Covariant time-ordered perturbation theory

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By reformulating Kadyshevsky's perturbation theory for the S matrix in quantum field theory, a covariant version of time-ordered perturbation theory is obtained. The new graphical rules are just like those of time-ordered perturbation theory except for the replacement of three-momentum-conserving δ functions with covariant three-dimensional δ functions and the use of invariant denominators instead of energy denominators. The new rules are illustrated by deriving an approximate three-dimensional integral equation which describes the scattering of two identical scalar particles which interact by exchanging a different scalar particle.

I. INTRODUCTION

In the standard S-matrix perturbation theory used in quantum field theory,[1] the various terms in the series are represented by Feynman diagrams. As is well known, for these diagrams the total four-momentum is conserved at each vertex and the intermediate particles are virtual in the sense that they are off-the-mass shell. Integral equations for scattering amplitudes can be obtained by summing series of diagrams. The well-known Bethe-Salpeter equation[2] arises in this way. Since all four components of the four-momentum of an intermediate particle in a Feynman diagram are independent variables, the two-particle Bethe-Salpeter equation becomes a four-dimensional integral equation when the conservation of the total four-momentum is taken into account. One of the distinctive features of this equation is the appearance of a relative energy variable which is not present in the standard integral equation of nonrelativistic potential scattering, i.e., the Lippmann-Schwinger equation.[3]

A highly desirable feature of Feynman diagrams is that the corresponding contributions to the S-matrix are individually covariant. This is accomplished by lumping together intermediate states with different numbers of particles and antiparticles. A disadvantage of this is that when series of diagrams are summed to obtain integral equations, it is difficult to justify the omission of various types of diagrams, since there is no one-to-one relation between internal lines and intermediate states.

To a Feynman diagram with n vertices there corresponds $n!$ terms of time-ordered perturbation theory[4–6] (TOPT). Each of the $n!$ diagrams of TOPT looks like a Feynman diagram with a particular ordering of the vertices. At each vertex of a TOPT diagram the total three-momentum is conserved and the intermediate particles are real, i.e., on-the-mass shell. Energy denominators are associated with intermediate states, in contrast with Feynman diagrams where propagators correspond to internal lines.

Individual terms in TOPT are not covariant. In order to obtain a covariant result it is necessary to sum the diagrams corresponding to the various orderings of the vertices. Integral equations obtained by summing series of TOPT diagrams have fewer variables than those derived from Feynman diagrams, but of course there are problems with covariance.

Several years ago Kadyshevsky[7] developed an S-matrix perturbation theory which is a covariant reformulation of TOPT. In the diagrams of this theory appear the particles associated with the underlying quantum field theory, as well as so-called quasiparticles or spurions. The total four-momentum of the particles and quasiparticles is conserved at each vertex, and the physical particles in intermediate states are real. Due to the presence of the quasiparticles, the total four-momentum of the physical particles is not conserved at each vertex.

In contrast to TOPT, individual Kadyshevsky diagrams lead to covariant expressions. Summing series of these diagrams leads to integral equations with fewer variables than those derived from Feynman diagrams.[8,9] In particular, for two-particle scattering the integral equations are three-dimensional and no relative energy variable appears.

The appearance of the quasiparticles in the Kadyshevsky diagrams makes them much more complicated than Feynman diagrams or the diagrams of TOPT. The purpose of the present work is to reformulate Kadyshevsky's approach so as to eliminate the quasiparticles. By so doing a covariant version of TOPT is obtained whose graphical rules are much closer to the rules of TOPT than those of Kadyshevsky. In fact, the new rules are the same as those of TOPT except for the replacement of three-momentum conserving δ functions by covariant three-dimensional δ functions and the use of invariant denominators instead of energy denominators.

The covariant time-ordered perturbation theory (CTOPT) developed here becomes identical to TOPT in a special set of Lorentz frames, called λ frames. These frames arise as the result of introducing a timelike unit four-vector, denoted by λ, which assigns an invariant time direction. A λ frame is one in which $\lambda=(1,\mathbf{0})$. If the λ vector is chosen to be parallel to the total four-momentum of the system, a λ frame is the same as a c.m. frame.

