Relativistic effects in three-body bound states

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We formulate relativistic and nonrelativistic two-particle dynamics in such a manner that the two-body binding energies are the same for both. We then formulate and solve the relativistic Faddeev equations for a simple *s*-wave potential (Malfliet-Tjon V). The relativistic effects are small (about 3%) and reduce the three-body binding energy. The expectation value of the relativistic energy operator with the nonrelativistic wave function is a fairly good approximation, but approximate expressions involving expansions in powers of the momentum are shown to be quite unreliable.

I. INTRODUCTION

Nuclear many-body dynamics is in essence a phenomenology based on a two-body dynamics fitted to two-body data. More fundamental theories motivate and guide the construction of two-body potentials to some extent. Multibody forces are expected to be nonvanishing but relatively small. The nonrelativistic many-body theory of nuclei so constructed has been quite successful, but quantitative discrepancies remain, especially in the triton binding energy and in the description of nuclear saturation properties.[1,2] Relativistic effects may be significant since the velocities of nucleons in nuclei can be of the order of one-third of the velocity of light. Such velocities are sufficiently large for the requirements of Lorentz invariance to produce effects of the order of 10%. A consistent treatment of the interaction with high energy probes also requires Lorentz invariant dynamics. The purpose of this paper is an investigation of the dynamical consequences of the requirements of Lorentz invariance for the ground-state properties of three-body systems. Since the validity of widely used expansions in powers of the momenta is suspect, we investigate a simple model for which exact numerical computations are possible with a moderate effort. The "relativistic effects" associated with strong fields, comparable in strength to the nucleon mass, which are the principal feature of relativistic mean field formalisms[3] are another matter and not the subject of this paper.

If the starting point is a relativistic field theory the nonrelativistic limit is specified by the static potentials together with the nonrelativistic kinetic energies.[4] This comparison of relativistic and nonrelativistic theory may show large relativistic effects which are not relevant to the question: To what extent does Lorentz invariance require modifications of the conventional nuclear many-body dynamics? To answer this question we need to start with a relativistic and a nonrelativistic two-body dynamics which both fit the two-body data. The approach first proposed by Bakamjian and Thomas[5] is well suited for that purpose. It consists of constructing the Poincaré invariant dynamics by introducing the interactions in the two-body mass operator. Starting from this construction it has been shown[6–9] that it is possible to arrive at a consistent Poincaré invariant many-body dynamics based on phenomenological two-body interactions.

Given a phenomenological relativistic dynamics that fits the two-body data, the construction of a corresponding nonrelativistic theory which fits the same data, is by no means unique. We will use this ambiguity to keep relativistic effects small. On the two-body level a particularly simple correspondence is established by the following observation.[10] Since the mass operator h_0 of two noninteracting nucleons is $h_0=2(\mathbf{k}^2+M_N^2)^{1/2}$, the operator $H\equiv h^2/4M_N\text{-}M_N$ has exactly the same form as the nonrelativistic two-body Hamiltonian after elimination of the center-of-mass motion, provided we define the mass operator h of the interacting system in terms of h_0 and the nonrelativistic potential V_{NR} by

$$h^2=h_0^2+4M_N V_{NR} \; . \tag{1.1}$$

The eigenvalue of H is $M_D^2/4M_N\text{-}M_N$, which differs from the deuteron energy $E_D=M_D-2M_N$ by $E_D^2/4M_N$. Therefore, the only relativistic effect in the deuteron is the tiny adjustment in the potential required by the shift $E_D^2/4M_N$ in the eigenvalue. The preceding relation between relativistic and nonrelativistic two-body dynamics was used in a study of relativistic effects in three- and four-body models[11] and in the binding energy of nuclear matter.[10] This calculation showed significant relativistic effects in individual partial waves, but the net effects on the saturation curves turned out to be negligible. Important three-body correlations[2] were not included in this study, and the importance of relativistic effects in the three-body correlations is unknown.

The construction of the relativistic many-body Hamiltonians requires the two-body mass operator h rather than its square. The relation between relativistic and nonrelativistic dynamics specified by Eq. (1.1) thus requires the extraction of the square root of the right-hand side of Eq. (1.1), which presents serious computational difficulties. These difficulties can be avoided by suitable linear relations between h and V_{NR}, which will be described in detail in Sec. II.

