Convergence of distorted wave methods: Theory and a simple example

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The distorted wave Born series for inelastic scattering can be formulated as arising from the iteration of an integral equation. The small parameter determining the convergence of the series can then be identified as the spectral radius of the integral kernel. Such a formulation allows a detailed investigation of what controls the accuracy of the distorted-wave Born approximation. Several important questions can then be answered, such as the following: what are the relevant dimensionless parameters, what physics is important, and what is the best distortion potential to use. We consider these questions in a simple, analytically solvable two-channel model.

I. INTRODUCTION

The distorted-wave Born approximation[1] (DWBA) is one of the most commonly used methods in all of reaction theory. It is highly successful in nuclear,[2] atomic,[3] and chemical physics.[4] Despite this success, there has been little analysis of what physics controls the accuracy of the DWBA. In this paper and in subsequent papers in this series, we investigate the convergence of the distorted-wave Born series (DWBS).

This problem is of considerable importance in modern nuclear reaction theory. In the 1960's, when nuclear physicists were first beginning to study nuclear reactions to specific states, the DWBA was an adequate tool. At that point, it included a substantial phenomenological component. Both the potentials and the effective interactions were considered as "free" parameters to be fixed by experiment—ideally through experiments other than the one being analyzed.

As more reactions are studied, and as the theory of the optical potential becomes more sophisticated, more complex rationalizations and modifications of the DWBA have become required. Poorly justified nonlocalities or multistep processes are invoked to explain away failures to fit the data. The choice of which of many possible calculations to do almost plays the role of a new phenomenological free parameter. In order to avoid this undesirable ambiguity, a theory of corrections must be developed to accompany the DWBA. We want to do this in a way which increases our physical understanding of how a process is controlled.

We study the DWBA by adopting and extending the methods Weinberg applied to the undistorted Born series.[5] These methods rely on the mathematics of functional analysis and the associated theory of operators. A number of excellent texts exist describing these methods.[6] Principally, Weinberg learned that the convergence of the Born series is determined by the presence of "quasiparticles," eigenstates of the Lippmann-Schwinger integral kernel.

An analogous method may be applied to the DWBA by formulating the approximation as the inhomogeneous term of an integral equation. The multistep series is the Neumann series obtained by iterating the equation. For this paper, we restrict our considerations to inelastic scattering. This case is simpler than that of rearrangement since there is a unique asymptotic Hamiltonian. The more interesting case of rearrangement can be handled in a similar way but is more complex. Note that the oft-cited claim of Greider and Dodd[7] that such a formulation is impossible is not correct. For a discussion of this point, see Redish[8] or Bencze and Redish.[9]

The basic idea of the method is as follows. Suppose we have a transition operation, T, which satisfies the integral equation

$$T = B + KT \ . \tag{1}$$

The matrix elements of the operator B represent our first approximation. Corrections are obtained by iteration. Assume that K has a complete set of eigenvectors, $|n\rangle$, with eigenvalues λ_n. The iteration yields the expansion

$$T|i\rangle = \sum_n (1+\lambda_n+\lambda_n^2+\lambda_n^3+\cdots)|n\rangle\langle n|B|i\rangle \tag{2}$$

corresponding to a multistep series. The convergence is determined by the size of the largest eigenvalue. If the parameter defined by

$$\eta = \max_{n \ni \langle n|B|i\rangle \neq 0} |\lambda_n| \tag{3}$$

is less than unity, then the multistep series will necessarily converge. (The convergence of the n sum is guaranteed by the compactness of the kernel.) If $|\lambda_n|$ is greater than unity for any n for which $\langle n|B|i\rangle$ is nonzero, then the series will diverge.[10] We refer to η as the convergence parameter for the multistep series. More precisely, it is the spectral radius of the kernel of the integral equation.

In this paper we show how to use this convergence parameter to study the DWBA and its multistep series. We specifically analyze a simple analytically solvable two-channel model. Although the model is not particularly realistic, it illustrates clearly how the method is applied. Some of the results are instructive for guiding the analysis of more realistic (and more complicated) problems. Useful results have been obtained and have been presented

previously.[10–12] These more realistic problems will be discussed in detail in later papers in this series.

The paper has the following structure. In Sec. II, we review Weinberg's quasiparticle method for the Born series and apply it to the distorted-wave Born series for potential scattering (a one-channel model). In Sec. III, we introduce the two-channel separable model, solve it, and apply our analysis to the distorted wave series. Three natural distortion potentials are considered: the folding potential, the optical potential, and the minimal potential. The minimal potential is that distortion potential which minimizes the convergence parameter. Because of the simple structure of the model we are able to find the minimal potential analytically. Our analysis of the results is given in Sec. IV, and our conclusions in Sec. V.

II. ONE-CHANNEL MODEL

In this section we analyze perturbative treatments of a one-channel separable potential model. This is the simplest possible example. It permits a detailed analysis of the Born series and the distorted-wave multistep series.

A. The Born series

We begin by discussing the convergence of the Born series for potential scattering. In a given partial wave the series diverges if and only if the kernel has an eigenvector with an eigenvalue of norm greater than unity. This divergence may be cured by subtracting out a part of the full interaction. One technique to do this is the Schmidt method[13] which Weinberg[5] calls the quasiparticle method. This involves solving the problem with a multiterm separable potential—one term for each eigenvector with large eigenvalue. Corrections for the potential minus the separable terms are calculated perturbatively.

Consider a scattering system which is described by a Hamiltonian H which can be split up into an unperturbed part H_0 and an interaction V. We assume that H_0 has only continuum eigenstates with energy greater than zero. It will usually be the kinetic energy operator, $-\hbar^2\nabla^2/2m$. The resolvent for the system is $G(z)=(z-H)^{-1}$, while the free resolvent is $G_0(z)=(z-H_0)^{-1}$. The scattering operator can be obtained from $G(z)$ by the operator relation

$$T(z)=V+VG(z)V\ , \tag{4}$$

and satisfies the Lippmann-Schwinger (LS) equation

$$T(z)=V+VG_0(z)T(z)\ . \tag{5}$$

When matrix elements are taken, this becomes an integral equation of the second kind for the function $\langle \mathbf{q}\,|\,T(z)\,|\,\mathbf{k}\rangle$. (The vector $\mathbf{k}$ and the complex number z are fixed parameters in this equation.)

If the LS equation is iterated, the Born series is obtained:

$$T(z)=V+VG_0(z)V+VG_0(z)VG_0(z)V+\cdots\ . \tag{6}$$

This series will converge at an energy E if the eigenvalues $\lambda_i(E)$ of the kernel operator $G_0(E+i\epsilon)V$ all lie within the unit circle. We have added a small positive complex part to the energy E to guarantee outgoing wave boundary conditions. The limit $\epsilon\to 0$ is to be taken at the end of the calculation. We abbreviate this prescription by writing E^+ for $E+i\epsilon$. (For a more detailed discussion see Ref. 14.)

The physics controlling possible divergences is easily established by considering the eigenvalue equations. If $G_0(E+i\epsilon)V$ has an eigenvalue λ, then there exists an eigenvector $|f\rangle$ satisfying

$$G_0(E^+)V\,|f\rangle=\lambda\,|f\rangle\ . \tag{7}$$

Multiplying this by $(E-H_0)/\lambda$ yields the differential form of the eigenvalue equation

$$(E^+-H_0-V/\lambda)\,|f\rangle=0\ . \tag{8}$$

Since E is fixed and λ to be found, the problem is the following: given a fixed energy E, adjust the strength of the potential to put a "bound state" at E. This is referred to as the *Sturmian problem.* The inverse potential strength, $\lambda(E)$, is a function of the energy, E, and is referred to as the *Sturmian eigenvalue.* The meaning of "bound state" is determined by the boundary conditions in Eq. (7). If E is negative, then $|f\rangle$ corresponds to a normalizable state and behaves like a decaying exponential at infinity. If E is positive, then $|f\rangle$ must contain only outgoing waves.

Whenever the Hamiltonian $H=H_0+V$ has a bound state at an energy $-\epsilon_i$, then the Sturmian problem has a solution at that energy with Sturmian eigenvalue unity:

$$\lambda_i(-\epsilon_i)=1\ . \tag{9}$$

As we change the energy continuously from its starting value, the coupling constant must change continuously in order to put the bound state at the new energy. The Sturmian eigenvalue will move away from one.

Let us now increase E until it becomes positive. A real potential cannot have pure outgoing wave solutions, so $\lambda_i(E)$ must move into the complex plane making $V/\lambda_i(E)$ complex. In general, each bound state for the two-body problem (and for the two-body problem in which V is replaced by $-V$) leads to a Sturmian eigenvalue $\lambda_i(E)$ which follows a trajectory through the complex plane as the energy E ranges along the real axis.

