i(\mathbf{r}')]\psi_f(\mathbf{r}') , \quad (2.40b)$$

again assuming that the sequence of infinite integrals (2.37b) is uniformly convergent as $\epsilon\to 0$. Similarly we deduce

$$\mathscr{I}[G_i^{(+)}(\mathbf{r},\mathbf{r}';E),\Phi_f(\mathbf{r}')] = -\lim_{\epsilon\to 0} i\epsilon \int d\mathbf{r}' G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\Phi_f(\mathbf{r}') , \quad (2.41a)$$

which, if Eq. (2.17b) holds, implies

$$-\lim_{\epsilon\to 0} i\epsilon \int d\mathbf{r}' G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\Phi_f(\mathbf{r}')=0 . \quad (2.41b)$$

Of course, in Eq. (2.39)

$$\mathscr{I}[G_i^{(+)}(\mathbf{r},\mathbf{r}';E),\psi_f(\mathbf{r}')] = \lim_{R\to\infty}\lim_{\epsilon\to 0}\mathscr{I}[G_i(\mathbf{r},\mathbf{r}';E+i\epsilon),\psi_f(\mathbf{r}')]_R \quad (2.42)$$

from Eq. (2.37a), and similarly for Eqs. (2.25), (2.29), and (2.30). To our knowledge, equations such as (2.25) [and (2.29), (2.39), or (2.41a)], relating the limit as $\epsilon\to 0$ of a product like $i\epsilon G_i(E+i\epsilon)\Psi(E)$ to the limit as R approaches infinity of a surface integral (over the surface of radius R in $\mathbf{r}'$ space) involving $G_i^{(+)}$, have not been explicitly stated previously.[27] Similarly, results such as Eqs. (2.26b) and (2.41b) apparently have not been categorically stated in the literature.[28] Equations (2.33b) and (2.40a) are illustrations of Lippmann's identities.[4] In other words, the foregoing analysis has established a connection between Gerjuoy's surface integrals at infinity and the $\epsilon \to 0$ limits involved in Lippmann's identities.

In a future paper[17] we will also demonstrate, by explicit calculation, that Eqs. (2.26), (2.30), (2.40a), and (2.41b) hold in the McGuire three-particle model, thereby further confirming the validity of the assumptions (about convergence, behavior of Green's functions at infinity, etc.) made in deriving those equations, as well as in deriving the Eqs. (2.17), whose verification in the McGuire model we announced above. In addition, a future paper will seek to verify the pair of Eqs. (2.33a) and (2.33b) in the McGuire model, although—as in the case of Eq. (2.15) discussed earlier—Eqs. (2.33) hardly need verification since they are derived on the sole difficult-to-challenge assumption that for $\epsilon > 0$ the left-hand side of Eq. (2.32) vanishes as $R \to \infty$.

C. Implications of LS equation (2.6b)

Applying the operator $(H_i - E - i\epsilon)$ to both sides of Eq. (2.6b), we obtain

$$(H_i - E - i\epsilon)\Psi = -i\epsilon\psi_i - (H_i - E - i\epsilon)[G_i(E+i\epsilon)V_i\Psi] , \tag{2.43}$$

where the meaning of the condensed notation is obvious from Eq. (2.6a); of course, H_i in Eq. (2.43) operates on $\mathbf{r}$, not $\mathbf{r}'$. Because $G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)$ is quadratically integrable over $\mathbf{r}'$ for any $\epsilon > 0$ and finite $\mathbf{r}$, the infinite integral on the right-hand side of Eq. (2.6b) is expected to converge uniformly for any $\epsilon > 0$ in a domain $|\mathbf{r}| < \rho$, ρ bounded. Moreover, the uniform convergence property should be retained if G_i under the integral sign is replaced by ∇G_i, where the gradient operates on $\mathbf{r}$. It follows that the differential operator on the right-hand side of Eq. (2.43) can be taken under the integral sign,[29] yielding, by virtue of Eq. (2.2),

$$(H_i - E - i\epsilon)\Psi = -i\epsilon\psi_i - V_i\Psi , \tag{2.44a}$$

or

$$(H - E - i\epsilon)\Psi = -i\epsilon\psi_i . \tag{2.44b}$$

The foregoing manipulations have shown that solutions Ψ to Eq. (2.6b), if any, must satisfy Eqs. (2.44). But Eq. (2.44b) for Ψ, like Eq. (2.2) for $G_i(E+i\epsilon)$, must have a unique solution, because the complex quantity $E+i\epsilon$ cannot belong to the spectrum of the Hermitian operators H or H_i. Therefore Eq. (2.44b) is solved by

$$\Psi(E+i\epsilon) = -i\epsilon G(E+i\epsilon)\psi_i(E) , \tag{2.45a}$$

i.e., by

$$\Psi(\mathbf{r};E+i\epsilon) = -i\epsilon \int d\mathbf{r}' G(\mathbf{r},\mathbf{r}';E+i\epsilon)\psi_i(\mathbf{r}';E) , \tag{2.45b}$$

where $G(E+i\epsilon) \mp (H - E - i\epsilon)^{-1}$ is the unique quadratically integrable Green's function satisfying

$$(H - E - i\epsilon)G(\mathbf{r};\mathbf{r}';E+i\epsilon) = \mathbf{I} \equiv \delta(\mathbf{r}-\mathbf{r}') . \tag{2.46}$$