Most studies of relativistic corrections use expansions in powers of the momenta. Such expansions in powers of unbounded operators are prima facie suspect. This problem is aggravated by the general feature that net relativistic effects involve cancellations between positive and negative contributions. Exact three-body calculations are therefore essential for a reliable picture. Since realistic three-body calculations involve considerable complexities, exact calculations with oversimplified model potentials can give valuable initial insight. In this paper we present the results obtained with a simple s-wave potential, the Malfliet-Tjon-V (MT-V) potential,[12] from which we obtain relativistic potentials by the procedure described in Sec. II. In Sec. III, we discuss the relativistic Faddeev equation and our method of solution. The numerical procedure is described in the Appendix. Our results and conclusions are summarized in Sec. IV.

II. RELATIVISTIC AND NONRELATIVISTIC TWO-BODY DYNAMICS

Following Bakamjian and Thomas,[5,13] the relativistic two-body dynamics is specified in terms of the two-body mass operator h,

$$(\mathbf{k}' \mid h \mid \mathbf{k}) = 2E(\mathbf{k})\delta(\mathbf{k}'-\mathbf{k}) + v(\mathbf{k}',\mathbf{k}) , \tag{2.1}$$

where $E(k)=(m^2+k^2)^{1/2}$, and k is the relative momentum while the total momentum vanishes. The bound-state Schrödinger equation has therefore the form

$$h\chi_D(\mathbf{k}) = \chi_D(\mathbf{k})M_D , \tag{2.2}$$

and the deuteron vertex $\Gamma_D = v\chi_D$ satisfies the eigenvalue equation

$$\Gamma_D(\mathbf{k}) = \int d^3k' v(\mathbf{k},\mathbf{k}')[M_D - 2E(\mathbf{k}')]^{-1}\Gamma_D(\mathbf{k}') . \tag{2.3}$$

The two-nucleon scattering is described by the relativistic Lippman-Schwinger equation[10,14]

$$t(\mathbf{k},\mathbf{k}_0) = v(\mathbf{k},\mathbf{k}_0) + \int d^3k' \frac{v(\mathbf{k},\mathbf{k}')t(\mathbf{k}',\mathbf{k}_0)}{\omega(\mathbf{k}_0)-\omega(\mathbf{k}')+i\epsilon} , \tag{2.4}$$

where $\omega(\mathbf{k})=2E(\mathbf{k})$, and $t(\mathbf{k},\mathbf{k}_0)$ is the half-off-shell transition matrix.

In order to establish the connection to a corresponding nonrelativistic dynamics we define[14,15]

$$\widetilde{V}(\mathbf{k}',\mathbf{k}) = [E(\mathbf{k}')/m]^{1/2} v(\mathbf{k}',\mathbf{k})[E(\mathbf{k})/m]^{1/2} . \tag{2.5}$$

The virtue of this definition is that it allows us to transform the relativistic Eqs. (2.2)–(2.4) without approximation into equations, which have almost the nonrelativistic form. Let

$$\widetilde{T}(\mathbf{k},\mathbf{k}_0) = [E(\mathbf{k})/m]^{1/2} t(\mathbf{k},\mathbf{k}_0)[E(\mathbf{k}_0)/m]^{1/2} . \tag{2.6}$$

Using the definitions (2.5) and (2.6), multiplication of Eq. (2.4) on both sides by the appropriate factors produces the desired integral equation for $\widetilde{T}(\mathbf{k},\mathbf{k}_0)$,

$$\widetilde{T}(\mathbf{k},\mathbf{k}_0) = \widetilde{V}(\mathbf{k},\mathbf{k}_0) + \int d^3k' \frac{\widetilde{V}(\mathbf{k},\mathbf{k}')\widetilde{T}(\mathbf{k}',\mathbf{k}_0)}{\mathbf{k}_0^2/m - \mathbf{k}'^2/m + i\epsilon} \times \left[1 + \frac{E(\mathbf{k}_0)-E(\mathbf{k}')}{2E(\mathbf{k}')} \right] . \tag{2.7}$$