The qualitative behavior of the trajectory may be understood on the basis of simple physical arguments and is sketched in Fig. 1. Assume that V is purely attractive. What potential strength $g=1/\lambda(E)$ is required for gV to have a bound state at energy E? If we begin with $E=-\infty$, it is clear that g must be infinite. As the energy becomes less negative, a weaker potential is required, and g moves in along the positive real axis. If there is a physical bound state of V at an energy $-\epsilon$, g becomes 1 at that energy. For negative energies greater than this, g decreases further. When the energy becomes positive, the outgoing wave condition requires that the potential have a source term, so g acquires a negative imaginary part. As the energy goes to infinity, the wave function oscillates more rapidly. Since the functional dependence of the potential stays fixed, the wave function averages the potential to zero, effectively seeing less of it. As a result, to maintain a bound state, the potential strength must again

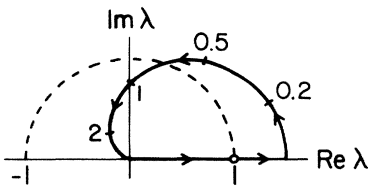

FIG. 1. Trajectory of the Sturmian eigenvalue $\lambda(E)$ in (16) for the attractive separable potential $V=|h\rangle\Lambda_0\langle h|$. The arrows show the direction of the trajectory as E increases and the value of the energy parameter α appears at several points along the trajectory. The Born series begins to diverge at the bound state (small circle) and still diverges until it returns to the unit circle. [Here, the dimensionless strength is taken to be $C_0=-1.5$, but $\lambda(E)$ scales with C_0.]

increase. We therefore expect the trajectory to head back to infinity as the energy goes to $+\infty$. The trajectory of the Sturmian eigenvalue, $\lambda(\epsilon)$, is obtained by taking the reciprocal of g. As E runs from negative infinity to positive infinity, $\lambda(E)$ starts from 0, runs along the real axis out past 1 (if the original potential has a bound state), and then loops into the upper half plane and finally back towards zero.

Different kinds of potentials have different sets of trajectories. For a local potential which is purely attractive and which supports N bound states, there will be an infinite number of trajectories. This is because such a potential can be made to support an arbitrarily large number of bound states if its coupling constant is increased sufficiently. Of these, the N corresponding to the bound states of the original potential will leave the real axis outside the unit circle. The others will leave the real axis before λ_i reaches 1. In general, these trajectories will occur entirely within the unit circle and will therefore not cause the Born series to diverge at any energy. Each bound state will cause the Born series to diverge for a range of energies ranging from 0 to some positive energy E^i_{crit}, when its trajectory reenters the unit circle.

If the potential is local and contains repulsive as well as attractive parts, there will be negative eigenvalues, corresponding to the bound states of the potential with the sign reversed. These eigenvalues are extremely important in cases where the potential has a strong repulsive core, such as in nuclear or molecular physics.[15]

The simplest potential we can study is the single term separable potential, $V=|h\rangle\Lambda_0\langle h|$, where the form factor $\langle k|h\rangle$ does not change sign. In this case, there can only be a single bound state and therefore only a single Sturmian trajectory. We consider this example explicitly to demonstrate contour plots of the convergence parameter in the space of the dimensionless control parameters.

We choose to study the specific case of the S-wave Yamaguchi potential.[16] In momentum space this potential has the form

$$\langle \mathbf{k}'|V|\mathbf{k}\rangle = h(k')\Lambda_0 h(k)\ , \tag{10}$$

where $h(k)$ is the function

$$h(k)=(k^2+\beta^2)^{-1}\ . \tag{11}$$

For this potential, the unique Sturmian eigenvalue can be calculated by a quadrature. Normalizing the Sturmian eigenvector by

$$\langle h|f\rangle = 1\ , \tag{12}$$

Eq. (7) easily yields

$$\lambda=\Lambda_0\langle h|G_0(E+i\epsilon)|h\rangle\ . \tag{13}$$

The matrix element may be evaluated by contour integration in momentum space to yield the explicit form

$$\lambda=\frac{2m\pi^2}{\hbar^2}\frac{\Lambda_0}{\beta(k+i\beta)^2}\ , \tag{14}$$

where we have written the energy as $E=(\hbar k)^2/2m$.

Since the convergence parameter of the Born series given by Eq. (3) is itself dimensionless, we must be able to express it in terms of dimensionless combinations of the input parameters that control the problem: k, m, Λ_0, and the range $R=\beta^{-1}$. The relevant dimensionless combinations from expression (14) are

$$\begin{aligned}\alpha&=kR\ ,\\ C_0&=(2m\pi^2R^2/\hbar^2)(\Lambda_0 R)\ .\end{aligned} \tag{15}$$

The parameter α is proportional to the ratio of the radius of the potential to the wavelength of the incident particle. For a fixed potential its square is proportional to the energy. From Eq. (14), we can see that the units of Λ_0 are energy/length. The quantity $\Lambda_0 R=\Lambda_0/\beta$ corresponds to the potential strength for a local potential. The parameter C_0 is proportional to the potential strength divided by the kinetic energy given by the uncertainty principle when a particle is confined to a sphere of volume R. For brevity, we refer to α as the *energy parameter* and to C_0 as the *strength parameter.*

In terms of these dimensionless variables, the eigenvalue and the convergence parameter take the simple forms

$$\lambda=C_0/(\alpha+i)^2\ , \tag{16}$$

$$\eta=|\lambda|=C_0/(1+\alpha^2)\ . \tag{17}$$

The convergence parameter is contour plotted as a function of the dimensionless model parameters in Fig. 2. The divergent region ($\eta>1$) is shaded. We refer to the line between the convergent and divergent regions as the *catastrophe boundary.* It is determined by the equation $\eta=1$.

Starting with any divergent point, the series can be made to converge by going to a high enough energy (larger values of α). In the large α limit, the condition for convergence becomes

$$|C_0|/\alpha^2<1 \tag{18}$$

or

$$\pi^2|\Lambda_0|R/E<1\ . \tag{19}$$

These conditions are analogous to those obtained by other authors[17,18] for the validity of the Born approximation at high energy. For local potentials they found the conditions $kR\gg 1$ and $k\gg V/\hbar v$, where V is the potential

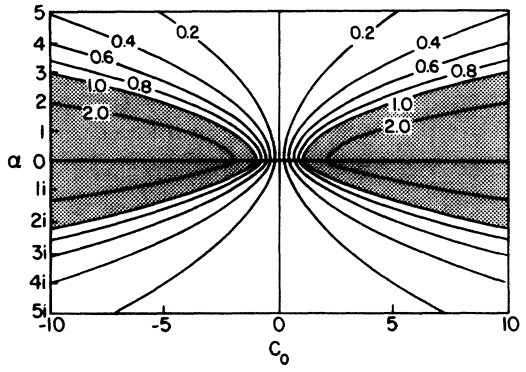

FIG. 2. Contour map of the convergence parameter, η, for the Born series with the one-channel separable Yamaguchi potential plotted as a function of the control parameters $C_0=2m\pi^2R^3\Lambda_0/\hbar^2$ and $\alpha=kR$. The divergent region is shaded.

strength and v is the particle's velocity. Together these yield $V/E \ll 1$. Note that our condition $\eta<1$ with η given by Eq. (17) is potential dependent and sharper than these high energy conditions. It yields the precise catastrophe boundary between the divergent and the convergent regions.

From Fig. 2 we observe that if C_0 is small enough, the Born series will converge for all energies (values of α). This is a consequence of the relation between the bound states of a potential and the convergence of the Born series. The boundary occurs when C_0 is sufficiently small that the potential will fail to support a bound state.

To see this in more detail, let us consider the condition for the existence of a bound state and the condition for the convergence of the Born series in the k plane ($k=\alpha/R$). The boundary between convergence and divergence of the Born series is determined by the equation $\eta=1$. Setting $k=k_0+ik_1$, we obtain the equation for the circle in the k plane shown in Fig. 3. Inside the shaded region the Born series diverges.

A bound state occurs at an energy $-\epsilon$ when $\lambda(-\epsilon)=1$. Taking $\epsilon=\hbar^2k^2/2m$ and setting $k=ik_B$ in (22), we find

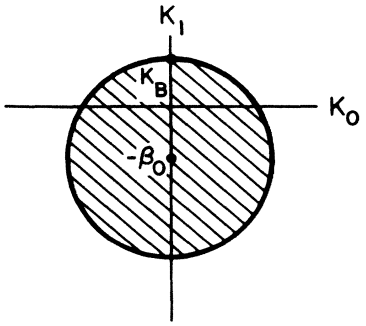

FIG. 3. The values of complex momentum k_0+ik_1, for which the Born series diverges. The potential range is $R=1/\beta_0$ and the strength is fixed to produce a bound state at the energy $\epsilon=-\hbar^2k_B^2/2m$. The Born series diverges inside the shaded circle.

$$k_BR=(-C_0)^{1/2}-1\ . \tag{20}$$

The bound state lies in the k plane at the intersection of the catastrophe boundary and the positive imaginary k axis. We see from Eq. (20) that if $|C_0|<1$, the circle will not cross the real axis and the potential will fail to support a bound state. Both the entire real and the positive imaginary k axes will lie in the convergent region. The Born series will therefore converge for all energies.