Moreover, we can verify that $\Psi(E+i\epsilon)$ of Eq. (2.45a) does satisfy Eq. (2.6b), because this $\Psi(E+i\epsilon)$ satisfies Eq. (2.44a), which can be rewritten as

$$(H_i - E - i\epsilon)(\Psi - \psi_i) = -V_i\Psi . \tag{2.47}$$

Equation (2.6a) now follows from the fact that $G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)$ is the unique quadratically integrable solution of Eq. (2.2), just as Eq. (2.45a) followed from Eq. (2.44b). Alternatively, if we are concerned that the right-hand side of Eq. (2.47) cannot be regarded as a known function, we can employ our usual manipulations, namely multiply Eq. (2.47) on the left-hand side by G_i, multiply Eq. (2.2) on the left-hand side by

$$\Phi(E+i\epsilon) \equiv \Psi(E+i\epsilon) - \psi_i(E) , \tag{2.48}$$

etc. Thus we obtain, as in Eq. (2.27),

$$\int^R d\mathbf{r}' G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)V_i(\mathbf{r}')\Psi(\mathbf{r}';E+i\epsilon) = \mathscr{I}[G_i(E+i\epsilon),\Phi(E+i\epsilon)]_R - \Phi(\mathbf{r};E+i\epsilon) . \tag{2.49}$$

Since the surface integral on the right-hand side vanishes as $R \to \infty$ for $\epsilon > 0$, Eq. (2.49) immediately reduces to Eq. (2.6b).

We conclude that the complex energy LS equation (2.6b) has a unique solution $\Psi(E+i\epsilon)$, in contrast to the real energy versions (2.3) or (2.4) of the LS equation, whose solutions are nonunique for reasons which have been discussed. We further conclude that the quantity $\hat{\Psi}(E)$ on the left-hand side of Eq. (2.5) also has been uniquely defined since (assuming the limit exists) it is the limit of a uniquely specified sequence of functions $\Psi(E+i\epsilon)$.

That Eq. (2.45a) yields a unique solution to the complex energy LS equation (2.6b) has been known for a long time.[6] Moreover, Eq. (2.44b) also can be rewritten as

$$(H - E - i\epsilon)\Psi = (H_i - E - i\epsilon)\psi_i = (H - E - i\epsilon)\psi_i - V_i\psi_i , \tag{2.50a}$$

or

$$(H - E - i\epsilon)\Phi = -V_i\psi_i , \tag{2.50b}$$

where $\Phi \equiv \Phi(E+i\epsilon)$ of Eq. (2.48). Consequently, we now legitimately obtain, as in Eq. (2.44b),

$$\Phi(E+i\epsilon) = -G(E+i\epsilon)V_i\psi_i(E) , \tag{2.51a}$$

or

$$\Psi(\mathbf{r};E+i\epsilon) = \psi_i(E) - \int d\mathbf{r}' G(\mathbf{r},\mathbf{r}';E+i\epsilon)V_i(\mathbf{r}')\psi_i(\mathbf{r}';E) . \tag{2.51b}$$

Alternatively, the analogous manipulations which yielded Eq. (2.49) now yield

$$\int^R d\mathbf{r}' G(\mathbf{r},\mathbf{r}';E+i\epsilon)V_i(\mathbf{r}')\psi_i(\mathbf{r}') = \mathscr{I}[G(E+i\epsilon),\Phi(E+i\epsilon)]_R - \Phi(\mathbf{r};E+i\epsilon) , \tag{2.51c}$$

from which Eq. (2.51a) is obtained in the limit $R\to\infty$.

Equation (2.51b) is a *formula*, not an integral equation, for $\Psi(\mathbf{r};E+i\epsilon)$, which we will henceforth denote by $\Psi_i(E+i\epsilon)$ to signify that it corresponds to incident wave $\psi_i(E)$. The corresponding limit defined by Eq. (2.5) is

$$\hat{\Psi}_i(E)=\psi_i(E)-\lim_{\epsilon\to 0}\int d\mathbf{r}'G(\mathbf{r};\mathbf{r}',E+i\epsilon)V_i(\mathbf{r}')\psi_i(\mathbf{r}') \tag{2.52a}$$

$$=\psi_i(E)-\int d\mathbf{r}'G^{(+)}(\mathbf{r},\mathbf{r}';E)V_i(\mathbf{r}')\psi_i(\mathbf{r}')\ , \tag{2.52b}$$

where, as in Eq. (2.7),

$$G^{(+)}(E)=\lim_{\epsilon\to 0}G(E+i\epsilon)\quad(\epsilon>0)\ , \tag{2.53}$$

and where, as previously, we assume the sequence of infinite integrals on the right-hand side of Eq. (2.52a) is uniformly convergent as $\epsilon\to 0$, so that—as in Eq. (2.20a)—interchange of the order of integration and $\lim \epsilon\to 0$ is justified. Note that we can take the limit in (2.52a) with confidence, although we could not do so in Eq. (2.6d), because the integral in Eq. (2.6d)—unlike the integral in Eq. (2.52a)—involves the unknown function $\Psi(\mathbf{r}';E+i\epsilon)$ whose limiting properties we seek to establish.