One of the nice features of the formalism developed here is that it leads more directly to covariant few particle integral equations than Kadyshevsky's approach. This aspect of the CTOPT will not be fully developed here, but will be the subject of future publications. Here we shall be content with a simple example.

The covariant formal expansion of the S matrix is developed in Sec. II. The graphical rules are presented in Sec. III in the context of a simple field theory in which scalar "nucleons" emit and absorb scalar "mesons." The rules are illustrated by deriving an approximate integral equation for "nucleon-nucleon" scattering. A brief discussion and suggestions for future work is given in Sec. IV.

II. BASIC FORMALISM

We begin by considering the S operator defined by

$$S=1+\sum_{n=1}^{\infty}(-i)^n\int d^4x_1d^4x_2\cdots d^4x_nH_I(x_1)\theta[\lambda\cdot(x_1-x_2)]H_I(x_2)\theta[\lambda\cdot(x_2-x_3)]\cdots\theta[\lambda\cdot(x_{n-1}-x_n)]H_I(x_n)\,, \tag{1}$$

where $H_I(x)$ is a Lorentz invariant Hamiltonian density, $\theta(\tau)$ is the unit step function, and $\lambda=(\lambda_0,\boldsymbol{\lambda})$ is a timelike unit vector with the properties

$$\lambda^2=1,\ \ \lambda_0>0\,. \tag{2}$$

Since λ is timelike, we can always find a frame in which $\lambda=(1,\mathbf{0})$. We will call such a frame a λ frame, and will distinguish quantities evaluated in these frames by a subscript λ.

It is well known[1] that there exists a unitary operator $U(a,b)$ which transforms the Hamiltonian density according to

$$U(a,b)H_I(x)U(a,b)^{-1}=H_I(ax+b)\,, \tag{3}$$

where the set of parameters symbolized by a and b occur in the coordinate transformation

$$x'^{\mu}=a^{\mu}{}_{\nu}x^{\nu}+b^{\mu}\,. \tag{4}$$

It should be kept in mind that we are working in the interaction representation, so that $U(a,b)$ involves the *free-field* momentum and angular momentum operators. If in each of the integrals occurring in (1) we make a transformation according to

$$x=a(\lambda)x_\lambda\,, \tag{5}$$

where the components of x_λ are evaluated in a λ frame, then from (3) it follows that

$$S=U[a(\lambda)]S_\lambda U[a(\lambda)]^{-1}\,, \tag{6}$$

with

$$S_\lambda=1+\sum_{n=1}^{\infty}(-i)^n\int d^4x_{1\lambda}d^4x_{2\lambda}\cdots d^4x_{n\lambda}H_I(x_{1\lambda})\theta(t_{1\lambda}-t_{2\lambda})H_I(x_{2\lambda})\theta(t_{2\lambda}-t_{3\lambda})\cdots\theta(t_{n-1\lambda}-t_{n\lambda})H_I(x_{n\lambda})\,. \tag{7}$$

Equation (7) is a well-known expression for the S operator,[5] and it is equally well known that it can be rewritten in the form

$$S_\lambda=1+\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int d^4x_{1\lambda}d^4x_{2\lambda}\cdots d^4x_{n\lambda}T[H_I(x_{1\lambda})H_I(x_{2\lambda})\cdots H_I(x_{n\lambda})]\,, \tag{8}$$

where T symbolizes the time-ordering operation. This operation is Lorentz invariant as long as

$$[H_I(x),H_I(x')]=0 \text{ for } (x-x')^2<0\,, \tag{9}$$

which in turn is a consequence of microcausality. By putting (8) into (6) and using (3) and (5), we find that

$$S=1+\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int d^4x_1d^4x_2\cdots d^4x_n \times T[H_I(x_1)H_I(x_2)\cdots H_I(x_n)]\,. \tag{10}$$

Thus we have shown that Eq. (1) is equivalent to the standard expression for the S operator given by Eq. (10). It is well known[5] that the Feynman diagram approach to quantum field theory can be developed by using Wick's theorem to work out the matrix elements of the terms in (10). These diagrams have the virtue of being individually covariant. This is in contrast to time-ordered perturbation theory [which can be derived[10] from (7)] where it is necessary to combine the time-ordered diagrams corresponding to a particular Feynman diagram so as to obtain an explicitly covariant result. We will see that when the matrix elements of the terms in (1) are evaluated, a generalization of time-ordered perturbation theory is obtained in which the time-ordered diagrams are individually covariant.