B. The distorted-wave Born series

One way to cure the divergence of the Born series is the distorted wave method. This approach divides the potential into a piece V_1 that is easily solvable and a residual potential $V_2=V-V_1$ that is treated perturbatively.

Using the two-potential formula,[19,9] the full T operator may be expressed in the form

$$T=T_1+\Omega_1^{(-)\dagger}T_2\Omega_1^{(+)}\ , \tag{21}$$

where

$$\begin{aligned} T_1&=V_1+V_1G_0T_1\ ,\\ T_2&=V_2+V_2G_1T_2\ . \end{aligned} \tag{22}$$

The wave operators are given by

$$\Omega_1^{(\pm)}=1+G_0(E\pm i\epsilon)T_1(E\pm i\epsilon) \tag{23}$$

and the resolvent operators are $G_i=(E-H_i)^{-1}$, $i=0,1,2$.

We solve for T_1 (and Ω_1) exactly and solve for T_2 perturbatively. Treating T_2 perturbatively in (22) gives the DWBA

$$T\cong T_1+\Omega_1^{(-)\dagger}V_2\Omega_1^{(+)}\ . \tag{24}$$

Iterating (22) yields the multistep or distorted-wave Born series (DWBS)

$$T=T_1+\Omega_1^{(-)\dagger}(V_2+V_2G_1V_2+\cdots)\Omega_1^{(+)}\ . \tag{25}$$

The convergence of this series is governed by the eigenvalues of the operator G_1V_2:

$$G_1(E+i\epsilon)V_2\,|f_n\rangle=\lambda_n(E)\,|f_n\rangle\ . \tag{26}$$

The relation of these eigenvalues to those of the undistorted problem is more complex than in the quasiparticle case because of the presence of G_1. Suppose that $\lambda_n^B(E)$ are the Sturmian eigenvalues of the Born series kernel G_0V. Then the eigenvectors and eigenvalues satisfy

$$(E-H_0-V/\lambda_n^B)\,|f_n\rangle=0\ . \tag{27}$$

The Sturmian eigenvectors, $|g_n\rangle$, and eigenvalues, λ_n, of the distorted wave kernel satisfy

$$(E-H_0-V_1-V_2/\lambda_n)\,|g_n\rangle=0\ . \tag{28}$$

The distorted wave case is more complex since the unperturbed Hamiltonian $h=H_0+V_1$ may have bound states. If it has a bound state at energy $E=-\epsilon$, then the Sturmian eigenvalues of the distorted wave kernel will be singular there, i.e.,

$$\lambda(-\epsilon)=\infty\ . \tag{29}$$

These singular points will turn out to be of considerable importance in controlling the accuracy of the DWBA.

Let us consider a specific example in order to illustrate the relation between the Born series trajectories and those for the DWBS. Suppose we simply split our potential V into two parts of the same form. (This example is instructive, if not particularly useful.) We take

$$V=V_1+V_2=wV+(1-w)V\ , \tag{30}$$

where w is a constant between 0 and 1. A comparison of (27) and (28) shows that the DWB and Born eigenvalues are related by

$$\lambda(w)=[(1-w)\lambda^B]/(1-w\lambda^B)\ . \tag{31}$$

Note that we have suppressed the energy dependence and that $\lambda(w=0)=\lambda^B$. We observe that even if λ^B is very large, the choice $w=\frac{1}{2}$ will reduce $\lambda(w)$ to the neighborhood of unity. Note, however, the possibility that if λ^B is on the order of unity, some choice of w may result in λ being very large. This is not a physical phenomena associated with the full problem, but is rather a reflection of our choice of the unperturbed Hamiltonian.

In order to be specific, we consider V to be the separable Yamaguchi potential considered in Sec. II A above. Combining (16) and (31) we find that the eigenvalue of G_1V_2 has the form

$$\lambda(w)=(1-w)C_0/[(\alpha+i)^2-wC_0]\ . \tag{32}$$

Note that this eigenvalue diverges at an energy $-\epsilon$ when the distortion potential supports a bound state at that energy. From Eq. (16), we see that this occurs for a potential of strength parameter wC_0 when $wC_0=(\alpha+i)^2$.

Here it is easier to analyze the trajectory of the coupling constant, $g=1/\lambda(w)$, which is

$$\begin{aligned} &g(w)=-w/(1-w)+(\alpha+i)^2/[(1-w)C_0]\ , \\ &g(0)=(\alpha+i)^2/C_0\ . \end{aligned} \tag{33}$$

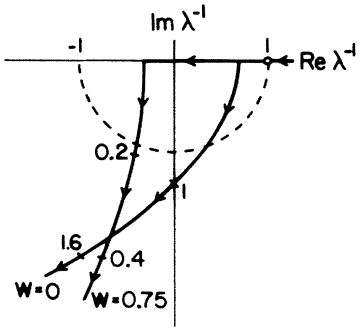

FIG. 4. Trajectories of the coupling constant $g(w)=1/\lambda(w)$ in (33), where $\lambda(w)$ is the distorted-wave eigenvalue for the one-channel separable potential with $C_0=-1.5$. Arrows indicate the direction of the trajectories as energy increases from $-\infty$ to $+\infty$ and the value of the energy parameter α is given at several points on the trajectories. The distorted wave series begins to diverge at the bound state energy (small circle) of the full potential and continues to diverge until it leaves the shaded unit circle. (The coupling constant with $w=0$ is just the Born coupling constant.)

We see that the effect of introducing a distortion (a nonzero value of w) is to slide the g trajectory to the left by $-w/(1-w)$ and to increase the rate at which the trajectory is traversed. This means that the trajectory will in general leave the divergent circle at a lower energy when w is between zero and one. This is illustrated in Fig. 4.

III. SEPARABLE MULTICHANNEL MODEL

In this section we discuss a separable multichannel model for the inelastic scattering of a projectile by a complex target without rearrangement. Our canonical example is the scattering of a nucleon by a nucleus when the nucleus can occupy a (finite) number of possible excited states. We begin with an N-channel model, but later restrict our considerations to two.

When our two-channel model is made specific by the choice of S-wave Yamaguchi form factors we have a model which is analytically solvable for both the elastic and inelastic amplitudes. This has a considerable advantage as it allows us to investigate in detail the important question: What is the appropriate distortion potential to use in the calculation of a DWBA? We investigate three possible choices of distorting potential. The first is the folding potential. This is simply the diagonal part of the Hamiltonian coupling matrix and is an obvious first choice in this model. The second is the optical potential. This is the diagonal potential which gives the exact elastic scattering. Since the model is solvable the optical potential can be found analytically. This is the potential which is most frequently assumed to be the correct potential to use in the DWBA.

Finally, we ask the question: What distortion potential would yield the most rapid convergence of the DWBS? We calculate the convergence parameter for the DWBS with a general distorting potential and then choose the distortion to minimize η.

A. N-channel model

Consider the scattering of an elementary projectile (labeled 0) from an A-particle bound state (with the particles labeled 1 through A). The asymptotic Hamiltonian has the form

$$H_0=p^2/2M+H_A\ , \tag{34}$$

where p and M are, respectively, the relative projectile-target momentum and the system's reduced mass. H_A is the internal Hamiltonian of the target. Its eigenfunctions are $\phi_\beta(\xi_A)$ with energies ϵ_β. The argument ξ_A represents $A-1$ internal relative coordinates of the particles 1 to A.

Let the exact scattering wave function of the $A+1$ particle system be $|\mathbf{p}^{(+)}\phi_0\rangle$ where the labeling indicates that the boundary conditions have the projectile incident in a plane wave with momentum $\mathbf{p}$ on the target in its ground state ($\beta=0$). This satisfies the Schrödinger equation

$$(H_0+V)\,|\mathbf{p}^{(+)}\phi_0\rangle=E\,|\mathbf{p}^{(+)}\phi_0\rangle\ , \tag{35}$$

where V is the interaction between the projectile and the target,

$$V=\sum_{j=1}^{A} v_{0j} , \tag{36}$$

and E is the energy

$$E=p^2/2M+\epsilon_0 . \tag{37}$$

We now expand the exact wave function in terms of the target eigenstates

$$\langle \mathbf{r},\xi_A \mid \mathbf{p}^{(+)}\phi_0\rangle=\sum_{\beta} u_\beta(\mathbf{r})\phi_\beta(\xi_A) . \tag{38}$$

The coordinate $\mathbf{r}$ is the distance between the projectile and the c.m. of the target. It is the coordinate conjugate to the relative momentum, $\mathbf{p}$. Inserting the expression (38) into the Schrödinger equation, multiplying by $\phi_\alpha(\xi_A)^*$, and integrating over the coordinates of the target yields the coupled channel equations:

$$(E-\epsilon_\alpha+\hbar^2\nabla^2/2M)u_\alpha(\mathbf{r})=\sum_{\beta}\int d\mathbf{r}' V_{\alpha\beta}(\mathbf{r},\mathbf{r}')u_\beta(\mathbf{r}') , \tag{39}$$

where the coupling potentials are given by the folding matrix elements:

$$V_{\beta\delta}(\mathbf{r},\mathbf{r}')=\langle \mathbf{r},\phi_\beta \mid V \mid \mathbf{r}',\phi_\delta\rangle . \tag{40}$$

We now choose a specific model. For simplicity of calculation we restrict our potential matrix to be an S-wave multichannel separable interaction of the form

$$(\mathbf{V})_{\beta\delta}=\mid h\rangle(\mathbf{\Lambda})_{\beta\delta}\langle h \mid , \tag{41}$$

where $\mathbf{\Lambda}$ is a constant matrix. We choose $\mid h\rangle$ to be the Yamaguchi form (11). The restriction to the S wave should not be considered to be too unrealistic, as the S wave usually provides the severest test of the DWBA.