The right-hand side of Eq. (2.52b) is the formula[7,30] for the unique solution Ψ_i of the Schrödinger equation (2.8) whose incoming part is ψ_i and whose scattered part obeys the outgoing condition[7]

$$\mathscr{I}[G^{(+)}(\mathbf{r},\mathbf{r}';E),\Phi_i(\mathbf{r}';E)]=0\ . \tag{2.54}$$

This assertion is readily verified via the usual manipulations of

$$(H-E)G^{(+)}(E)=\delta(\mathbf{r}-\mathbf{r}') \tag{2.55a}$$

and of Eq. (2.8) in the rewritten form

$$(H-E)\Phi=-V_i\psi_i\ . \tag{2.55b}$$

Thus the function $\hat{\Psi}_i(E)$ given by Eq. (2.52b), and obtained via the limit $\epsilon\to 0$ in Eq. (2.5), must be identical with the function $\Psi_i(E)$ just identified, since both functions have the same *formula*. Equation (2.54) is the uniquely correct interpretation of the phrase "the scattered part Φ_i of Ψ_i is everywhere outgoing," as has been discussed.[7] In fact, Eq. (2.54) is immediately obtained if we make the usual assumption that the order of limits $R\to\infty$ and $\epsilon\to 0$ can be interchanged in Eq. (2.51c).

We now know that the limit $\epsilon\to 0$ in Eq. (2.5) yields the function $\Psi_i(E)$ satisfying Eq. (2.52b) *and* the Schrödinger equation, i.e., we now know that the function $\Psi(E+i\epsilon)$ in the integral on the right-hand side of Eq. (2.6c) does have a well-defined presumably well-behaved limit $\hat{\Psi}(E)=\Psi_i(E)$ solving the Schrödinger equation. Thus, making our usual assumption that the sequence of infinite integrals on the right-hand side of Eq. (2.6c) converges uniformly as $\epsilon\to 0$, we now can conclude that the limit on the right-hand side of Eq. (2.6c) indeed is meaningful and yields

$$\lim_{\epsilon\to 0}[G_i(E+i\epsilon)V_i\Psi(E+i\epsilon)]=\int d\mathbf{r}'\lim_{\epsilon\to 0}[G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)V_i(\mathbf{r}')\Psi(\mathbf{r}';E+i\epsilon)]$$
$$=\int d\mathbf{r}'[\lim_{\epsilon\to 0}G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)]V_i(\mathbf{r}')\left[\lim_{\epsilon\to 0}\Psi(\mathbf{r}';E+i\epsilon)\right]=G_i^{(+)}(E)V_i\Psi_i(E)\ . \tag{2.56a}$$

We remark that the incorrect claim[14] (mentioned in the Introduction to this paper), to the effect that Faddeev-type equations may have nonunique scattering solutions, apparently stemmed[15] from failure to recognize the importance of assuring [as we have attempted to assure in deriving Eq. (2.56a)] that limits as $\epsilon\to 0$ exist and that the order of integration and limit as $\epsilon\to 0$ can be interchanged.

Equation (2.56a) implies that Eq. (2.6b) actually reduces to Eq. (2.3) or Eq. (2.4) (which have been shown to be equivalent under our uniform convergence assumption). In other words, the $\Phi_i(E)$ which is the scattered part of the just-identified $\Psi_i(E)$ and which satisfies Eq. (2.54) also must satisfy

$$\Phi_i(\mathbf{r};E)=-\int d\mathbf{r}'G_i^{(+)}(\mathbf{r},\mathbf{r}';E)V_i(\mathbf{r}')\Psi_i(\mathbf{r}';E)\ . \tag{2.56b}$$

But earlier we saw that whenever a solution to the Schrödinger equation satisfies the LS equation (2.4), i.e., whenever its scattered part satisfies Eq. (2.56b), then Eq. (2.16) must hold. And indeed, from Eq. (2.49), assuming interchange of $\lim R\to\infty$ and $\lim\epsilon\to 0$ is justified, we immediately do infer Eq. (2.17a) for $\Phi\equiv\Phi_i(E)$ just discussed.

Note, however, that the immediately preceding argument now has *deduced*— not *assumed* as previously stated in connection with Eq. (2.17a)—that the same Φ_i which satisfies the precise outgoing condition (2.54) (involving the Green's function for the complete Hamiltonian H) also satisfies the outgoing condition in the form of (2.17a) (involving the Green's function in the i channel). On the other hand, just as we *deduced* (no assumptions were made) Eq. (2.15) from Eqs. (2.1) and (2.10), so we can *deduce* without assumptions about uniform convergence, etc.,

$$\mathscr{I}[G^{(+)},\Psi_i]=\Psi_i \tag{2.57a}$$

from Eqs. (2.8) and (2.55a). Equations (2.57a) and (2.54) imply

$$\mathscr{I}[G^{(+)},\psi_i]=\Psi_i\ . \tag{2.57b}$$

Similarly we infer, for the solution Ψ_f of the Schrödinger equation which is incoming only in the f channel,

$$\mathscr{I}[G^{(+)},\Psi_f]=\Psi_f\ , \tag{2.58a}$$

$$\mathscr{I}[G^{(+)},\psi_f]=\Psi_f\ . \tag{2.58b}$$

Therefore, if we believe Eq. (2.17d), a function (here symbolized by Ψ_f) which apparently satisfies the outgoing condition involving the Green's function $G_i^{(+)}$ can have an incoming part and, as Eq. (2.58a) shows, need not satisfy the outgoing condition involving the Green's function $G^{(+)}$; indeed, one such Ψ_f is the solution to Schrödinger's equation having incoming part ψ_f and everywhere outgoing part Φ_f obeying

$$\mathscr{I}[G^{(+)}(\mathbf{r},\mathbf{r}';E),\Phi_f(\mathbf{r}';E)]=0\ , \tag{2.59a}$$

the f channel analogue of Eq. (2.54). Equation (2.59a), as usual, is integrated over the surface at infinity in $\mathbf{r}'$ space, for fixed $\mathbf{r}$.