It is straightforward to show that (1) can be rewritten in the form

$$S=1+\sum_{n=1}^{\infty}(-i)^n\int d\tau_1d\tau_2\cdots d\tau_nV_\lambda(\tau_1)\theta(\tau_1-\tau_2)V_\lambda(\tau_2) \times\theta(\tau_2-\tau_3)\cdots\theta(\tau_{n-1}-\tau_n)V_\lambda(\tau_n)\,, \tag{11}$$

where

$$V_\lambda(\tau)=\int d^4x\, H_I(x)\delta(\tau-\lambda\cdot x)\,, \tag{12a}$$

$$=\int d^4x\, H_I(x+\lambda\tau)\delta(\lambda\cdot x)\,. \tag{12b}$$

In going from (12a) to (12b), we have used (2). It is clear that (11) can also be obtained from the equations

$$i\frac{\partial S(\tau,\tau_0)}{\partial \tau}=V_\lambda(\tau)S(\tau,\tau_0) , \quad (13a)$$

$$S(\tau_0,\tau_0)=1 , \quad (13b)$$

$$S=S(\infty,-\infty) . \quad (13c)$$

These equations, which supply a covariant reformulation of the interaction picture, have been previously derived by Kadyshevsky,[7] but in a less direct manner. This formulation can also be obtained by specializing the spacelike surfaces in the Tomonaga-Schwinger equation[11,12] to those defined by $\lambda\cdot x=\tau$.

From (12) and (3), it follows immediately that

$$V_\lambda(\tau)=U(\lambda\tau)V_\lambda U(\lambda\tau)^{-1} , \quad (14)$$

where

$$V_\lambda=V_\lambda(0)=\int d^4x\, H_I(x)\delta(\lambda\cdot x) . \quad (15)$$

Since $U(\lambda\tau)$ describes a space-time translation, it is given by

$$U(\lambda\tau)=e^{i\lambda\cdot P\tau} , \quad (16)$$

where P^μ is the *free* four-momentum operator.

Although it is somewhat beside the point, it is interesting to note that the operator S_s defined by

$$S_s(\tau,\tau_0)=U(\lambda\tau)^{-1}S(\tau,\tau_0)U(\lambda\tau_0) \quad (17)$$

satisfies the equation

$$i\frac{\partial S_s(\tau,\tau_0)}{\partial\tau}=(\lambda\cdot P+V_\lambda)S_s(\tau,\tau_0) , \quad (18)$$

where we have used (13a), (16), and (14). This gives a covariant reformulation of the Schrödinger picture.

Upon putting (14) into (11) and using the identity[10]

$$(-i)\int_{-\infty}^{\tau} d\tau' e^{i(\lambda\cdot P-\lambda\cdot Q)\tau'}=\frac{e^{i(\lambda\cdot P-\lambda\cdot Q)\tau}}{\lambda\cdot Q+i\epsilon-\lambda\cdot P} \underset{\tau\to\infty}{\longrightarrow} -2\pi i\delta(\lambda\cdot P-\lambda\cdot Q) , \quad (19)$$

we find

$$(S-1)|Q\rangle=-2\pi i\delta(\lambda\cdot P-\lambda\cdot Q) \times\left[V_\lambda+V_\lambda\frac{1}{\lambda\cdot Q+i\epsilon-\lambda\cdot P}V_\lambda+V_\lambda\frac{1}{\lambda\cdot Q+i\epsilon-\lambda\cdot P}V_\lambda\frac{1}{\lambda\cdot Q+i\epsilon-\lambda\cdot P}V_\lambda+\cdots\right]|Q\rangle , \quad (20)$$

where $|Q\rangle$ is any free state with a well-defined total four-momentum Q, i.e.,

$$P^\mu|Q\rangle=Q^\mu|Q\rangle . \quad (21)$$

Equation (20) is the basic result. In a λ frame, i.e., where $\lambda=(1,\mathbf{0})$, $\lambda\cdot Q=E$, $\lambda\cdot P=H_0$, and $V_\lambda=H_1$ where E, H_0, and H_1 are the total energy, the free Hamiltonian, and the interaction Hamiltonian, respectively; so in this frame (20) reduced to time-ordered perturbation theory. In the next section we will give rules for the graphical interpretation of (20) when V_λ arises from a simple field theory.