The basic assumption that is supposed to make the DWBA work is that there are many small inelastic channels. They contribute coherently to yield a strong effect on the elastic channel. We therefore consider a model in which we neglect the couplings between the inelastic channels and only permit elastic-inelastic couplings.

This leaves us with a bordered coupling matrix:

$$\mathbf{\Lambda}=\begin{bmatrix} \Lambda_0 & \Lambda_1 & \Lambda_2 & \cdots & & \Lambda_{N-1} \\ \Lambda_1 & \Lambda_0 & 0 & 0 & \cdots & 0 \\ \Lambda_2 & 0 & \Lambda_0 & 0 & \cdots & 0 \\ & 0 & 0 & \cdots & & \\ \vdots & \vdots & \vdots & & & \vdots \\ \Lambda_{N-1} & 0 & 0 & \cdots & & \Lambda_0 \end{bmatrix} . \tag{42}$$

The eigenvalue equation for the Born kernel G_0V is

$$G_0(E^+)\mid h\rangle\mathbf{\Lambda}\langle h \mid \mathbf{u}\rangle=\lambda \mid \mathbf{u}\rangle , \tag{43}$$

where $\mid \mathbf{u}\rangle$ is the vector with components $\langle \mathbf{r} \mid u_\beta\rangle = u_\beta(\mathbf{r})$. Upon left multiplication by $\langle h \mid$ this becomes

$$\sum_{\gamma}\langle h \mid G_0(E^+-\epsilon_\beta)\mid h\rangle\Lambda_{\beta\gamma}c_\gamma=\lambda c_\beta , \tag{44}$$

where we have written $c_\beta=\langle h \mid u_\beta\rangle$.

To simplify matters further, we make the following assumption. Since we are primarily interested in open-channel phenomena, we consider the excitation energies to be much less than the kinetic energy of the projectile. We therefore set all the excitation energies to be zero. Another reason for this simplification is to permit us to investigate the physics of a small number of control parameters at a time. Even in our two-channel simplified model we will have three dimensionless control parameters corresponding to the energy, diagonal coupling strength, and off-diagonal coupling strength. We will briefly consider the dependence of the convergence parameter on the excitation energy in Sec. IV.

For this simplified problem the eigenvalues are determined by the secular equation

$$\det(\lambda I-G_0V)=0 . \tag{45}$$

The N eigenvalues are easily determined to be

$$\begin{aligned}\lambda&=\Lambda_0\langle h \mid G_0(E)\mid h\rangle \quad N-2 \text{ times} \\ &=\left[\Lambda_0\pm\left[\sum_{\gamma}\Lambda_\gamma^2\right]^{1/2}\right]\langle h \mid G_0(E)\mid h\rangle .\end{aligned} \tag{46}$$

The convergence of the Born series is determined by the eigenvalue of maximum norm. For the case of the bordered matrix, this is necessarily one of the last two eigenvalues. We therefore obtain the interesting result that for this example, the convergence of the Born series is controlled in a manner which is identical to that of a two-channel model in which the effective coupling between the elastic and the single inelastic channel is given by the rms coupling strength

$$\Lambda=\left[\sum_{\gamma}\Lambda_\gamma^2\right]^{1/2} . \tag{47}$$

For the DWBS we will only consider distortions which are diagonal and have the same functional form as V. The conclusions for the DW case will therefore be analogous. As a result, we restrict our consideration here to the two-channel problem with coupling matrix

$$\mathbf{\Lambda}=\begin{bmatrix}\Lambda_0 & \Lambda \\ \Lambda & \Lambda_0\end{bmatrix} . \tag{48}$$

The distortion potentials we study have the form

$$\mathbf{V}_1=\mid h\rangle\begin{bmatrix}L & 0 \\ 0 & L\end{bmatrix}\langle h \mid . \tag{49}$$

We could allow a more general distortion potential by permitting the strengths to differ in the two channels, but we choose this form in keeping with our approximation that the excitation energy is zero and with our desire to handle only a small number of parameters at a time. We note that this two-channel problem is not equivalent to the many-channel problem we began with, although many of the features are similar. If we start with many channels (say $\beta=0,\ldots,N-1$) and are interested in the transition to one particular channel (say $\beta=1$), then eliminating channels $2,\ldots,N-1$ formally will result in complex potentials in the matrix (48). This complicates matters fur-

ther by introducing additional control parameters. We ignore them for now, but note that absorption to the unobserved channels is an important part of the physics that makes the DWBA work in nuclear physics. This is discussed in Sec. IV.

The choice of a distortion potential $\mathbf{V}_1$ leads to the separation of the full potential strength matrix into two parts:

$$\mathbf{\Lambda}=\mathbf{\Lambda}_1+\mathbf{\Lambda}_2=\begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix}+\begin{bmatrix} \Lambda_0-L & \Lambda \\ \Lambda & \Lambda_0-L \end{bmatrix} . \tag{50}$$

We treat the potential $V_1=|h\rangle\mathbf{\Lambda}_1\langle h|$ exactly and calculate the effects of $V_2=|h\rangle\mathbf{\Lambda}_2\langle h|$ perturbatively. To determine the convergence rate of this series we have to solve the eigenvalue problem:

$$\mathbf{G}_1(E^+)\mathbf{V}_2|\mathbf{f}\rangle=\lambda|\mathbf{f}\rangle . \tag{51}$$

If we multiply this equation by $\langle h|$ and solve for the matrix elements of G_1 via the resolvent equation, we can easily obtain the eigenvalues

$$\lambda_\pm=[(\Lambda_0-L)\pm\Lambda]\langle h|G_1(E^+)|h\rangle , \tag{52}$$

$$\langle h|G_1(E^+)|h\rangle = \langle h|G_0(E^+)|h\rangle/[1-L\langle h|G_0(E^+)|h\rangle] . \tag{53}$$

We again introduce a convenient set of dimensionless parameters:

$$\begin{aligned} &\alpha=kR , \\ &C_0=2M\pi^2\Lambda_0R^3/\hbar^2 , \\ &C=2M\pi^2\Lambda R^3/\hbar^2 , \\ &w=L/\Lambda_0 . \end{aligned} \tag{54}$$

We refer to α as the *energy parameter,* C_0 as the *diagonal potential parameter,* C as the *coupling parameter,* and w as the *distortion parameter.* In terms of these dimensionless parameters the eigenvalues become

$$\lambda_\pm=[(1-w)C_0\pm C]/[(\alpha+i)^2-wC_0] . \tag{55}$$

The convergence parameter is then

$$\eta=\max|\lambda_\pm| . \tag{56}$$

In the next three subsections we consider the implications of this formula for three different choices of distortion potential.

B. Folding potential distortion

In choosing a distortion potential, we want to put as much of the dynamics as possible into the diagonal operator $\mathbf{V}_1$, since we solve for its effects to all orders. In the simple two-channel model described above, an obvious first choice is to take the distortion potential equal to the diagonal part of the potential, i.e., $L=\Lambda_0$. This gives

$$\mathbf{\Lambda}_1=\begin{bmatrix} \Lambda_0 & 0 \\ 0 & \Lambda_0 \end{bmatrix} . \tag{57}$$

This leaves the potential to be treated perturbatively as purely off-diagonal:

$$\mathbf{\Lambda}_2=\begin{bmatrix} 0 & \Lambda \\ \Lambda & 0 \end{bmatrix} . \tag{58}$$

We refer to this choice of $\mathbf{V}_1$ as the folding potential, since the diagonal matrix elements of $\mathbf{V}$ as obtained by "folding" the two-body potential over the distribution of nucleons in the target. Explicitly, if v_{01} is a local potential, then the diagonal matrix elements have the local form:

$$\begin{aligned} &v_{\beta\beta}(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}')v_\beta(\mathbf{r}) , \\ &v_\beta(\mathbf{r})=\int d\mathbf{r}''v_{01}(\mathbf{r}-\mathbf{r}'')\rho_\beta(\mathbf{r}'') , \end{aligned} \tag{59}$$

where $\rho_\beta(\mathbf{r}'')$ is the density of target nucleons at the point $\mathbf{r}''$ when the nucleus is in the state ϕ_β.