We shall show that Eq. (2.59a) (along with our customary assumption of uniform convergence of infinite integrals as $\epsilon\to 0$) implies Eq. (2.17b), which—along with Eq. (2.17c)—yields (2.17d). For this purpose it is convenient to interchange $\mathbf{r}$ and $\mathbf{r}'$ in Eq. (2.59a), i.e., to rewrite Eq. (2.59a) in the form

$$\mathscr{I}[G^{(+)}(\mathbf{r}',\mathbf{r};E),\Phi_f(\mathbf{r};E)] =\mathscr{I}[G^{(+)}(\mathbf{r},\mathbf{r}';E),\Phi_f(\mathbf{r};E)]=0\ , \tag{2.59b}$$

where now in Eq. (2.59b) we are integrating over the surface at infinity in $\mathbf{r}$ space for fixed $\mathbf{r}'$; we shall use the second equality in Eq. (2.59b), obtained from the first recalling that the Green's function $G^{(+)}(\mathbf{r},\mathbf{r}')$, like $G_i^{(+)}(\mathbf{r},\mathbf{r}')$, is symmetric in $\mathbf{r}$ and $\mathbf{r}'$.

It is well known that[7,18]

$$G=G_i-G_iV_iG=G_i-GV_iG_i\ , \tag{2.60a}$$

where G and G_i are evaluated at complex energy $E+i\epsilon$, $\epsilon>0$. Correspondingly, at real energy E

$$G^{(+)}=G_i^{(+)}-G_i^{(+)}V_iG^{(+)}=G_i^{(+)}-G^{(+)}V_iG_i^{(+)} \tag{2.60b}$$

along with

$$\mathscr{I}[G^{(+)}(\mathbf{r},\mathbf{r}'';E),G_i^{(+)}(\mathbf{r}',\mathbf{r}'';E)]=0\ , \tag{2.60c}$$

where the surface integral now is evaluated at infinity in $\mathbf{r}''$ space. Equations (2.60a)–(2.60c) can be derived via the manipulations and uniform convergence assumption we regularly have employed and which no longer need be detailed. Equations (2.59b) and (2.60b) imply

$$\mathscr{I}[G_i^{(+)}(\mathbf{r},\mathbf{r}';E),\Phi_f(\mathbf{r};E)]=\mathscr{I}[A(\mathbf{r},\mathbf{r}';E),\Phi_f(\mathbf{r};E)]\ , \tag{2.61a}$$

where we define

$$A(\mathbf{r},\mathbf{r}';E)=\int d\mathbf{r}''G^{(+)}(\mathbf{r},\mathbf{r}'';E)V_i(\mathbf{r}'')G_i^{(+)}(\mathbf{r}'',\mathbf{r}';E)\ . \tag{2.61b}$$

Correspondingly, define

$$A(\mathbf{r},\mathbf{r}';E+i\epsilon)=\int d\mathbf{r}''G(\mathbf{r},\mathbf{r}'';E+i\epsilon)V_i(\mathbf{r}'') \times G_i(\mathbf{r}'',\mathbf{r}';E+i\epsilon)\ . \tag{2.61c}$$

The quantity $A(\mathbf{r},\mathbf{r}';E+i\epsilon)$ obviously satisfies

$$(H-E-i\epsilon)A(\mathbf{r},\mathbf{r}';E+i\epsilon)=V_i(\mathbf{r})G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\ , \tag{2.62}$$

using Eq. (2.46) and arguing—as in the derivation of Eq. (2.44a) from Eq. (2.43)—that the operator $(H-E-i\epsilon)$ can be taken under the integral sign in Eq. (2.61c). We also have the analogue of Eq. (2.50b), namely

$$(H-E-i\epsilon)\Phi_f(\mathbf{r};E+i\epsilon)=-V_f(\mathbf{r})\psi_f(\mathbf{r};E)\ . \tag{2.63}$$

Our usual manipulations now yield

$$\mathscr{I}[A(\mathbf{r},\mathbf{r}';E+i\epsilon),\Phi_f(\mathbf{r};E+i\epsilon)]_R=\int^R d\mathbf{r}\,\Phi_f(\mathbf{r};E+i\epsilon)V_i(\mathbf{r})G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)+\int^R d\mathbf{r}\,A(\mathbf{r},\mathbf{r}';E+i\epsilon)V_f(\mathbf{r})\psi_f(\mathbf{r};E)\ . \tag{2.64a}$$

In Eq. (2.64a) substitute Eq. (2.61c) and the f channel analogue of Eq. (2.51a). Then,

$$\begin{aligned}\mathscr{I}[A(\mathbf{r};\mathbf{r}';E+i\epsilon),\Phi_f(\mathbf{r};E+i\epsilon)]_R=&-\int^R d\mathbf{r}\int^\infty d\mathbf{r}''G(\mathbf{r},\mathbf{r}'';E+i\epsilon)V_f(\mathbf{r}'')\psi_f(\mathbf{r}'';E)V_i(\mathbf{r})G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\\&+\int^R d\mathbf{r}\int^\infty d\mathbf{r}''G(\mathbf{r},\mathbf{r}'';E+i\epsilon)V_i(\mathbf{r}'')G_i(\mathbf{r}'',\mathbf{r}';E+i\epsilon)V_f(\mathbf{r})\psi_f(\mathbf{r};E)\ .\end{aligned} \tag{2.64b}$$