III: AN EXAMPLE

For the interaction Hamiltonian density we take

$$H_I(x)=g:\psi^\dagger(x)\phi(x)\psi(x): , \quad (22)$$

where $\phi(x)$ and $\psi(x)$ are Hermitian and non-Hermitian scalar fields, respectively. Their Fourier decompositions are given by

$$\phi(x)=\int\frac{d^3k}{[(2\pi)^3 2\omega_k]^{1/2}}[a(k)e^{-ik\cdot x}+a^\dagger(k)e^{ik\cdot x}] , \quad k^2=\mu^2 , \quad (23)$$

and

$$\psi(x)=\int\frac{d^3p}{[(2\pi)^3 2E_p]^{1/2}}[b(p)e^{-ip\cdot x}+d^\dagger(p)e^{ip\cdot x}] , \quad p^2=m^2 , \quad (24)$$

where

$$\omega_k=(\mathbf{k}^2+\mu^2)^{1/2} , \quad E_p=(\mathbf{p}^2+m^2)^{1/2} . \quad (25)$$

When these decompositions are used in (22), and (22) is put into (15), we encounter

$$\delta_\lambda(q)=\frac{1}{(2\pi)^3}\int d^4x\, e^{iq\cdot x}\delta(\lambda\cdot x) , \quad (26a)$$

$$=\frac{1}{\lambda_0}\delta^3\left[\mathbf{q}-q_0\frac{\boldsymbol{\lambda}}{\lambda_0}\right] , \quad (26b)$$

where q is a linear combination of on-mass-shell four-momenta. From (26a), it follows that

$$\delta_\lambda(q)=\delta_{\lambda'}(q') \quad (27a)$$

with

$$\lambda'=a\lambda \text{ and } q'=aq , \quad (27b)$$

so that $\delta_\lambda(q)$ is a covariant three-dimensional δ function. If we choose the prime frame to be the λ frame, we see that

$$\delta_\lambda(q)=\delta^3(\mathbf{q}_\lambda) . \quad (28)$$

It will be convenient to have the relations between the components of a four-vector in a λ frame $(x_{0\lambda},\mathbf{x}_\lambda)$ and its components in a general frame $(x_0,\mathbf{x})$. It should be kept in mind that the λ frame is not unique, since given any λ frame another one can be obtained by carrying out a rotation of the spatial coordinates. A simple choice is obtained by choosing the parameters of the Lorentz transformation from the general frame to the λ frame to be

$$\beta=\lambda/\lambda_0,\quad \gamma=\lambda_0\ , \tag{29}$$

for which the relations become

$$x_{0\lambda}=\lambda_0 x_0-\boldsymbol{\lambda}\cdot\mathbf{x}=\lambda\cdot x\ , \tag{30a}$$

$$\mathbf{x}_\lambda=\mathbf{x}+\left[\frac{\mathbf{x}\cdot\boldsymbol{\lambda}}{\lambda_0+1}-x_0\right]\boldsymbol{\lambda}\ . \tag{30b}$$

It is important to note that $x_{0\lambda}$ is a Lorentz scalar and that the three-dimensional dot product

$$\mathbf{x}_\lambda\cdot\mathbf{y}_\lambda=(\lambda\cdot x)(\lambda\cdot y)-x\cdot y \tag{31}$$

is also. If we choose $\lambda=Q/W$ where

$$W=+(Q\cdot Q)^{1/2}\ , \tag{32}$$

the three vectors $\mathbf{x}_\lambda$ are the same as the so-called "special vectors" introduced some time ago by Aaron, Amado, and Young.[13,14]

It is important to note that it follows from (28) and (30a) that

$$\delta(\lambda\cdot q)\delta_\lambda(q)=\delta^4(q)\ . \tag{33}$$

For the field theory we are considering the S matrix for a transition $\alpha\rightarrow\beta$ can be written

$$S_{\beta\alpha}=\delta_{\beta\alpha}-(2\pi)^4 i\delta^4(Q_\beta-Q_\alpha)\Omega_{\beta\alpha}T_{\beta\alpha}\ . \tag{34}$$