This choice corresponds to taking $w=1$. The resulting eigenvalues are

$$\lambda_\pm^f=\frac{\pm C}{(\alpha+i)^2-C_0} \tag{60}$$

which yields the convergence parameter

$$\eta^f=|C|/[(\alpha^2-C_0-1)^2+4\alpha^2]^{1/2} . \tag{61}$$

The convergence parameter is plotted as a function of the control variables α and C_0 in Fig. 5. Since η only depends on the coupling parameter linearly, we have chosen to do the plot for the value $C=-5$. Other values can easily be inferred by scaling. For purposes of comparison, we show the corresponding plot for the Born series ($w=0$) in Fig. 6.

The Born result is qualitatively similar to the one-channel model (Fig. 2), but the presence of the second channel results in the occurrence of a bound state even when the diagonal coupling strength, C_0, vanishes. This causes the two divergent regions to overlap. It still has the characteristic that as the diagonal coupling constant is increased, the range of divergent energies grows. The

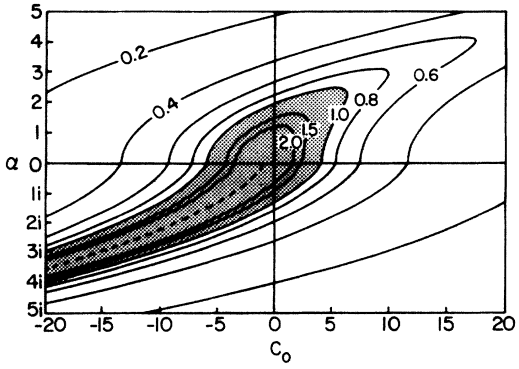

FIG. 5. Contour map of the convergence parameter, η^f [Eq. (61)], for the DWBS in the two-channel model using the folding distortion. The divergent region is shaded and $C=-5$.

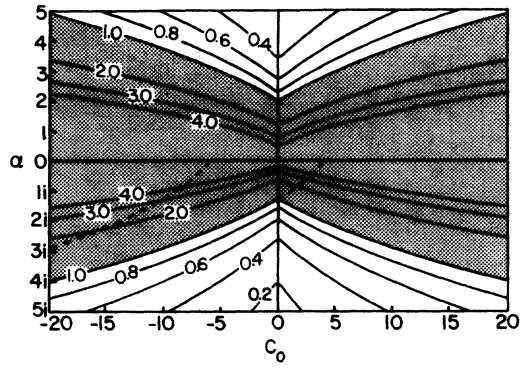

FIG. 6. Contour map of the convergence parameter, η^B [Eq. (62)], for the Born series in the two-channel model. The divergent region is shaded and $C=-5$.

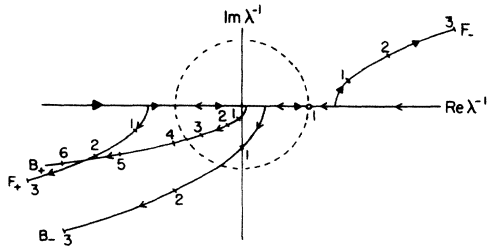

FIG. 7. Trajectories of the inverse eigenvalues for the Born case ($B_\pm$) and the folding distortion case ($F_\pm$). Arrows show the direction of the trajectories as energy increases from $-\infty$ to $+\infty$ and several values of the energy parameter α appear along trajectories. The Born series and the distorted wave series diverge when the corresponding trajectories lie inside the unit circle. (For the cases shown, the dimensionless strengths are $C_0=-8$ and $C=-5$.)

Born eigenvalues and the convergence parameter are given by

$$\lambda_\pm^B=\frac{C_0\pm C}{(\alpha+i)^2}\ ,$$
$$\eta^B=\frac{|C_0|+|C|}{\alpha^2+1}\ . \tag{62}$$

The distorted wave result differs qualitatively from this. Since the diagonal potential appears to all orders, increasing its value improves the convergence of the series.

We can understand what governs the character of the divergent region by considering the analytic structure of the convergence parameter, η. As in the one-channel case, the Sturmian eigenvalues have poles when the distorting potential has a bound state. Here, the distorting potential is identical to our one-channel separable potential, so it has a bound state when the λ of Eq. (16) is equal to 1, i.e., when

$$\alpha=i[(-C_0)^{1/2}-1]\ . \tag{63}$$

This trajectory is displayed as a dotted line in Fig. 5. Note that the upper half plane is labeled "α" and the lower half as "$i\alpha$." We are displaying the half-planes corresponding to α^2 positive and negative (positive and negative energies) but have chosen to use the "kR" scale rather than energy scale. Along the dotted line η is infinite. The shape and structure of the contours in this figure are controlled by the presence of this singularity. Along the catastrophe boundary at negative energies one of the eigenvalues equals 1 (the other equal -1) so these are the trajectories of the bound states of the full potential.

At large values of the energy parameter, the convergence condition for the DWBS simplifies to

$$|C|/\alpha^2<1 \text{ or } |\Lambda|R/E\ll 1\ . \tag{64}$$

The off-diagonal coupling strength controls the high energy behavior of the DWBS. Even at low energies, however, the series may also converge if the diagonal strength is large enough. For large C_0 the convergence condition becomes

$$|C|<|C_0+1|\ . \tag{65}$$

This condition has the expected behavior at low energy since the zero energy bound state of the distortion potential occurs at $C_0=-1$. If one is sufficiently far from this point, $|C/C_0|=|\Lambda/\Lambda_0|$ becomes the expansion parameter for the series. This behavior is unusual[12] and depends upon the fact that a separable potential may only have a single bound state. Although Λ/Λ_0 is often considered to be the small parameter that makes the DWBA work, in more realistic cases[12] this is not the relevant parameter: η always is.

In Fig. 7 we have plotted the trajectories of the inverse eigenvalues for the Born and folding distortion cases in the complex plane. For the Born series, the values at $E=0$ are $-1/(C_0+C)$. For the folding DWBS, they are $\pm(1+C_0)/C$. We see that if $|1+C_0|>|C|$, the trajectories in the inverse eigenvalue plane leave the divergent circle before the energy reaches zero. As a result the series will converge for all positive values of E.

C. Optical potential distortion

Although the choice of the folding potential makes the residual potential $\mathbf{V}_2$ purely off-diagonal, it is clear that it does not include all of the effects that could be included by a diagonal potential. The 2×2 scattering matrix, $\mathbf{T}_1$, which is produced by scattering from the folding potential, is diagonal, but its matrix elements do not agree with those of the exact scattering matrix, $\mathbf{T}$. So although $\mathbf{V}_2$ is purely off-diagonal, $\mathbf{T}_2$ is not.

We can choose our distortion potential to make $\mathbf{T}_2$ purely off-diagonal. To do this, we must find a diagonal potential whose scattering matrix is equal to the diagonal part of the exact multichannel scattering matrix. This potential is called the (generalized) optical potential.[20] This is in fact the distortion potential that is most commonly used in practice.[2]

The formal expression for the optical potential in terms of the many-particle operators is given by Feshbach[20] as

$$U=\langle\phi_0|V|\phi_0\rangle+\langle\phi_0|VQ(E^+-QHQ)^{-1}QV|\phi_0\rangle\ , \tag{66}$$

where Q is the projector on the inelastic channels,

$$Q=1-P=1-|\phi_0\rangle\langle\phi_0| \ . \tag{67}$$

The expression (66) may be evaluated explicitly[12] to give the strength of the optical potential

$$L_{\rm opt}=\Lambda_0+\Lambda\frac{\langle h\,|\,G_0(E-\epsilon_1)\,|\,h\rangle}{1-\Lambda_0\langle h\,|\,G_0(E-\epsilon_1)\,|\,h\rangle}\Lambda \ . \tag{68}$$

Neglecting the excitation energy of the excited state and expressing our result in terms of our dimensionless parameters, we get

$$w^{\rm opt}=L_{\rm opt}/\Lambda_0=1+(C^2/C_0)/[(\alpha+i)^2-C_0] \ . \tag{69}$$

It is convenient to separate this into real and imaginary parts:

$$\begin{aligned}&{\rm Re}(w^{\rm opt}C_0)=C_0+C^2(\alpha^2-1-C_0)/[(\alpha^2-1-C_0)^2+4\alpha^2] \ , \\ &{\rm Im}(w^{\rm opt}C_0)=-2\alpha C^2/[(\alpha^2-1-C_0)+4\alpha^2] \ .\end{aligned} \tag{70}$$

The reader may show that the scattering amplitude produced by this potential yields the exact elastic scattering amplitude for the two-channel model.