Next, interchange the dummy variables $\mathbf{r}$ and $\mathbf{r}''$ in the second integral on the right-hand side of Eq. (2.64b), and let $R\to\infty$. In that limit the left-hand side of Eq. (2.64b) should vanish for any $\epsilon>0$, recalling that, according to Eq. (2.60a), $A(\mathbf{r},\mathbf{r}';E+i\epsilon)$ is merely the difference between the two quadratically integrable Green's functions $G(E+i\epsilon)$ and $G_i(E+i\epsilon)$. Thus Eq. (2.64b) yields

$$\begin{aligned}&-\int d\mathbf{r}\int d\mathbf{r}''G(\mathbf{r},\mathbf{r}'';E+i\epsilon)V_f(\mathbf{r}'')\psi_f(\mathbf{r}'';E)V_i(\mathbf{r})G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)\\&\qquad+\int d\mathbf{r}''\int d\mathbf{r}\,G(\mathbf{r}'',\mathbf{r};E+i\epsilon)V_i(\mathbf{r})G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)V_f(\mathbf{r}'')\psi_f(\mathbf{r}'';E)=0\ ,\end{aligned} \tag{2.65a}$$

where $\mathbf{r}$ and $\mathbf{r}''$ both are integrated over all space.

Since $G(\mathbf{r}'',\mathbf{r})=G(\mathbf{r},\mathbf{r}'')$, both integrals in Eq. (2.65a) have the same integrand. Equation (2.56a) therefore means that the orders of integration over $\mathbf{r}$ and $\mathbf{r}''$ can be interchanged in the integrals on the right-hand side of Eq. (2.65a), as we expect from the previously discussed anticipated exponential decay of the Green's functions $G(E+i\epsilon)$ and $G_i(E+i\epsilon)$ at infinity. With the usual assumption that the infinite integrals are uniformly convergent as $\epsilon\to 0$, Eq. (2.65a) yields

$$-\int d\mathbf{r}\int d\mathbf{r}''G^{(+)}(\mathbf{r},\mathbf{r}'')V_f(\mathbf{r}'')\psi_f(\mathbf{r}'')V_i(\mathbf{r})G_i^{(+)}(\mathbf{r},\mathbf{r}')+\int d\mathbf{r}\int d\mathbf{r}''G^{(+)}(\mathbf{r},\mathbf{r}'')V_i(\mathbf{r}'')G_i^{(+)}(\mathbf{r}'',\mathbf{r}')V_f(\mathbf{r})\psi_f(\mathbf{r})=0\ , \tag{2.65b}$$

where we have reinterchanged the dummy variables $\mathbf{r}$ and $\mathbf{r}''$ in the second integral of Eq. (2.65b) [compare with Eq. (2.64b)]. But we also have the analogue of Eq. (2.62),

$$(H-E)A(\mathbf{r},\mathbf{r}';E)=V_i(\mathbf{r})G_i^{(+)}(\mathbf{r},\mathbf{r}';E)\ , \tag{2.66a}$$

which—if we mistrust differentiating under the integral sign of Eq. (2.61b) at real energies—can also be derived by remembering $A=(G_i^{(+)}-G^{(+)})$ and making use of Eq. (2.55a) along with Eq. (2.10) in the form

$$(H-E)G_i^{(+)}=\delta(\mathbf{r}-\mathbf{r}')+V_iG_i^{(+)}\ . \tag{2.66b}$$

Equation (2.66a) and the real energy analogue of Eq. (2.63) for $\Phi_f(E)$ imply, via the usual manipulations, that the right-hand side of Eq. (2.61a) equals the left-hand side of Eq. (2.65b) equals zero. We conclude that Eq. (2.17b) must hold. Moreover, since the foregoing proof of Eq. (2.17b) starting from Eq. (2.59a) did not require $f\neq i$ (i.e., we could have chosen $f=i$), the proof also has demonstrated that Eq. (2.54) implies Eq. (2.17a), confirming the argument immediately beneath Eq. (2.56b).

The above proof that Eq. (2.59a) implies Eq. (2.17b) completes our mathematical analysis, except for the Appendix. Before summarizing our results, however, we must add a very important proviso. When the incident wave ψ_i represents three or more independently incident particle aggregates, i.e., when ψ_i represents a collision of the form

$$a+b+c\rightarrow d+e+\cdots\ , \tag{2.67a}$$

the integral $G^{(+)}V_i\psi_i$ need not be convergent, even for short range potentials;[20] here, $a,b,c,\ldots$ denote the various individual fundamental (i.e., noncomposite) particles comprising the multiparticle system, or denote bound aggregates of those particles. Accordingly, in "three-body" collisions (2.67a) the interchange of the order of integration and $\lim\epsilon\rightarrow 0$, needed to infer Eq. (2.52b) from Eq. (2.52a), is not justified. It follows[20] that use of Eq. (2.54) to represent the physical condition "the scattered part Φ_i is everywhere outgoing" also is suspect for collisions of the type (2.67a).