A similar bound-state equation can be derived from Eq. (2.3)

$$\widetilde{\Gamma}_D(k) = \int d^3k' \frac{\widetilde{V}(\mathbf{k},\mathbf{k}')\widetilde{\Gamma}_D(\mathbf{k}')}{\widetilde{E}_D - \mathbf{k}'^2/m} + \int d^3k' \frac{m\widetilde{V}(\mathbf{k},\mathbf{k}')\widetilde{\Gamma}_D(\mathbf{k}')}{E(\mathbf{k}')[M_D+2E(\mathbf{k}')]} , \tag{2.8}$$

where $\widetilde{\Gamma}_D(k) = [E(\mathbf{k})/m]^{1/2}\Gamma_D(\mathbf{k})$ and

$$\widetilde{E}_D = M_D^2/4m - m = (M_D - 2m)[1 + (M_D - 2m)/4m] . \tag{2.9}$$

The difference between the eigenvalue $\widetilde{E}_D$ and the deuteron energy E_D defined as M_D-$2m$ is the tiny relativistic effect mentioned in the Introduction; $E_D^2/4M_\mathrm{N} \simeq 0.001$ MeV. The last terms of Eqs. (2.7) and (2.8) are small corrections to the nonrelativistic equations. Their presence is the price we have to pay for a simple relation between $\widetilde{V}$ and v.

We use Eqs. (2.7) and (2.8) to establish the desired correspondence of the relativistic and nonrelativistic dynamics. We assume

$$\widetilde{V}(\mathbf{k}',\mathbf{k}) = f V_\mathrm{NR}(\mathbf{k}',\mathbf{k}) , \tag{2.10}$$

where f is a constant factor determined by the requirement that the relativistic dynamics defined by Eqs. (2.1)–(2.4) and the ordinary nonrelativistic dynamics, defined by V_NR, both give the correct deuteron binding energy. If the last term in Eq. (2.8) was negligible, and we neglect the small difference $\widetilde{E}_D - E_D$, then we would have $f=1$. For the MT-V and Reid-soft core (RSC) potentials we find, respectively, $f=1.02025$ and 1.0487. This last factor is substantially smaller than the 1.08 obtained by a perturbation estimate in which $E(k)$ was approximated by M_N.[15] A first-order perturbation treatment which retains the correct k dependence of $E(k)$ still overestimates $(f-1)$ but only by 15%.

In Table I we summarize the results for the relativistic and nonrelativistic energies

TABLE I. Two-body energies. (See the text for notations.)

	RSC	MT
Binding energy	2.22	0.35
$E_\mathrm{kin}^\mathrm{NR}$	22.12	4.17
E_kin^R	21.63	4.12
$E_\mathrm{pot}^\mathrm{NR}$	−24.34	−4.52
E_pot^R	−23.85	−4.47
$(E_\mathrm{kin}^R)_\mathrm{NR}$	20.91	4.07
$(E_\mathrm{pot}^R)_\mathrm{NR}$	−22.63	−4.41
$\Delta E'_\mathrm{kin}$	−1.58	−0.14
ΔE_kin	−0.49	−0.05

TABLE II. Relativistic effects. (See the text for notations.)

Potentials	$\Delta E_{\rm pot}/E_{\rm pot}^{\rm NR}$	$\Delta E_{\rm kin}/E_{\rm kin}^{\rm NR}$	$\Delta E_B/E_B$
RSC two-body	−0.020	−0.022	0.00
MT two-body	−0.011	−0.012	0.00
MT three-body	−0.033	−0.036	−0.027

$$E_{\rm kin}^R = 2\langle E(k)-m\rangle_R;\quad E_{\rm kin}^{\rm NR} = \langle k^2/m\rangle_{\rm NR} \tag{2.11}$$

and

$$E_{\rm pot}^R = \langle v\rangle_R;\quad E_{\rm pot}^{\rm NR} = \langle V_{\rm NR}\rangle_{\rm NR}\ . \tag{2.12}$$