When the distorting potential is taken to be the optical potential, the distorted wave eigenvalues become

$$\lambda_{\pm}^{\rm opt}=\frac{-C^2\pm C[(\alpha+i)^2-C_0]}{[(\alpha+i)^2-C_0]^2-C^2} \ . \tag{71}$$

The contour plots of the convergence parameter for this case are shown in Fig. 8. As before, the structure of the contour plot is controlled by the presence of bound states of the distortion potential. For the case of the optical potential, these bound states are exactly the bound states of the full problem. We might have guessed that doing this would suffice to remove the divergences arising from the distortion potentials "wrong" bound states. Unfortunately, a more careful analysis shows that this is not the case. The bound states occur when the Born eigenvalues (62) equal 1, i.e., when

$$(\alpha+i)^2=C_0\pm C \ . \tag{72}$$

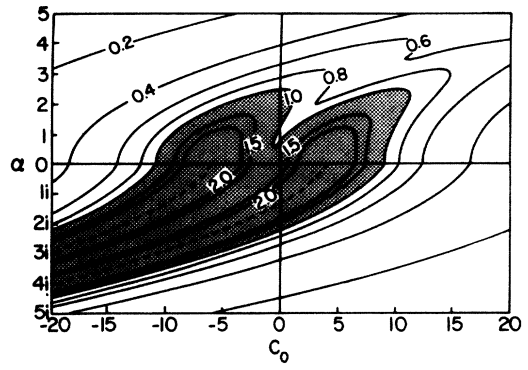

FIG. 8. Contour map of the convergence parameter, $\eta^{\rm opt}$ [see Eq. (71)], for the DWBS in the two-channel model using the generalized optical potential as the distortion. The divergent region is shaded and $C=-5$.

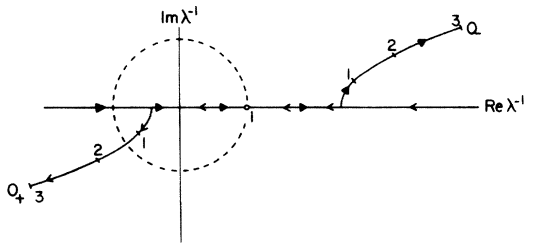

FIG. 9. Trajectories of the inverse eigenvalues of the DWB kernel in the two-channel model using optical potential distortion. Potential parameters and marked energy points are the same as in Fig. 7.

Substituting this into (71) we see that if the minus sign is chosen, $\lambda_+^{\rm opt}$ is finite, but $\lambda_-^{\rm opt}$ becomes infinite. The reverse occurs if (72) is satisfied with the plus sign. We therefore get two singular trajectories. These are shown as dotted lines in Fig. 8.

The result is that the convergence of the distorted wave series is worse when the optical potential is chosen as the distortion than it is when the folding potential is chosen. This relation can be further understood by considering the trajectories of the Sturmian coupling constants. We observe by a comparison of (60) and (71) that

$$(\lambda_{\pm}^{\rm opt})^{-1}=(\lambda_{\pm}^{f})^{-1}+1 \ . \tag{73}$$

This is illustrated in Fig. 9. Introducing the optical distortion displaces the folding potential's trajectories to the right. Although the convergence problems caused by one of the eigenvalues is eliminated, those caused by the other is exacerbated. The result is that the width of the divergent region is nearly doubled. For large values of α or of C_0 the conditions in (64) and (65) are again obtained, but larger values of α or C_0 are needed to achieve them.

D. Minimal distortion potential

We have seen that choosing the diagonal part of $\mathbf{V}$ to be the distortion potential makes $\mathbf{V}_2$ purely off-diagonal, but not $\mathbf{T}_2$. Choosing the distortion to be the optical potential makes $\mathbf{T}_2$ purely off-diagonal but expands the divergent region. What distortion potential should we choose? The criterion that we really want to satisfy is that the distortion potential includes "as much of the dynamics as is possible with a diagonal potential." This means that we want our higher-order corrections to be as small as possible (without requiring that we calculate the full solution as input).

One possible way to determine the distortion potential is to choose it to minimize the convergence parameter. This is the choice that will make the corrections converge as rapidly as possible within the structure of the DWBS. We consider η to be a functional of the distortion potential and find that V_1 which solves the variational condition

$$\frac{\delta\eta}{\delta V_1}=0 \ . \tag{74}$$

We refer to this distortion potential as the *minimal distortion potential.* For the example being considered here, this means that we want to find the value of w that solves

$$\eta^{\min} = \min_{w} \{\max[\,|\lambda_+(w)|\,,|\lambda_-(w)|\,]\}\ . \tag{75}$$

The solution of this min-max problem for our specific example is given in the Appendix. The result is that the optimum choice for w ($E>0$) is given by $w^{\min}$ where

$$\begin{aligned} &\mathrm{Re}(w^{\min}C_0)=C_0\ , \\ &\mathrm{Im}(w^{\min}C_0)=[4\alpha]^{-1}([(\alpha^2-1-C_0)^2+4\alpha^2-C^2] \\ &\qquad -\{[(\alpha^2-1-C_0)^2+4\alpha^2-C^2]^2 \\ &\qquad +16\alpha^2C^2\}^{1/2})\ . \end{aligned} \tag{76}$$

For negative energies, $w^{\min}=1$.

We note that the real part of $V_1^{\min}$ is equal to the folding potential for all energies and that the imaginary part is negative definite.

The contour map of the convergence parameter in the minimal case is shown in Fig. 10. For negative energies the picture is the same as for the folding potential. But for positive energies we observe that the convergence is improved compared to the previously considered distortions for all values of α and C_0. Indeed, as shown in the Appendix, the DWBS now converges for all values of C and C_0, and all real positive values of α (positive energies). (Note the convergence parameter no longer depends linearly on C, so the figure must be recalculated for each value of the coupling. Other values of C produce similar structures.)

Since the DWBS with folding distortion has a divergent region at positive energy, and since the minimal distortion potential has a real part equal to the folding potential, the improvement must come from the imaginary part. Let us consider this in more detail.

Let us suppose that the coupling parameter C is small. From (76) the imaginary part of the optimum distortion strength has the form

$$\mathrm{Im}(w^{\min}C_0)=[4\alpha]^{-1}[A-(A^2+16\alpha^2C^2)^{1/2}]\ , \tag{77}$$

where we have defined

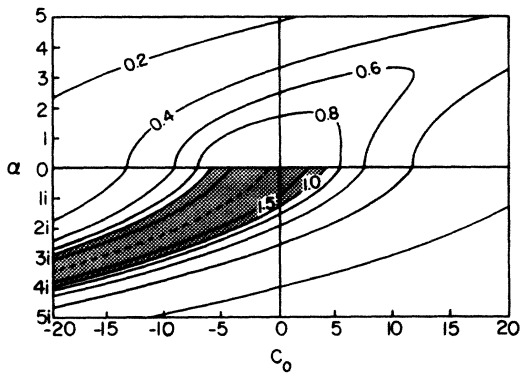

FIG. 10. Contour map of the convergence parameter $\eta^{\min}$, for the DWBS in the two-channel model using the minimal distortion potential. The divergent region is shaded and $C=-5$.

$$A=(\alpha^2-1-C_0)^2+4\alpha^2-C^2\ . \tag{78}$$

Approximating the square root in (77) by

$$(A^2+16\alpha^2C^2)^{1/2}\approx A(1+8\alpha^2C^2/A^2)\ , \tag{79}$$

we obtain

$$\begin{aligned} \mathrm{Im}(w^{\min}C_0)&=-2\alpha C^2/A \\ &=-2\alpha C^2/[(\alpha^2-1-C_0)^2+4\alpha^2-C^2]\ . \end{aligned} \tag{80}$$

To lowest order in C^2 this expression agrees with the imaginary part of the optical potential given in (70).

We have therefore shown, that to lowest order in the dimensionless coupling parameter, the minimal potential is equal to the folding potential plus the imaginary part of the optical potential. It is interesting to note that this prescription has occasionally been used in realistic calculations.[21]

IV. ANALYSIS OF RESULTS

We have considered in detail the convergence parameter governing the accuracy of the multistep distorted-wave series in a simple two-channel separable model. This analysis leads to a number of interesting, if tentative, conclusions. The primary results of this study are the following:

(1) The DWBS can be generated by the iteration of an integral equation of the Lippmann-Schwinger—type by using the operator two-potential formula.

(2) The convergence of the DWBS is governed by the spectral radius of the kernel of that integral equation. If the integral equation has been properly formulated (i.e., if the kernel is compact), this parameter can be identified with the norm of the kernel's largest eigenvalue.

(3) Dimensionless control parameters can be identified and the convergence parameter can be mapped as a function of the control parameters. The structure of this map gives guidance in the extraction of the physics that controls the accuracy of the DWBA.

(4) A study of the convergence parameter permits, in principle, a determination of the distortion potential that optimizes the convergence of the DWBS. In the model studied, it can be found in practice.

(5) The convergence of the DWBS is governed by the presence of bound states of the distortion potential.

(6) In the model studied, using the optical potential for distortions decreases the quality of the DWBA compared to using the folding potential. The best potential to use for distortions is approximately the folding potential plus the imaginary part of the optical potential.

The model used is not particularly realistic. Nonetheless, it gives some guidance for further study. In the rest of this section we consider the implications for physical problems. We begin in subsection A by considering the scales associated with nuclear systems in order to place our dimensionless parameters in perspective. In subsection B we consider some of the effects that were omitted to reduce the number of parameters treated. We analyze the implications of nonzero target excitation energies and of including absorption to other channels in the diagonal

potential. In subsection C we discuss briefly the other models that have been analyzed. These models give some insight into the limitations of the current model.