The just-mentioned convergence difficulties do not arise in all collisions of the form (2.67a), nor do they arise in all the integrals involving real energy Green's functions which we have been studying. Nevertheless, to avoid any possible convergence problems, it is prudent to restrict the analysis and conclusions of this paper to two-body collisions, i.e., to collisions of the form

$$a+b\rightarrow c+d+e+\cdots\ . \tag{2.67b}$$

For collisions of the form (2.67b), there are no convergence difficulties of the sort just mentioned (we reject[16] Benoist-Gueutal's criticisms,[11] as previously mentioned) and we see no reason to doubt our analysis. We are confident that the (free of convergence difficulties) demonstrated nonuniqueness of solutions to the real energy LS equation (2.4) for two-body collisions (2.67b) means that solutions to the real energy LS equation will be nonunique for three-body collisions (2.67a), or for the even more complicated collisions involving $n>3$ incident aggregates. We must admit, however, that for such collisions we have not modified our analysis to avoid or rectify the aforesaid convergence difficulties.

III. SUMMARY

The results we have obtained may be very briefly summarized as follows. If we make the plausible assumption that infinite integrals which arise naturally in the LS equation and equations related thereto—e.g., the integrals found in Eqs. (2.3) or (2.51b), involving the Green's functions $G_i(E+i\epsilon)$ or $G(E+i\epsilon)$—converge uniformly as $\epsilon\rightarrow 0$, then the three versions (2.3), (2.4), and (2.6b) of the LS equations are identical in the limit $\epsilon\rightarrow 0$. Under the same assumption, it can be inferred that one solution of the real energy LS equation (2.4) will be the solution Ψ_i of the Schrödinger equation whose scattered part Φ_i [defined by Eq. (2.9)] satisfies the precise outgoing condition (2.54). The same Φ_i will satisfy the analogous condition (2.17a) involving the i channel Green's function $G_i^{(+)}$; as Eq. (2.17b) shows, however, the same outgoing condition involving $G_i^{(+)}$ also is satisfied by the everywhere outgoing scattered part Φ_f [i.e., a Φ_f satisfying the precise outgoing condition (2.54)] of a solution to the Schrödinger equation having incoming part ψ_f in any channel f which is a rearrangement of the i channel. Thus, to reach Eq. (2.17d) and the consequent conclusion that Eq. (2.4) has nonunique solutions, only the single additional assumption (2.17c) need be made; both Eqs. (2.17a) and (2.17b) have now been derived. Equation (2.17c) is especially plausible, since it depends only on the very reasonable notion that at infinity $G_i^{(+)}$ has negligibly small projections on bound state eigenfunctions characterizing the f channel. Furthermore, still assuming uniform convergence, various surface integrals of the form $\mathscr{I}(X,Y)$ defined by Eq. (2.13), in which now $X\equiv G_i^{(+)}(E)$, can be related to the limit as $\epsilon\rightarrow 0$ of products $-i\epsilon\int G_i(E+i\epsilon)Y$, as in Eq. (2.25), for example; such relations can be used, e.g., to connect the Lippmann identity result (2.40a) with the corresponding value of the surface integral on the left-hand side of Eq. (2.39).[31]

Finally, and very importantly, although the existence of functions $\Psi_f(E)$ satisfying Eq. (2.17d) implies the real energy LS equation (2.4) does not have unique solutions, and although the complex energy LS equation (2.6b) reduces to Eq. (2.4) in the $\lim\epsilon\rightarrow 0$ when uniform convergence is assumed, the function $\hat{\Psi}_i(E)$ defined by Eq. (2.5) in terms of the unique $\Psi_i(E+i\epsilon)$ satisfying Eq. (2.6a) for $\epsilon>0$ is itself unique and is the desired solution $\Psi_i(E)$ of the Schrödinger equation satisfying the precise outgoing condition (2.54).[32] However, the uniqueness of this limit $\hat{\Psi}_i(E)$ does not mean that conventional methods of solving the real energy LS equation (2.4)—which methods generally do not involve obtaining the aforementioned lim$\epsilon \to 0$ of $\Psi_i(E+i\epsilon)$—will yield the desired solution $\Psi_i(E)$ without explicit imposition of the precise outgoing boundary conditions (2.54); even explicit imposition of the outgoing condition (2.17a) will not suffice to make the solution to Eq. (2.4) unique. The necessary boundary condition can be achieved, however, by considering the "basic set" of LS equations proposed by Glöckle,[33] consisting of Eq. (1.1) and the set of homogeneous equations.

$$\Psi_f = -G_i^{(+)} V_i \Psi_f \tag{3.1}$$

for all rearrangement channels f of the incident channel i.

Of course, the foregoing summary remains subject to the proviso stated at the end of Sec. II.

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APPENDIX: EQUATION (2.20a) AND UNIFORM CONVERGENCE

Consider Eq. (2.20a). The volume element $d\mathbf{r}'$ in configuration space can always be written as $d\mathbf{r}' = r'^n dr' d\mathbf{v}'$, where n is an integer depending on the dimensionality of the configuration space and $d\mathbf{v}'$ is the element of the solid angle in configuration space. For instance, with three three-dimensional particles in the laboratory system, $n=8$; in the McGuire model of three one-dimensional particles in the center of mass system, $n=1$. The essential point is that $d\mathbf{v}'$ always is integrated over a finite angular range; the infinite range of integration in the integrals of Eq. (2.20a) is completely reflected in the fact that dr' is integrated from 0 to ∞.