The subscripts R or NR on the angular brackets $\langle\ \rangle$ indicate whether the expectation value was computed with relativistic or nonrelativistic wave functions. In order to exhibit the quality of various approximations we also show the expectation values of the relativistic operators with the nonrelativistic wave functions,

$$(E_{\rm kin}^R)_{\rm NR} = 2\langle E(k)-m\rangle_{\rm NR}:\ (E_{\rm pot}^R)_{\rm NR} = \langle v\rangle_{\rm NR} \tag{2.13}$$

as well as the kinetic energy difference

$$\Delta E_{\rm kin} = E_{\rm kin}^R - E_{\rm kin}^{\rm NR}$$

and the approximation

$$\Delta E'_{\rm kin} = -\langle k^4\rangle_{\rm NR}/(4m^3)\ .$$

Both the relativistic kinetic and potential energies are smaller in magnitude than the corresponding nonrelativistic energies. The inadequacy of an expansion in powers of the momentum is manifest (as seen in the large difference between $\Delta E'_{\rm kin}$ and $\Delta E_{\rm kin}$). Since the nonrelativistic (relativistic) wave functions decrease for large p as p^{-4} ($p^{-7/2}$), the expectation values $\langle p^n\rangle$ diverge for $n>4$ (3).

Taking the expectation values of the relativistic operators with the nonrelativistic wave functions involves errors of the same magnitude as the relativistic effects in the kinetic and potential energies. In Table II we show the fractional changes in the kinetic and potential energies

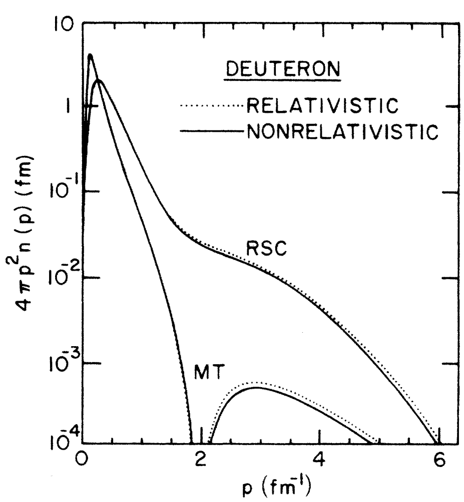

FIG. 1. Weighted deuteron momentum distributions $4\pi p^2 n(p)$ from nonrelativistic (solid curves) and relativistic (dashed curves) calculations. MT and RSC denote, respectively, the Malfliet-Tjon and Reid-soft-core potentials.

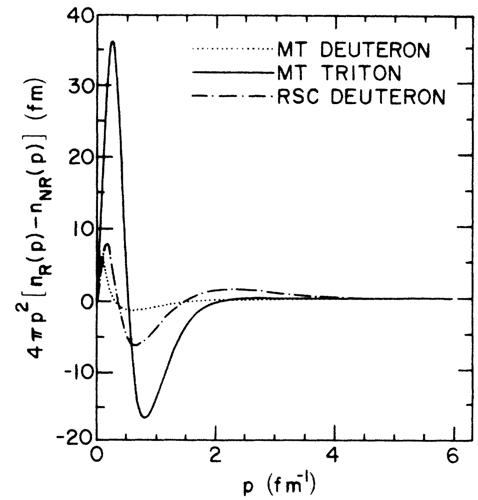

FIG. 2. Weighted differences between relativistic and nonrelativistic momentum distributions calculated from the Malfliet-Tjon (MT) potential for the deuteron and the triton, and from the Reid-soft-core (RSC) potential for the deuteron.

$$\Delta E_{\rm pot}/E_{\rm pot}^{\rm NR} = (E_{\rm pot}^R - E_{\rm pot}^{\rm NR})/E_{\rm pot}^{\rm NR}$$

and $\Delta E_{\rm kin}/E_{\rm kin}^{\rm NR}$. The momentum distributions for both potentials shown in Fig. 1 do not reveal any dramatic effects. On the logarithmic plot the relativistic effects show up only for momenta larger than 2 fm^{-1}. Figure 2 shows the difference between the relativistic and nonrelativistic momentum distributions for both potentials.

We have verified that the relativistic and nonrelativistic models so constructed are approximately phase-shift equivalent. For energies ≤ 50 MeV the phase shifts of the relativistic potential are more repulsive by less than 1%.