A. Scales

We are somewhat restricted in our considerations of the dimensionless parameters for physical systems by the nature of the model. Only a few coupled separable calculations exist and they are not very realistic. The local models discussed in Refs. 10 and 12 (and to be described in future papers) permit more extensive comparisons. Nonetheless, the exercise will give us some feeling for the order of magnitude of the relevant dimensionless parameters in nuclear systems.

The model we choose to analyze is from the study of King and MacKellar.[22] In this work the authors test various theories of stripping using a three-body model consisting of a neutron (n), a proton (p), and a two-state core (C). The pair interactions are separable and their parameters can be converted into dimensionless form. The n-p interaction is a one-term Yamaguchi S-wave potential (the deuteron D state is ignored) with inverse range $\beta=1.02$ fm^{-1} and strength adjusted to produce a binding energy of 2.225 MeV. This is the same form as the one-state model analyzed in Sec. II. Expressing their parameters in our normalization yields:

$$C_0(\mathrm{np})=-3.8\ . \tag{81}$$

Their nucleon (N)-core interaction has the same form as our two-state model analyzed in Sec. III. The binding energy of the nucleon-core ground state is adjusted to be 7.0 MeV and the range parameter is taken as $\beta=0.848$ fm^{-1}. Their coupling strengths yield the dimensionless parameters:

$$\begin{aligned} &C_0(\mathrm{NC})=-2.29\ , \\ &C(\mathrm{NC})=-0.72\ . \end{aligned} \tag{82}$$

This is a fairly strong coupling since the ground state spectroscopic factor with these parameters turns out to be only 65%.

Our energy parameter α becomes $(2mER^2/\hbar^2)^{1/2}$. For the case of the n-p interaction this is $0.11E^{1/2}$ where E is the energy in the laboratory frame (target particle at rest) in MeV. Thus for laboratory energies up to about 100 MeV the energy parameter α stays less than 1. At 400 MeV α is still only 2.

For the case of the nucleon-core interaction the energy parameter has the value $0.26E^{1/2}$. Thus, α becomes of the order of 1 at 15 MeV, is about 3 when E is 135 MeV, and is nearly 5 at 350 MeV.

For the local models considered in Ref. 12, the dimensionless strength parameters have the form

$$C_0=2mV_0R^2/\hbar^2\ , \tag{83}$$

where V_0 is the potential depth and R is the range. For a nuclear well of depth 40 MeV and a radius of 5 fm, this parameter has a value of nearly -50. Coupling strengths are substantially less than the full diagonal potential, the largest values being on the order of a few MeV. For nuclear physics, we therefore expect to find parameter ranges:

$$\begin{aligned} &\alpha\cong 0 \text{ to } 5\ , \\ &C_0\cong 0 \text{ to } -50\ , \\ &C\cong 0 \text{ to } -5\ . \end{aligned} \tag{84}$$

B. Extensions of the model

In our analysis of the two-channel model we ignored the excitation energy of the core's excited state and the absorption of flux into other states not observed. In this subsection we briefly discuss the extension of the model to include these factors. We restrict both discussions to the case of the folding potential distortion.

Consider the N-channel model with nonzero excitation energies ϵ_β and the bordered interaction matrix (42). The distorted-wave eigenvalue equation for the folding distortion is

$$\langle h\mid \mathbf{G}_1(E^+)\mid h\rangle\mathbf{\Lambda}_2\langle h\mid \mathbf{u}\rangle=\lambda\langle h\mid \mathbf{u}\rangle\ , \tag{85}$$

where now the Green's function becomes the diagonal matrix

$$\langle h\mid \mathbf{G}_1(E^+)\mid h\rangle_{\beta\alpha}=\langle h\mid G_1(E-\epsilon_\beta)\mid h\rangle\delta_{\beta\alpha}\ . \tag{86}$$

The secular equation

$$\det(\lambda\mathbf{I}-\langle \mathbf{G}_1\rangle\mathbf{\Lambda}_2)=0 \tag{87}$$

has the roots

$$\lambda_\pm=\pm\left[\sum_{\beta=1}^{N-1}\Lambda_{0\beta}\langle h\mid G_1(E^+)\mid h\rangle\Lambda_{\beta 0}\times\langle h\mid G_1(E^+-\epsilon_\beta)\mid h\rangle\right]^{1/2}\ . \tag{88}$$

The matrix elements of G_1 are given by (53). In addition to diverging when E equals the binding energy of the distortion potential [since $G_1(E)$ diverges there], the eigenvalues will now also diverge when $E-\epsilon_\beta$ is equal to the bound state energy of the distortion potential [since $G_1(E-\epsilon_\beta)$ diverges there].

In the case of the Born series, we get G_0 rather than G_1. Since $G_0(z)$ has no poles we will not get divergent eigenvalues, but the convergence of the series will in general be worsened because replacing E by $E-\epsilon_\beta$ leads to a lower effective energy.

For the case of our two-channel model, if the excited state has an excitation energy ϵ, Eq. (88) reduces to

$$\lambda_\pm=\pm[\Lambda^2\langle h\mid G_1(E)\mid h\rangle\langle h\mid G_1(E-\epsilon)\mid h\rangle]^{1/2}\ . \tag{89}$$

This leads to the conclusion that for nonzero excitation energies, the convergence parameter with the folding distortion becomes

$$\eta^f(\alpha,C_0,C,\epsilon)=[\eta^f(\alpha,C_0,C)\eta^f(\alpha',C_0,C)]^{1/2}\ , \tag{90}$$

where α' is the value of α appropriate to the energy $E-\epsilon$. Thus, in this model, η^f with a nonzero excitation energy

is simply the mean proportional between the η^f's in the zero-excitation-energy model at the elastic and inelastic values of the asymptotic momenta. Estimates of this effect can therefore be read directly off the graphs.

Another important factor is the presence of other channels than the one observed. If we begin with the N-channel problem and formally eliminate channels 2 to $N-1$, we will obtain a two-channel problem with effective (complex) potentials. The primary result will be to introduce an absorptive part into the diagonal potentials. We can see what effect this has on the eigenvalues by simply replacing the real diagonal coupling strength Λ_0 by a complex strength $\Lambda_0(1+i\zeta)$. The dimensionless constant C_0 becomes $C_0(1+i\zeta)$. Performing this replacement in the form (60) for the eigenvalues for the DWBS with the folding distortion, we find:

$$\eta^f(\zeta)=\{(\eta_0^f)^2/[1+s(\eta_0^f)^2]\}^{1/2}\,,$$
$$s=-\zeta C_0(4\alpha-\zeta C_0)/|C|^2\,. \tag{91}$$

We have written η_0^f for η^f ($\zeta=0$), the convergence parameter given by Eq. (61). Since in general C_0 is negative and ζ is positive (i.e., ζC_0 is negative corresponding to absorption of flux), s is positive and $\eta^f(\zeta)$ will be smaller than η_0^f, resulting in an improved convergence of the DWBS.

C. Other models

Two more realistic models have been considered in conjunction with this work and some preliminary results have been presented.[10,12] These will be discussed in detail in two papers now in preparation. The first considers a model of Chase, Wilets, and Edmonds.[23] In this model, the nucleus has a rotational band of states. The projectile interacts with the nucleus with a local square well potential and coupling is provided by surface delta interactions. In the second model, the coupling is instead provided by square wells. This is similar to the Lane model of (p,n) reactions.[24]

Many of the qualitative features of the simple two-channel separable model also hold for local models. For example, the convergence parameter is still dominated by the infinities caused by the presence of bound states of the distortion potential. The primary difference between the local and the separable models is that the local potentials may have many bound states. Instead of finding a single "divergence mountain" centered on or near $C_0=0$ (as in Figs. 5, 8, and 10), we find a series of ridges centered around those values of C_0 which produce a zero energy bound state of the distortion potential. The behavior as α increases is similar to that of the separable two-channel case, but there is no simplification as C_0 becomes large and negative. As C_0 grows, the accuracy of the DWBA oscillates.[10,23] These models make clear that the use of C/C_0 as the "small parameter which makes the DWBA work" only is correct for the separable model and that more complex statements are required for local potentials.

V. CONCLUSIONS

An important failing in the theory of nuclear reactions is the lack of a clear theory of reaction mechanisms. In the theory of direct reactions, we may define a reaction mechanism as the specification of the set of important channels and arrangements,[25] plus a statement of which processes can be treated perturbatively and which must be treated to all orders. Such information must be contained in the Hamiltonian for the system, since it contains all the information about the system.

