# Exclusion effect of a strong, short-range, attractive real potential

M. A. Alberg
*Department of Physics, Seattle University, Seattle, Washington 98122*
*and Institute for Nuclear Theory, Department of Physics, FM-15, University of Washington, Seattle, Washington 98195*

L. Wilets
*Institute for Nuclear Theory, Department of Physics, FM-15, University of Washington, Seattle, Washington 98195*

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The effects of strong, short-range attraction in exotic atoms are considered in a simplified model. The model consists of a spherical box of radius $b$ which contains a strong, short-range, real potential well of radius $a$. Its eigenvalue spectrum has two parts: negative energy, strongly bound "nuclear" states, and positive energy "atomic" states. It is found that the primary effect of the attractive potential on positive energy states is to exclude the wave function from the well. The eigenvalues of these states, identified with experimentally observed atomic states, are thereby shifted upward to the values expected for a smaller box of radius $(b-a)$. Implications for analysis of exotic atom spectra are discussed.

Experimental work on exotic atoms formed by kaons and antiprotons has shown that the energy levels of the lowest atomic states are shifted upward from the values expected for hydrogenlike atoms.[1,2] This result was surprising, since the two-body $\bar{K}N$ and $\bar{N}N$ interactions are strongly attractive.[3,4] Krell[5] showed that this effective repulsion was due to absorption in the nucleus, i.e., exclusion of the atomic wave function from the nuclear region by the imaginary part of the optical potential. He also observed oscillations in energy level shifts and widths as the real part of the potential was increased. Koch *et al.*[6] explained these nonlinear effects in terms of a model in which a complex square well of nuclear size was contained within a spherical box of atomic size. For strong absorption the eigenstates of this system separated into inner states (confined to the nuclear region) and outer states (confined to the atomic region outside of the well). The effective repulsion observed by Krell corresponded to the exclusion of outer states from the nuclear region. The oscillations in energy level shifts and widths were shown to be due to weak coupling between inner and outer states, which resulted in a "level-crossing" effect as an increase in the real part of the potential caused the energy spectrum of inner states to drop relative to the spectrum of outer states.

We present in this Brief Report the results of a model calculation for a strong, short-range, attractive *real* potential which shows similar nonlinear effects. We find that such a potential also excludes the wave function of a positive energy state from the inner (nuclear) region, except for energies very close to "resonance," as defined below. Off resonance, the eigenvalues of positive energy states, which are to be identified with experimentally observed atomic states, are shifted upward relative to the values expected if there were no attractive nuclear potential. Based on observations in the atomic region (by which we mean the region external to the nucleus), one would conclude that the effect of a strongly attractive, real, short-range potential is primarily repulsive. Such nonlinear effects are well known in scattering theory, but it is instructive to study the analogous phenonemon for bound states. We recognize, of course, that the optical potential for an exotic atom will in general have both real and imaginary parts, but wish to point out that exclusion of a particle from the nuclear region can occur even in the absence of absorptive or repulsive terms in the potential.

We consider the $s$-wave states of a particle in a spherical box of radius $b$ which contains an attractive square well of radius $a$ and real depth $V_R$. The Schrödinger equation for the particle is (using $\hbar=2m=c=1$)

$$\left[-\frac{d^2}{dr^2}+V(r)\right]u(r)=Eu(r) \ . \tag{1}$$

The potential $V(r)=-V_R$, $0<r<a$ (inner, nuclear region), $V(r)=0$, $a<r<b$ (outer, atomic region), $V(r)\to\infty$, $r>b$. The wave functions for positive energy states are

$$u(r)=\begin{cases} A\sin kr, & 0<r<a, \quad k^2=E+V_R \ , \\ \sin K(b-r), & a<r<b, \quad K^2=E \ . \end{cases} \tag{2}$$

Continuity of the logarithmic derivative at $r=a$ requires

$$k\cot ka=-K\cot K(b-a) \ . \tag{3}$$

Solutions to Eq. (3) were found numerically. The results are presented in Fig. 1 for a representative case ($b=1.0$ fm, $a=0.1$ fm). For $V_R=0$ the spherical box solutions $K^{(n)}=n\pi/b$, $n=1,2,3\ldots$ are found. As $V_R$ increases the energy of each level decreases monotonically, and its outer wave number $K$ decreases in a highly nonlinear way. The ground-state energy drops quickly through zero to negative values, for which the outer wave function is $u(r)=\sinh|K|(b-r)$. Such negative energy states are confined to the nuclear region. For excited states $K$ is found to be relatively constant for broad "plateau" regions, with rapid transitions between plateaus. The values of $K$ in plateau regions are given approximately by the solutions $K_{\mathrm{pl}}^{(n)}=n\pi/(b-a)$, $n=1,2,3\ldots$ indicated by dashed lines in Fig. 1. Thus, except for transitions between plateau regions, the particle appears to be confined to the outer re-

 