III. RELATIVISTIC FADDEEV EQUATION

For both relativistic and nonrelativistic dynamics the Hamiltonian for a three-particle system with vanishing total momentum can be explicitly written as

$$H = H_0 + \tfrac{1}{2}\sum_{i\neq j} V_{ij}\ . \tag{3.1}$$

Their difference is in the momentum dependence of the kinetic energy and in the relation of the operators V_{ij} to the two-body dynamics discussed in Sec. II. In the nonrelativistic case we have, of course,

$$H_{\rm NR} = \sum_i (p_i^2/2m) + \tfrac{1}{2}\sum_{ij}(V_{\rm NR})_{ij}\ , \tag{3.2}$$

while in the relativistic case

$$H_0 = \sum_i [(m^2+\mathbf{p}_i^2)^{1/2} - m] \tag{3.3}$$

and the relation between V_{ij} and the two-body mass operator $h_{ij} = \omega(\mathbf{k}_{ij}) + v_{ij}$

$$V_{ij} = \{[\omega(\mathbf{k}_{ij}) + v_{ij}]^2 + \mathbf{p}_{ij}^2\}^{1/2} - [\omega(\mathbf{k}_{ij})^2 + \mathbf{p}_{ij}^2]^{1/2} \tag{3.4}$$

is determined by the cluster separability requirement.[16] The sum of the momenta p_i must vanish,

$$\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0 \tag{3.5a}$$

and p_{ij} is the total momentum of the subsystem ij,

$$\mathbf{p}_{ij}=\mathbf{p}_i+\mathbf{p}_j \; . \tag{3.5b}$$

The relative momentum $\mathbf{k}_{ij}$ of the ij subsystem is in the rest frame of that cluster equal to one-half the momentum difference. For equal mass particles it is therefore related to the momenta p_i and p_j by[13]

$$\mathbf{k}_{ij}\equiv\mathbf{k}(\mathbf{p}_i,\mathbf{p}_j) \equiv\frac{1}{2}\left\{\mathbf{p}_i-\mathbf{p}_j-\mathbf{p}_{ij} \times\frac{E_i-E_j}{(E_i+E_j)+[(E_i+E_j)^2-(\mathbf{p}_i+\mathbf{p}_j)^2]^{1/2}}\right\} . \tag{3.6}$$

The last term exhibits the relativistic effect in the definition of the relative momentum. The constraints of relativistic invariance require that the two-body interaction v_{ij} be independent of the total momentum p_{ij} of the cluster ij. The mass operator h_{ij} depends therefore only on the relative momentum $\mathbf{k}_{ij}$. Once the two-particle mass operator h_{ij} is defined, the three-body Hamiltonian H is then completely determined by Eqs. (3.1) and (3.4).

Our objective now is to solve the three-particle bound state problem defined by

$$H\,|\,\Phi_B\rangle=|\,\Phi_B\rangle E_B \; , \tag{3.7}$$

where H is given in Eq. (3.1). Since the formal structure of the Hamiltonian (3.1) is the same for relativistic and nonrelativistic dynamics, the formal derivation of the Faddeev equations is also the same in both cases. Following the Faddeev method, we define three components $|\,\phi_i\rangle$ by

$$|\,\phi_i\rangle=(E-H_0)^{-1}V_{jk}\,|\,\Phi_B\rangle, \quad i,j,k \text{ cyclic} \; . \tag{3.8}$$

It follows from Eqs. (3.7) and (3.8) that

$$|\,\Phi_B\rangle=|\,\phi_1\rangle+|\,\phi_2\rangle+|\,\phi_3\rangle \tag{3.9}$$

and

$$|\,\phi_i\rangle=\frac{1}{E_B-H_0-V_{jk}}V_{jk}(\,|\,\phi_j\rangle+|\,\phi_k\rangle), \quad i,j,k \text{ cyclic} \; . \tag{3.10}$$

For three identical particles, the component $|\,\phi_1\rangle$ is represented by a function $\phi_1(\mathbf{p}_1;\mathbf{p}_2,\mathbf{p}_3)$, symmetric in the last two arguments, and the representations of all three components are related to each other by cyclic permutation of the three momenta. Instead of the three coupled equations (3.10) we need to consider only one of them, for instance,

$$|\,\phi_1\rangle=\frac{1}{E_B-H_0-V_{23}}V_{23}(\,|\,\phi_2\rangle+|\,\phi_3\rangle) \; . \tag{3.11}$$