Now introduce, to simplify the notation,

$$Y^{(+)}(\mathbf{r},r';E) = \int d\mathbf{v}' r'^n G_i^{(+)}(\mathbf{r},\mathbf{r}';E) V_i(\mathbf{r}') \Psi(\mathbf{r}';E) \,, \tag{A1a}$$

$$Y(\mathbf{r},r';E+i\epsilon) = \int d\mathbf{v}' r'^n G_i(\mathbf{r},\mathbf{r}';E+i\epsilon) V_i(\mathbf{r}') \Psi(\mathbf{r}';E) \,. \tag{A1b}$$

Because $d\mathbf{v}'$ is integrated over a finite angular range only, the integrals (A1) may be presumed to converge. Thus Eq. (2.20a) takes the form

$$\lim_{\epsilon\to 0} \int_0^\infty dr' Y(\mathbf{r},r';E+i\epsilon) = \int_0^\infty dr' Y^{(+)}(\mathbf{r},r';E) \,. \tag{A2}$$

The assertion that the integral on the right-hand side of Eq. (A2) converges at large r' for specified $\mathbf{r},E$ means precisely the following. There exists a number N, depending on $\mathbf{r},E$ such that—given any $\eta > 0$ however small—one can find an L_0 depending on η for which

$$\left| N(\mathbf{r};E) - \int_0^L dr' Y^{(+)}(\mathbf{r},r';E) \right| < \eta \tag{A3}$$

whenever $L > L_0(\eta)$. The value assigned to the integral on the right-hand side of Eq. (A2) is, of course, the number $N(\mathbf{r},E)$ in Eq. (A3). But granted this assignment [which now provides a definition of the previously undefined expression on the right-hand side of (A2)] has been made, introduction of the symbol $N(\mathbf{r},E)$ is superfluous; one may as well symbolize this number by the original expression on the right-hand side of (A2). With this understanding, (A3) can be rewritten as

$$\left| \int_0^\infty dr' Y^{(+)}(\mathbf{r},r';E) - \int_0^L dr' Y^{(+)}(\mathbf{r},r';E) \right| < \eta \quad \text{if } L > L_0(\eta) \,. \tag{A4a}$$

Similarly, the assertion that the integral on the left-hand side of (A2) converges at large r' for specified $\mathbf{r},E$ and $\epsilon > 0$ means

$$\left| \int_0^\infty dr' Y(\mathbf{r},r';E+i\epsilon) - \int_0^L dr' Y(\mathbf{r},r';E+i\epsilon) \right| < \eta \quad \text{if } L > L_\epsilon(\eta) \,, \tag{A4b}$$

where the subscript in $L_\epsilon(\eta)$ makes explicit the dependence on ϵ (as well as on η) of the smallest allowed upper limit in the second integral under the absolute value sign. Of course, in general both L_0 and L_ϵ depend also on $\mathbf{r},E$, but in the subsequent discussion $\mathbf{r},E$ will be held fixed.

The set of integrals on the left-hand side of (A2) is said to be uniformly convergent[23] in a domain about $\epsilon=0$ if there exists an $\epsilon_m > 0$ such that—given any $\eta > 0$— for any ϵ in the open interval $0 < \epsilon \le \epsilon_m$ one can find an L_m depending on η but independent of ϵ for which

$$\left| \int_0^\infty dr' Y(\mathbf{r},r';E+i\epsilon) - \int_0^L dr' Y(\mathbf{r},r';E+i\epsilon) \right| < \eta \quad \text{if } L > L_m(\eta),\ 0 < \epsilon \le \epsilon_m \,. \tag{A5}$$

In Eq. (A5) we have followed customary practice and excluded the point $\epsilon=0$ because, in general, a sequence $Y(\mathbf{r},r';E+i\epsilon)$ appearing in a relation like (A2) may not be well defined when $\epsilon=0$, at which value of ϵ, therefore, a limiting relation such as (A3) is required to prescribe $Y(\epsilon=0)$ and to assign it a sensible value. This difficulty does not arise, however, when $Y(E+i\epsilon)$ has the specific definition (A1b).

To prove Eq. (A2) it is necessary to show that—given any $\eta_1 > 0$—there exists an $\epsilon_1(\eta_1) \le \epsilon_m$ such that (now dropping the awkward and unnecessary variables $\mathbf{r},E$)

$$\left| \int_0^\infty dr' Y^{(+)}(r') - \int_0^\infty dr' Y(r';\epsilon) \right| < \eta_1 \quad \text{if } 0 < \epsilon < \epsilon_1(\eta_1) \le \epsilon_m \,. \tag{A6}$$

In (A6) one must keep in mind the definitions of the infinite integrals therein, as explained following Eq. (A3). Thus one cannot immediately write

$$\int_0^\infty dr' Y^{(+)}(r') - \int_0^\infty dr' Y(r';\epsilon) = \int_0^\infty dr' [Y^{(+)}(r') - Y(r';E)] \,. \tag{A7a}$$

However, one can write

$$\int_0^\infty dr' Y^{(+)}(r') - \int_0^\infty dr' Y(r';\epsilon) = \left[\left[\int_0^\infty dr' Y^{(+)}(r') - \int_0^L dr' Y^{(+)}(r') \right] - \left[\int_0^\infty dr' Y(r';\epsilon) - \int_0^L dr' Y(r';\epsilon) \right] + \left[\int_0^L dr' Y^{(+)}(r') - \int_0^L dr' Y(r';\epsilon) \right] \right] . \quad \text{(A7b)}$$

in the last set of large parentheses in (A7b) it is legitimate to write

$$\int_0^L dr' Y^{(+)}(r') - \int_0^L dr' Y(r';\epsilon) = \int_0^L dr' [Y^{(+)}(r') - Y(r';\epsilon)] . \quad \text{(A8)}$$