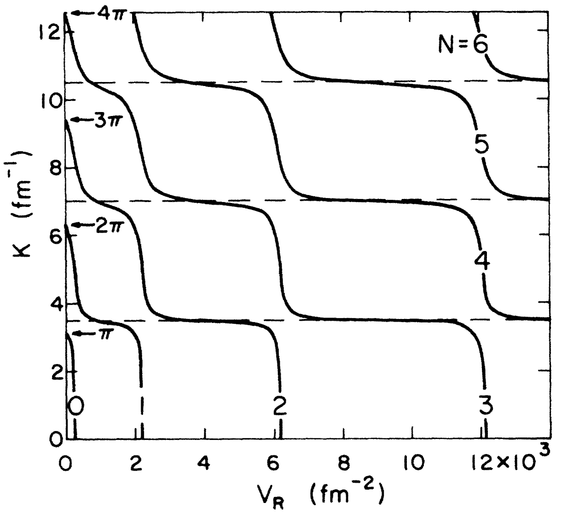


FIG. 1. The outer wave number $K$ as a function of the depth of the real potential $V_R$, for the case $b=1.0$ fm, $a=0.1$ fm. The values $K^{(n)}=n\pi/b$, $n=1,2,3\ldots$ are the solutions for $V_R=0$. The dashed lines represent values of $K$ which correspond to a box of size $(b-a)$, viz., $K_{\rm pl}^{(n)}=n\pi/(b-a)$, $n=1,2,3\ldots$. The number of nodes in the wave function for each state is labeled by $N$. The energy is given by $K^2$. Negative energy states, which correspond to imaginary $K$, do not appear in this figure.

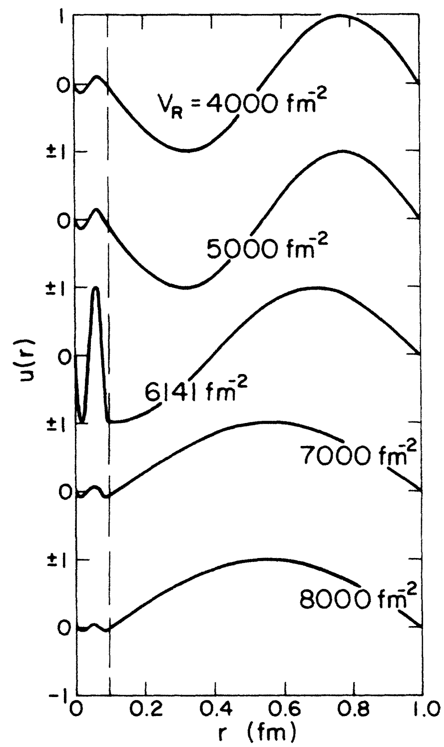


FIG. 2. The wave function $u(r)$ in the $N=3$ state for different values of the real potential $V_R$. Resonance occurs for $V_R=V_{\rm tr}=6141$ fm$^{-2}$. The dashed line indicates the edge of the inner well at $a=0.1$ fm.

gion, a box of size $(b-a)$.

Examination of the wave functions (Fig. 2) provides an explanation of these effects. We consider the three-node state ($N=3$). For $V_R=4000$ and 5000 fm$^{-2}$, on the plateau defined by $K_{\rm pl}^{(2)}=2\pi/(b-a)$, the wave function in the outer region looks like a one-node state. The amplitude of the wave function in the inner region is quite small ($A\sim 0.1$). The particle is essentially excluded from the inner region, and increasing $V_R$ has very little effect on $K$. However, at resonance, for which $ka=(N_i+\frac{1}{2})\pi$ and $K(b-a)=(N_0+\frac{1}{2})\pi$, the amplitudes of the wave functions in the inner and outer regions are equal. Here $N_i$ and $N_0$ are the number of nodes in the inner and outer regions, respectively. In a transition betwen plateaus $K$ passes through such a resonance, as shown by the wave function for $V_R=6141$ fm$^{-2}$ ($N_i=2$, $N_0=1$). Then on the next plateau the wave function looks like a nodeless state in the outer region, as shown for $V_R=7000$ and 8000 fm$^{-2}$. Again, $A$ is very small, and the particle is effectively excluded from the inner region.

In Fig. 3 we plot $u(a)$, the value of the wave function at the edge of the inner well, for the $N=3$ state. It is seen that except near resonance, $u(a)\approx 0$ and the particle appears to be in a state of $N_0$ nodes in the outer region. For most values of the potential $V_R$ the particle is essentially confined to the atomic region in an eigenstate which corresponds to the spectrum of states associated with a box of radius $(b-a)$.