To solve Eq. (3.11), it is most convenient to choose the following representation in which H_0 is diagonal:

$$H_0\,|\,\mathbf{k},\mathbf{p}\rangle_1=E(\mathbf{k},\mathbf{p})\,|\,\mathbf{k},\mathbf{p}\rangle_1 \; , \tag{3.12}$$

$$E(\mathbf{k},\mathbf{p})=[\omega^2(k)+\mathbf{p}^2]^{1/2}+(m^2+p^2)^{1/2} \; , \tag{3.13}$$

where

$$|\,\mathbf{k},\mathbf{p}\rangle_1\equiv|\,\mathbf{k}_{23},\mathbf{p}_1\rangle\equiv|\,\mathbf{k}_{23}\rangle\times|\,\mathbf{p}_1\rangle \; ;$$

$\mathbf{p}=\mathbf{p}_1$ is the momentum of the particle 1, and $\mathbf{k}=\mathbf{k}_{23}$ is the relative momentum of the (23) pair [as defined by Eq. (3.6)].

An explicit representation of Eq. (3.11) is

$$\phi_1(\mathbf{k},\mathbf{p})=\int\int\int\int\left\langle\mathbf{k},\mathbf{p}\left|\frac{1}{E_B-H_0-V_{23}}V_{23}\right|\mathbf{k}',\mathbf{p}'\right\rangle_1({}_1\langle\mathbf{p}',\mathbf{k}'\,|\,\mathbf{k}'',\mathbf{p}''\rangle_2+{}_1\langle\mathbf{p}',\mathbf{k}'\,|\,\mathbf{k}'',\mathbf{p}''\rangle_3)\phi_1(\mathbf{k}'',\mathbf{p}'')d^3k'd^3p'd^3k''d^3p'' \; . \tag{3.14}$$

The two terms in Eq. (3.14) are identical because of the symmetry under interchange of the particles 2 and 3. The transformation matrices ${}_1\langle p,k\,|\,k',p'\rangle_2$ can easily be obtained from

$${}_1\langle\mathbf{k},\mathbf{p}\,|\,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\rangle=\delta(\mathbf{p}-\mathbf{p}_i)\delta[\mathbf{k}-\mathbf{k}(\mathbf{p}_j,\mathbf{p}_k)]/N(\mathbf{p}_j,\mathbf{p}_k), \quad \mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=0 \tag{3.15}$$

where $k(p_j,p_k)$ is defined by Eq. (3.6) and N is the square root of the Jacobian,

$$N(p_j,p_k)=\left[\frac{\partial(p_j,p_k)}{\partial(k,p_{jk})}\right]^{1/2}=\left[\frac{4E_jE_k}{M^0_{jk}(E_j+E_k)}\right]^{1/2} , \tag{3.16}$$

with

$$M^0_{jk}=[(E_j+E_k)^2-\mathbf{p}^2_{jk}]^{1/2} \; . \tag{3.17}$$

By using Eq. (3.15), we have

$$\begin{aligned}{}_1\langle\mathbf{p},\mathbf{k}\,|\,\mathbf{k}',\mathbf{p}'\rangle_2&=\int d^3p_1\int d^3p_2\int d^3p_3\,{}_1\langle\mathbf{p},\mathbf{k}\,|\,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\rangle\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\langle\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\,|\,\mathbf{k}',\mathbf{p}'\rangle_2\\&=\delta[\mathbf{k}-\mathbf{k}(\mathbf{p}',-\mathbf{p}-\mathbf{p}')]\delta[\mathbf{k}'-\mathbf{k}(-\mathbf{p}-\mathbf{p}',\mathbf{p})]/[N(\mathbf{p}',-\mathbf{p}-\mathbf{p}')N(-\mathbf{p}-\mathbf{p}',\mathbf{p})] \; .\end{aligned} \tag{3.18}$$

A characteristic difficulty of the relativistic dynamics stems from the complicated nonlinear relation between V_{23} and v_{23}, shown in Eq. (3.4). We overcome this technical difficulty by expressing the operator $(E_B-H_0-V_{23}+i\epsilon)^{-1}V_{23}$ in terms of the half-off-shell matrix t_{23}, which can be easily computed using Eqs. (2.1)–(2.4).