Here Q_α and Q_β are the total initial and final four-momenta, respectively. According to (33) the four-dimensional δ function can also be written as

$$\delta^4(Q_\beta-Q_\alpha)=\delta(\lambda\cdot Q_\beta-\lambda\cdot Q_\alpha)\delta^3(\mathbf{Q}_{\beta\lambda}-\mathbf{Q}_{\alpha\lambda})\ . \tag{35}$$

The quantity $\Omega_{\beta\alpha}$ is a product of external factors — one for each particle in the initial and final states. The factor for a ψ-field particle is $(2\pi)^{-3/2}(2E_p)^{-1/2}$, while for a ϕ-field particle it is $(2\pi)^{-3/2}(2\omega_k)^{-1/2}$. These factors arise because the states α and β are defined with a conventional noncovariant norm, e.g., $\langle\mathbf{p}'\,|\,\mathbf{p}\rangle=\delta^3(\mathbf{p}'-\mathbf{p})$. The matrix element $T_{\beta\alpha}$ is a Lorentz-invariant function of the incoming and outgoing momenta.

The results of the covariant perturbation theory obtained from (20), (15), and (22) can be summarized in a set of graphical rules for constructing $T_{\beta\alpha}$:

(a) Draw all possible ordered diagrams for the transition $\alpha\rightarrow\beta$. That is, draw each nth order Feynman diagram $n!$ times, ordering the n vertices in every possible way in a sequence running from right to left, with lines for the particles in the initial state α and the final state β entering on the right and leaving on the left, respectively. Label each line with an *on-mass-shell* momentum p ($p^2=m^2$ or $p^2=\mu^2$).

(b) For every internal line include a factor

$$(2\pi)^{-3}(2E_p)^{-1},\quad (\psi \text{ particle}) \tag{36a}$$

or

$$(2\pi)^{-3}(2\omega_k)^{-1}\ (\phi \text{ particle})\ . \tag{36b}$$

(c) For every vertex except the last, include a factor

$$(2\pi)^3\delta_\lambda(\Delta q)\ , \tag{37}$$

where Δq is the total four-momentum leaving the vertex minus the total four-momentum entering the vertex. The factor for the last vertex is already included in (34). Of course, a factor of g is associated with each vertex.

(d) For every intermediate state, i.e., a set of lines between any two adjacent vertices, include a denominator

$$(\lambda\cdot Q_\alpha+i\epsilon-\lambda\cdot Q_\gamma)^{-1}\ , \tag{38}$$

where Q_γ is the total four-momentum of the intermediate state.

(e) Integrate the product of these factors over all internal *three-momenta,* and sum the result over all diagrams to obtain $T_{\beta\alpha}$.

These rules are exactly the same as those for time-ordered perturbation theory[6] except for the replacement of a three-momentum conserving δ function by δ_λ and the use of the invariant denominator (38) instead of an energy denominator. In a λ frame the rules for the covariant theory give exactly the same results as the time-ordered theory. As pointed out above, individual diagrams arising from (20) are covariant, however the result does depend on λ. In order to get a result independent of λ it is necessary to sum the $n!$ ordered diagrams corresponding to a Feynman diagram.

In order to illustrate the graphical rules and to demonstrate their simplicity as compared to the earlier formulation[7] based on (1), we shall consider the scattering of two ψ particles, i.e., the particles created by the $b^\dagger$'s. The initial and final states are given by

$$\begin{aligned}|\,\alpha\rangle&=b^\dagger(\mathbf{p}_1)b^\dagger(\mathbf{p}_2)\,|\,0\rangle\ ,\\ |\,\beta\rangle&=b^\dagger(\mathbf{p}_1')b^\dagger(\mathbf{p}_2')\,|\,0\rangle\ .\end{aligned} \tag{39}$$

We begin by considering the diagram in Fig. 1. The vertical line here and in subsequent diagrams indicates the intermediate states that must be taken into account. According to the rules, Fig. 1 gives the result

$$V_{\beta\alpha}^{(1)}=g^2\int\frac{d^3k}{(2\pi)^3 2\omega_k}\frac{(2\pi)^3\delta_\lambda(p_2'+k-p_2)}{\lambda\cdot Q+i\epsilon-\lambda\cdot(p_1+k+p_2')}\ , \tag{40}$$

where

$$Q=p_1'+p_2'=p_1+p_2\ . \tag{41}$$

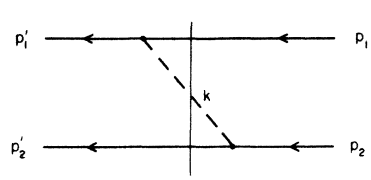

FIG. 1. Exchanged particle with one ordering.