Why then are plateaus so broad and transitions so narrow? Resonance occurs for values of the potential $V_R$ given by

$$V_{\rm tr}=\left[\left(\frac{N_i+\frac{1}{2}}{a}\right)^2-\left(\frac{N_0+\frac{1}{2}}{b-a}\right)^2\right]\pi^2 . \quad (4)$$

A measure of the transition region $\Delta V_{\rm tr}$ between two levels of $K_{\rm pl}^{(n+1)}$ and $K_{\rm pl}^{(n)}$ is found by considering the slope $dK/dV_R$ at $V_{\rm tr}$, as shown in Fig. 4:

$$\Delta V_{\rm tr}\equiv\frac{K_{\rm pl}^{(n)}-K_{\rm pl}^{(n+1)}}{(dK/dV_R)_{V_{\rm tr}}} , \quad (5)$$

or

$$\Delta V_{\rm tr}=\left(n+\frac{1}{2}\right)\left(\frac{\pi}{b-a}\right)^2\left(\frac{2b}{a}\right) . \quad (6)$$

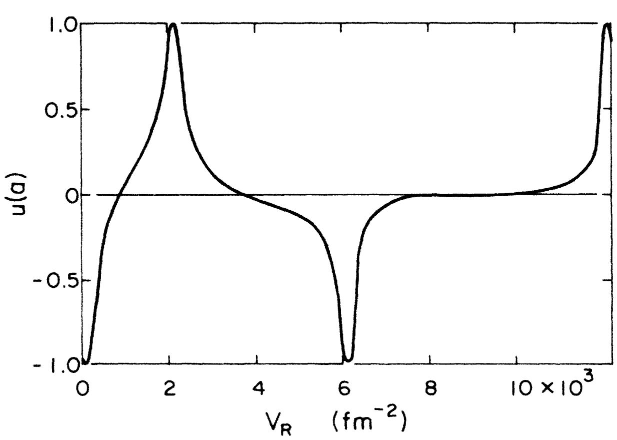


FIG. 3. The dependence of $u(a)$, the value of the wave function at the edge of the inner well, on $V_R$, the depth of the real potential.

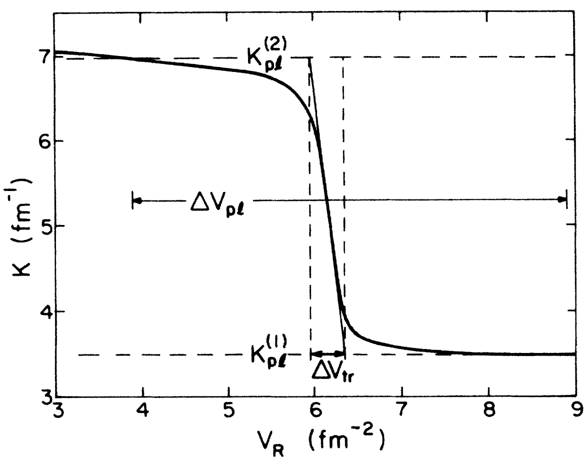


FIG. 4. The outer wave number $K$ in the transition region between the $n=2$ and $n=1$ plateaus for the $N=3$ state. A measure of the transition region is given by $\Delta V_{tr} = (K_{pl}^{(n)} - K_{pl}^{(n+1)})/(dK/dV)_{V_{tr}}$.

As seen in Figs. 1 and 4, this region is very narrow relative to $\Delta V_{pl}$, the distance between midplateaus. A midplateau is defined by $ka=(N_i+1)\pi$ and $K(b-a)=(N_0+1)\pi$, for which $V_R$ is given by

$$V_{pl}=\left[\left(\frac{N_i+1}{a}\right)^2-\left(\frac{N_0+1}{b-a}\right)^2\right]\pi^2 , \tag{7}$$

with $N=N_0+N_i+1$ (one node occurs at $r=a$). Thus

$$\Delta V_{pl}=2\pi^2\left(\frac{(N-n+\frac{1}{2})}{a^2}+\frac{(n+\frac{1}{2})}{(b-a)^2}\right) \tag{8}$$

is the change in $V_R$ between midplateaus which correspond to $K_{pl}^{(n+1)}$ and $K_{pl}^{(n)}$. Then

$$\frac{\Delta V_{tr}}{\Delta V_{pl}}=\frac{(n+\frac{1}{2})ab}{[(N-n+\frac{1}{2})(b-a)^2+(n+\frac{1}{2})a^2]} . \tag{9}$$

For short-range potentials, $b >> a$, and

$$\frac{\Delta V_{tr}}{\Delta V_{pl}}\approx\frac{(n+\frac{1}{2})}{(N-n+\frac{1}{2})}\frac{a}{b} . \tag{10}$$

Therefore transition regions are very narrow relative to plateau regions, and the spectrum of plateau regions will dominate experimental measurements. Note that for a fixed $N$, transition regions become relatively sharper as $n$ decreases, as seen in Fig. 1.

In summary, we find that a strong, short-range, attractive real potential divides the spectrum of a spherical box into two parts: a spectrum of negative energy, strongly bound states confined to the nuclear region, and a spectrum of positive energy atomic states which are effectively excluded from the region of strong attraction. This exclusion results in an eigenvalue spectrum of positive energy states which is shifted upward relative to the spectrum which corresponds to no attraction. Experimental measurements of these atomic states will reflect an apparent repulsion which occurs in the absence of repulsive or absorptive terms in the potential. It has been hoped that exotic atom spectra can be used to study two-body interactions of strongly interacting particles. Clearly, the exclusion effect should be taken into account in determining the nature of the short-range potentials involved.

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