Moreover, because dv' is integrated over a finite range, and because Eq. (2.7) holds, it is easy to show

$$\lim_{\epsilon\to 0} Y(r',\epsilon) = Y^{(+)}(r') . \quad \text{(A9a)}$$

It follows that—given any $\eta_2 > 0$—we can find an $\epsilon_2(\eta_2) \le \epsilon_m$ such that

$$| Y^{(+)}(r') - Y(r';\epsilon) | < \eta_2 \ \text{ if } 0 < \epsilon < \epsilon_2(\eta_2) \le \epsilon_m . \quad \text{(A9b)}$$

Now choose η in (A4a) equal to $\eta_1/6$, where η_1 is the assigned value on the right-hand side of (A6). Then, for the term in the first set of large parentheses on the right-hand side of (A7b),

$$\left| \int_0^\infty dr' Y^{(+)}(r') - \int_0^L dr' Y^{(+)}(r') \right| < \eta_1/6 \quad \text{if } L > L_0(\eta_1/6) . \quad \text{(A10a)}$$

Similarly, choose η in (A5) equal to $\eta_1/6$. Then, for the term in the second set of large parentheses on the right-hand side of (A7b),

$$\left| \int_0^\infty dr' Y(r';\epsilon) - \int_0^L dr' Y(r';\epsilon) \right| < \eta_1/6 \quad \text{if } L > L_m(\eta_1/6),\ 0 < \epsilon \le \epsilon_m . \quad \text{(A10b)}$$

Next let L, which was not specified in (A7b), be fixed at some value consistent with both (A10a) and (A10b), i.e., L is fixed at some value $L(\eta_1/6)$ exceeding the larger of $L_0(\eta_1/6)$ and $L_m(\eta_1/6)$. The important point is that, because of the postulated uniform convergence, Eqs. (A10a) and (A10b) can be simultaneously satisfied by appropriate choice of an $L(\eta_1/6)$ independent of ϵ. Finally, in (A9b), choose $\eta_2 = \eta_1/6L$, where $L \equiv L(\eta_1/6)$. Since L is a finite (though possibly very large) number independent of ϵ, this choice of η_2 is legitimate. Then, for the term in the third set of large parentheses in (A7b), using (A8),

$$\begin{aligned} \left| \int_0^L dr' Y^{(+)}(r') - \int_0^L dr' Y(r';\epsilon) \right| &= \left| \int_0^L dr' [Y^{(+)}(r') - Y(r';\epsilon)] \right| \\ &\le \int_0^L dr' \, | Y^{(+)}(r') - Y(r';\epsilon) | \\ &< \int_0^L dr' \frac{\eta_1}{6L} = \frac{\eta_1}{6} \quad \text{if } 0 < \epsilon < \epsilon_2(\eta_1/6L) \le \epsilon_m . \end{aligned} \quad \text{(A10c)}$$

Hence if $\epsilon < \epsilon_2(\eta_1/6L)$, Eqs. (A7b) and (A10) imply

$$\left| \int_0^\infty dr' Y^{(+)}(r') - \int_0^\infty dr' Y(r';\epsilon) \right| < \frac{\eta_1}{6} + \frac{\eta_1}{6} + \frac{\eta_1}{6} = \frac{\eta_1}{2} < \eta_1 \quad \text{if } 0 < \epsilon < \epsilon_2(\eta_1/6L) \le \epsilon_m . \quad \text{(A11)}$$

Equation (A11) demonstrates that the desired inequality (A6) will hold, provided $\epsilon_1(\eta_1)$ in (A6) is chosen $\le \epsilon_2(\eta_1/6L)$, Q.E.D.

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[24]The proof is adapted from E. Gerjuoy, University of Pittsburgh Space Research Coordinating Center Report No. 129, 1970 (unpublished).

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[26]As Hobson points out, the failure of Eq. (2.7) at the isolated singularity is inconsequential because the fact that this singularity is integrable means that the contribution near $\mathbf{r}'-\mathbf{r}$ is negligible for all integrals in Eq. (2.21a).

[27]It has been argued that $\lim i\epsilon G_i(\mathbf{r},\mathbf{r}';E+i\epsilon)$ and $\{G_i^{(+)}(\mathbf{r},\mathbf{r}';E)T-[TG_i^{(+)}(\mathbf{r},\mathbf{r}';E)]\}$ [where $T\equiv T(\mathbf{r}')$ is the kinetic energy operator of Eq. (2.13)] are equivalent operators in $\mathbf{r}'$ space, a result which immediately would imply Eqs. (2.25), (2.29), (2.39), and (2.41a). W. Tobocman, case Western Reserve Physics Department Report, Oct. 1981 (unpublished).

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[32]That $\Psi_i(E+i\epsilon)$ is a unique solution of Eq. (2.6a) for $\epsilon>0$, and that its limit as $\epsilon\to0$ then is a unique $\Psi_i(E)$ satisfying the real energy LS equation (2.4) and the outgoing condition (2.54) appear to be the conclusions reached by Mukherjee; if these were his only conclusions, we would not quarrel with him. See Ref. 10.

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