The invariance of the result is obvious. In order to do the integral we use the fact that the volume element $d^3k/2\omega_k$ is an invariant and carry out the integration in a λ frame with the help of (28). We get

$$V^{(1)}_{\beta\alpha}=\frac{g^2}{2\lambda\cdot k[\lambda\cdot Q+i\epsilon-\lambda\cdot(p_1+k+p'_2)]} \tag{42}$$

with

$$\mathbf{k}_\lambda=\mathbf{p}_{2\lambda}-\mathbf{p}'_{2\lambda}=\mathbf{p}'_{1\lambda}\mathbf{p}_{1\lambda}\ , \tag{43}$$

where we have used (30a) and the fact that the particles are on the mass shell. The second expression for $\mathbf{k}_\lambda$ follows from the conservation of three-momentum in the λ frame [see (35)]. Using the same analysis we find that Fig. 2 leads to the expression

$$V^{(2)}_{\beta\alpha}=\frac{g^2}{2\lambda\cdot k[\lambda\cdot Q+i\epsilon-\lambda\cdot(p'_1+k+p_2)]}\ , \tag{44}$$

where $\mathbf{k}_\lambda$ is again given by (43). It should be noted that (42) and (44) depend only on $\mathbf{k}^2_\lambda$. With the help of (31), (41), and (43), we find that

$$V^{(1)}_{\beta\alpha}+V^{(2)}_{\beta\alpha}=\frac{g^2}{(p'_1-p_1)^2-\mu^2+i\epsilon}\ , \tag{45}$$

which is independent of λ. It is important to appreciate that it is necessary to assume (41) in order to obtain this result, i.e., summing the various time-ordered diagrams corresponding to a Feynman diagram gives an expression independent of λ only for a physical process.

We now consider the sum of (42) and (44) when we choose $\lambda=Q/W$ with W given by (32). The λ frame is now a c.m. frame so that the total three-momentum in the λ frame is zero, i.e., $\boldsymbol{\lambda}_\lambda=\mathbf{Q}_\lambda=\mathbf{0}$. If we use (30a) and the fact that the particles are on the mass shell (e.g., $\lambda\cdot k=k_{0\lambda}=\omega_{k_\lambda}$), we find that Figs. 1 and 2 combine to give

$$V(\mathbf{p}',\mathbf{p};Q)=\frac{g^2}{\omega_{k_\lambda}(W+i\epsilon-E_{p'_\lambda}-\omega_{k_\lambda}-E_{p_\lambda})}\ , \tag{46a}$$

with

$$\mathbf{k}_\lambda=\mathbf{p}'_\lambda-\mathbf{p}_\lambda\ , \tag{46b}$$

where $\mathbf{p}'$ and $\mathbf{p}$ are either $\mathbf{p}'_1$ and $\mathbf{p}_1$ or $\mathbf{p}'_2$ and $\mathbf{p}_2$. We have not used conservation of energy in the c.m. frame as this would simply lead us back to (45). Equation (46) gives an approximate quasipotential for ψ-ψ scattering. The three-momenta $\mathbf{p}'_\lambda$ and $\mathbf{p}_\lambda$ that appear in (46) are related to $\mathbf{p}'$ and $\mathbf{p}$, respectively, by (30b) with $\lambda=(Q_0/W,\ \mathbf{Q}/W)$. It is important to note that (46) contains only three-vector dot products in the λ frame [recall (25)] which according to (31) are invariants, thus the potential given by (46) is a Lorentz scalar.