The scattering states $|\mathbf{k}_{23},\mathbf{p}_1\rangle^{(+)}$ and the bound states $|\phi_D,\mathbf{p}_1\rangle$ of the 23 subsystem in the presence of the spectator 1 are eigenfunctions of h_{23} and p_1. They are also eigenfunctions of H_0+V_{23} since both H_0 and V_{23} commute with $\mathbf{p}_1=-\mathbf{p}_{23}$. It follows that these states must satisfy the identities

$$|\mathbf{k},\mathbf{p}\rangle^{(+)}=|\mathbf{k},\mathbf{p}\rangle+[\omega(\mathbf{k})-h_0-v+i\epsilon]^{-1}v\,|\mathbf{k},\mathbf{p}\rangle\ , \tag{3.19a}$$

$$=|\mathbf{k},\mathbf{p}\rangle+[\omega(\mathbf{k})-h_0+i\epsilon]^{-1}t\,|\mathbf{k},\mathbf{p}\rangle\ , \tag{3.19b}$$

$$=|\mathbf{k},\mathbf{p}\rangle+[E(\mathbf{k},\mathbf{p})-H_0-V+i\epsilon]^{-1}V\,|\mathbf{k},\mathbf{p}\rangle\ , \tag{3.19c}$$

$$=|\mathbf{k},\mathbf{p}\rangle+[E(\mathbf{k},\mathbf{p})-H_0+i\epsilon]^{-1}V\,|\mathbf{k},\mathbf{p}\rangle^{(+)}\ , \tag{3.19d}$$

and

$$|\phi_D,\mathbf{p}\rangle=(M_D-h_0)^{-1}v\,|\phi_D,\mathbf{p}\rangle =[E_D(\mathbf{p})-H_0]^{-1}V\,|\phi_D,\mathbf{p}\rangle\ , \tag{3.20}$$

where

$$E_D(\mathbf{p})=(M_D^2+\mathbf{p}^2)^{1/2}+(m^2+\mathbf{p}^2)^{1/2}\ .$$

We note that the construction of the complete set of states $|\mathbf{k},\mathbf{p}\rangle^{(+)}$, $|\phi_D,\mathbf{p}\rangle$ does not require an explicit matrix representation of the operator V. They can be computed from the matrix elements of v or t as seen in Eqs. (3.19) and (3.20). The same is true for the states obtained by operating with V or H_0+V on any state in this complete set. In particular we readily derive from Eqs. (3.19) and (3.20) the following identities needed for our computation:

$$V\,|\mathbf{k},\mathbf{p}\rangle^{(+)}=\frac{E(\mathbf{k},\mathbf{p})-H_0+i\epsilon}{\omega(k)-h_0+i\epsilon}t\,|\mathbf{k},\mathbf{p}\rangle =\frac{\omega(\mathbf{k})+h_0}{E(k,p)+H_0-2(m^2+p^2)^{1/2}}t\,|\mathbf{k},\mathbf{p}\rangle\ , \tag{3.21}$$

$$V\,|\phi_D,\mathbf{p}\rangle=\frac{E_D(\mathbf{p})-H_0}{M_D-h_0}v\,|\phi_D,\mathbf{p}\rangle\ , \tag{3.22}$$

$$(E_B-H_0-V+i\epsilon)^{-1}\,|\mathbf{k},\mathbf{p}\rangle^{(+)} =|\mathbf{k},\mathbf{p}\rangle^{(+)}[E_B-E(k,p)+i\epsilon]^{-1}\ , \tag{3.23}$$

and

$$(E_B-H_0-V)^{-1}\,|\phi_D,p\rangle=|\phi_D,p\rangle[E_B-E_D(p)]^{-1}\ . \tag{3.24}$$

We use the completeness relation to write the required matrix elements

$$\langle\mathbf{p},\mathbf{k}\,|\,(E_B-H_0-V)^{-1}V\,|\,\mathbf{k}',\mathbf{p}\rangle$$

in the form

$$\left\langle\mathbf{p},\mathbf{k}\left|\frac{1}{E_B-H_0-V}V\right|\mathbf{k}',\mathbf{p}'\right\rangle=\int d^3k_0\int d^3p_0\langle\mathbf{p},\mathbf{k}\,|\,\mathbf{k}_0,\mathbf{p}_0\rangle^{(+)}\frac{1}{E_B-E(\mathbf{k}_0,\mathbf{p}_0)+i\epsilon}{}^{(+)}\langle\mathbf{p}_0,\mathbf{k}_0\,|\,V\,|\,\mathbf{k}'\mathbf{p}'\rangle + \int\langle\mathbf{p},\mathbf{k}\,|\,\phi_D,\mathbf{p}_0\rangle\frac{d\mathbf{p}_0}{E_B-E_D(\mathbf{p}_0)}\langle\mathbf{p}_0,\phi_D\,|\,V\,|\,\mathbf{k}',\mathbf{p}'\rangle\ . \tag{3.25}$$