What we have shown here is that the information about the reaction mechanism can be extracted if the reaction is described by a set of (compact kernel) integral equations.[26] The information is contained in a relatively simple property of the operator kernel: its spectral radius, or the norm of the largest eigenvalue. In general, it is simpler to extract the largest eigenvalue of a matrix than to invert that matrix. Even with our formulation, the matrix must still be constructed. However, by specifying the parameter that governs the convergence of the series, new questions can be formulated and attacked in new ways. For example, we have asked: What is the distortion potential that minimizes the size of the convergence parameter? For the model above we have shown that this question can be answered analytically and leads to some interesting (and nonobvious) insights. It is important to consider whether this approach can be applied to more realistic formulations of reaction problems. A second important question is whether the identification of distorted-wave quasiparticles as the cause of divergence can permit us to develop an approximate theory for them that would not require the construction of the full matrix but would let us extract them directly.

Even if analytic techniques are not easily available for the extraction of quasiparticles, the technique of forming contour maps of the convergence parameter could be valuable for summarizing numerical experience with a specific class of model. The convergence parameter could then simply be read off the map in order to determine whether one should use a perturbative treatment or a full coupled-channel calculation.

The above procedure also allows us to treat the DWBA for inconsistencies. We consider the DWBA to be the first step of a multistep series. Occasionally it is treated as a way to parametrize the full transition operator. If the latter procedure is followed, parameters in the transition potential must be adjusted until the experimental data is fit. One must not then calculate further corrections.

These two views are in contradiction and may lead to inappropriate calculations. Some authors attempt to adjust parameters in the DWBA until the data is fit as well as possible. If the data cannot easily be fit then multistep corrections may be added. If the DWBA is considered as part of a series to which multistep corrections should be added, the parameters of the first-order term should not be adjusted to fit the data. The parameters of the DWBA (strengths, radii, coupling, etc.) imply the size of higher order corrections. If the parameters required to fit the inelastic data imply important higher order corrections then the approach is inconsistent.

It is our view that an approach of the type described herein should be a part of a complete reaction theory: one in which the diagonal potentials, coupling strengths, etc., are calculated from first principles. The procedure presented here then allows the control of corrections.

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APPENDIX

The minimal distortion potential satisfying Eq. (74) can be found analytically for the two-channel model considered in Sec. III. We determine the value $w^{\min}$ that yields the minimum of the convergence parameter η:

$$(\eta^{\min})^2=\min_{w}\{\max[\,|\lambda_+(w)|^2,|\lambda_-(w)|^2]\}\ . \tag{A1}$$

For positive energy, the norms squared of the eigenvalues in (55) are

$$|\lambda_\pm(w)|^2=\frac{C_0^2[(1-w_1\pm C/C_0)^2+w_2^2]}{(\alpha^2-1-w_1C_0)^2+(2\alpha-w_2C_0)^2}\ , \tag{A2}$$

where the energy E is proportional to α^2. Except for the singular point at $((\alpha^2-1)/C_0,2\alpha/C_0)$, these are continuous functions of $w=w_1+iw_2$ that each attain a single minimum of zero at $(1\pm C/C_0,0)$.

This is sufficient information to conclude that $w^{\min}$ must lie in the intersection of the surfaces $|\lambda_\pm|^2$. Equating the norms in (A2), we find that $w_1^{\min}=1$ and we are left with a minimization over w_2,

$$\frac{d}{dw_2}|\lambda_\pm(1+iw_2)|^2=0\ . \tag{A3}$$

This yields the optimum value for w_2:

$$w_2^{\min}=\frac{1}{4\alpha C_0}([(\alpha^2-1-C_0)^2+4\alpha^2-C^2] -\{[(\alpha^2-1-C_0)^2+4\alpha^2-C^2]^2 +16\alpha^2C^2\}^{1/2})\ . \tag{A4}$$

One can show that at this value of w_2

$$(\eta^{\min})^2=\frac{w_2^{\min}C_0}{w_2^{\min}C_0-2\alpha}\ .$$

Since $w_2^{\min}C_0\le 0$ in (A4), $\eta^{\min}<1$ when $\alpha>0$. Therefore the DWBS converges for all positive energies when we choose this minimal potential.

At negative energies

$$|\lambda_\pm(w)|^2=\frac{(C_0-C_0w_1\pm C)^2+C_0^2w_2^2}{[(\alpha_1+1)^2-w_1C_0]^2+C_0^2w_2^2}\ ,$$

where $\alpha\equiv i\alpha_1$. As we did for positive energies, we minimize this expression with respect to w_2 after setting $w_1=1$. However, in this case, we cannot improve upon the convergence with the folding potential. The minimal potential at negative energies is the folding potential with $w=1$. In the special case that $C^2>[(\alpha_1+1)^2-C_0]^2$, the series diverges with the folding potential. It will diverge less strongly for nonzero choices but it will not converge.

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[26]The integral equation must have compact kernel in order for it to be guaranteed that the operator has only discrete eigenvalues with only a finite number outside any given circle in the complex plane. For a discussion of compact-kernel formulations of reaction theory see Ref. 25 and references therein.

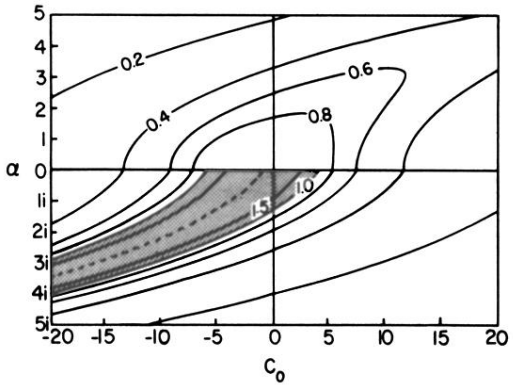

FIG. 10. Contour map of the convergence parameter $\eta^{\min}$, for the DWBS in the two-channel model using the minimal distortion potential. The divergent region is shaded and $C = -5$.

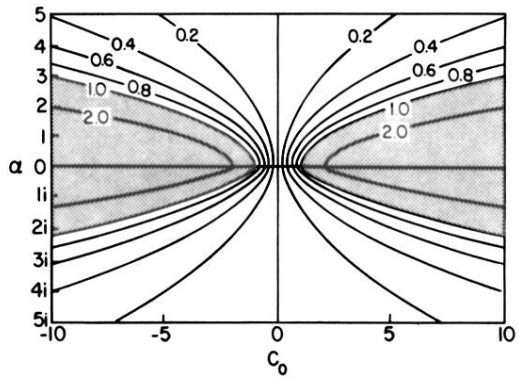

FIG. 2. Contour map of the convergence parameter, η, for the Born series with the one-channel separable Yamaguchi potential plotted as a function of the control parameters $C_0 = 2m\pi^2 R^3 \Lambda_0/\hbar^2$ and $\alpha = kR$. The divergent region is shaded.

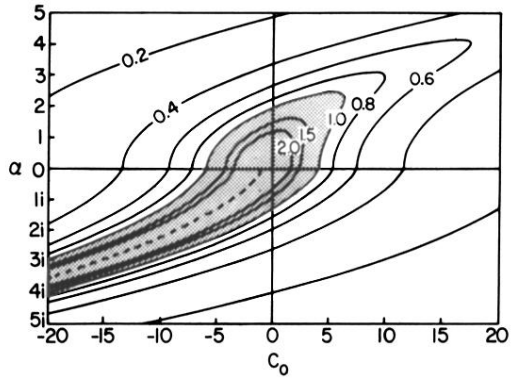

FIG. 5. Contour map of the convergence parameter, η^f [Eq. (61)], for the DWBS in the two-channel model using the folding distortion. The divergent region is shaded and $C = -5$.

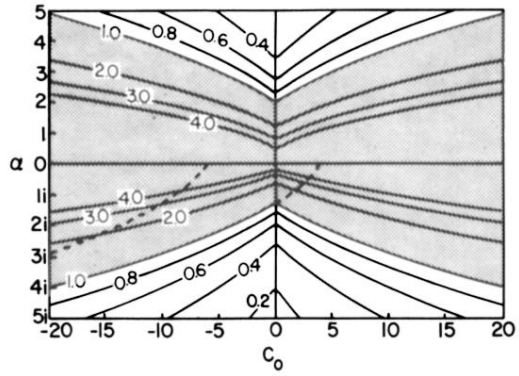

FIG. 6. Contour map of the convergence parameter, η^B [Eq. (62)], for the Born series in the two-channel model. The divergent region is shaded and $C = -5$.

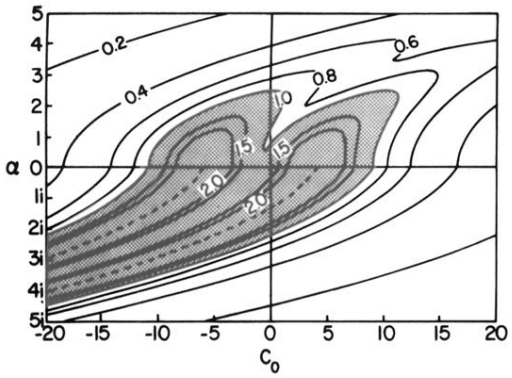

FIG. 8. Contour map of the convergence parameter, η^{opt} [see Eq. (71)], for the DWBS in the two-channel model using the generalized optical potential as the distortion. The divergent region is shaded and $C = -5$.