It is not difficult to show that if we sum the four diagrams given in Fig. 3 with $\lambda=Q/W$ we obtain the expression

$$\frac{1}{(2\pi)^3}\int V(\mathbf{p}',\mathbf{q};Q)\frac{d^3q}{2E_q}\frac{V(\mathbf{q},\mathbf{p};Q)}{2E_{q_\lambda}(W+i\epsilon-2E_{q_\lambda})}\ . \tag{47}$$

This immediately suggests that an approximate integral equation for ψ-ψ scattering is given by

$$T(\mathbf{p}',\mathbf{p};Q)=V(\mathbf{p}',\mathbf{p};Q)+\frac{1}{(2\pi)^3}\times\int V(\mathbf{p}',\mathbf{q};Q)\frac{d^3q}{2E_q}\frac{T(\mathbf{q},\mathbf{p};Q)}{2E_{q_\lambda}(W+i\epsilon-2E_{q_\lambda})}\ , \tag{48}$$

where

$$\mathbf{l}_\lambda=\mathbf{l}+\left[\frac{\mathbf{l}\cdot\mathbf{Q}}{Q_0+W}-E_l\right]\frac{\mathbf{Q}}{W}\ , \tag{49a}$$

with

$$\mathbf{l}=\mathbf{p}',\ \mathbf{p},\ \text{or}\ \mathbf{q}\ . \tag{49b}$$

The solution of (48) gives the ladder approximation for ψ-ψ scattering.

It is important to appreciate the fact that (48) is explicitly covariant, in fact, it involves only invariants. The quantity $d^3q/2E_q$ is an invariant volume element, and all of the other energies that appear [see (46) and (25)] contain only invariants of the form (31). If so desired, the three-vector dot products in the λ frame that arise in (48) can be expressed in terms of the momenta of any frame by

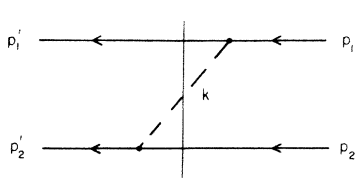

FIG. 2. Exchanged particle with other ordering.

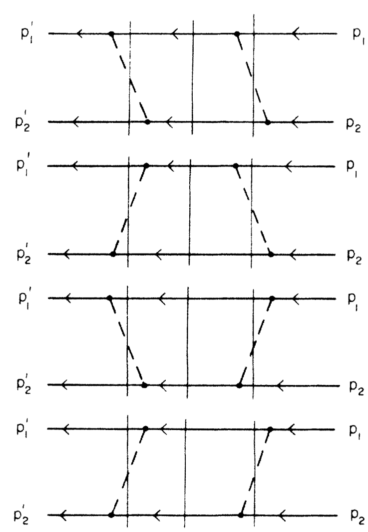

FIG. 3. Some two-particle exchange diagrams.

using (49). The explicit covariance of (48) is a general feature of integral equations obtained by summing series of diagrams in the CTOPT developed here.

IV. DISCUSSION

It is not anticipated that the graphical rules presented here will replace the use of Feynman diagrams in perturbation theory calculations, however for developing integral equations for few-particle systems CTOPT does have some advantages. In particular, for the two-body problem it leads directly to three-dimensional equations. It is not necessary to start with a four-dimensional equation and then reduce it to three dimensions, as is commonly done.[15–17] Also, since there is a one-to-one correspondence between internal lines and intermediate states in the diagrams developed here, integral equations obtained by summing a series of such diagrams can be characterized precisely in terms of the intermediate states allowed. In a sense, the formalism developed here makes it possible to apply the Tamm-Dancoff method in a covariant way.

It is interesting to note that the technique used by Bhalerao and Gurvitz[17] to reduce the two-body Bethe-Salpeter equation[2] to three-dimensional form leads back to the diagrams of TOPT. They have developed a method for summing these diagrams which leads to practical three-dimensional integral equations, however their equations are not explicitly covariant. By redoing their analysis with the approach developed here, explicit covariance can be restored.

Recently, the author[18] has carried out an analysis of a π-N-Δ field theory with static fermions using projection operator techniques. Exact two-particle and three-particle equations for π-N scattering have been derived. If the one-fermion irreducible part of the three-body interaction that appears is neglected a closed set of coupled nonlinear integral equations for all of the quantities of interest is obtained. These results are very encouraging, however the assumption of static fermions brings their general validity into question. By combining the formalism developed here with the projection operator techniques of Ref. 18, it should be possible to derive two-particle and three-particle equations for the π-N system starting from a covariant field theory. Hopefully the results obtained in the static limit will continue to have some validity when recoil is taken into account.

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