It follows from Eqs. (3.20)–(3.25) that

$$\left\langle\mathbf{p},\mathbf{k}\left|\frac{1}{E_B-H_0-V}V\right|\mathbf{k}',\mathbf{p}'\right\rangle=\delta(\mathbf{p}-\mathbf{p}')\frac{1}{E_B-E(\mathbf{k},\mathbf{p})}T(\mathbf{k},\mathbf{k}';\mathbf{p})\ , \tag{3.26}$$

where $[\omega(p)=2(m^2+\mathbf{p}^2)^{1/2}]$

$$T(\mathbf{k},\mathbf{k}';\mathbf{p})=\Bigg\{\frac{\omega(k)+\omega(k')}{E(k,p)+E(k',p)-\omega(p)}t^*(\mathbf{k}',\mathbf{k}) + \int d^3k_0\frac{\omega(k)+\omega(k_0)}{E(k_0,p)+E(k,p)-\omega(p)}t(\mathbf{k},\mathbf{k}_0)\frac{\omega(k_0)+\omega(k')}{E(k_0,p)+E(k',p)-\omega(p)}t^*(k',k_0) \times\left[\frac{1}{E(k_0,p)-E(k,p)}+\frac{1}{E_B-E(k_0,p)}\right] +\phi_D(\mathbf{k})\phi_D^*(\mathbf{k}')[E_B-E(\mathbf{k},\mathbf{p})][E_D(\mathbf{p})-E(\mathbf{k}',\mathbf{p})]/[E_B-E_D(\mathbf{p})]\Bigg\}\ . \tag{3.27}$$

We are now ready to assemble the Faddeev equations in their final form. Using the symmetry under interchange of particles 2 and 3, Eqs. (3.18) and (3.26), we can write Eq. (3.14) in the form

$$\phi(\mathbf{k},\mathbf{p})=\frac{2}{E_B-E(\mathbf{k},\mathbf{p})}\int d^3p'\frac{T[\mathbf{k},\mathbf{k}(\mathbf{p}',-\mathbf{p}-\mathbf{p}');\mathbf{p}]}{N(\mathbf{p}',-\mathbf{p}-\mathbf{p}')N(-\mathbf{p}-\mathbf{p}',\mathbf{p})}\times\phi[\mathbf{k}(-\mathbf{p}-\mathbf{p}',\mathbf{p}),\mathbf{p}'] , \quad (3.28)$$

where we have suppressed the subscript 1 of the amplitudes. The results which will be derived from Eq. (3.28) must be compared with results obtained from the well-known nonrelativistic equation

$$\phi_{\mathrm{NR}}(\mathbf{k},\mathbf{p})=\frac{2}{E_B-E_{\mathrm{NR}}(\mathbf{k},\mathbf{p})}\int d^3p' T_{\mathrm{NR}}(\mathbf{k},\mathbf{p}'+\tfrac{1}{2}\mathbf{p};\mathbf{p})\times\phi_{\mathrm{NR}}(\tfrac{1}{2}\mathbf{p}'-\mathbf{p},\mathbf{p}') , \quad (3.29)$$

where

$$E_{\mathrm{NR}}(\mathbf{k},\mathbf{p})=k^2/m+3p^2/4m ,$$

and T_{NR} is defined by

$$T_{\mathrm{NR}}(\mathbf{k},\mathbf{k}';\mathbf{p})=V_{\mathrm{NR}}(\mathbf{k},\mathbf{k}')+\int d^3k''\frac{V_{\mathrm{NR}}(\mathbf{k},\mathbf{k}'')T_{\mathrm{NR}}(\mathbf{k}'',\mathbf{k}';\mathbf{p})}{E_B-E_{\mathrm{NR}}(\mathbf{k}'',\mathbf{p})} . \quad (3.30)$$

Since the propagators of nonrelativistic and relativistic equations are quite different, one can expect that they will generate a different momentum distribution at least at high momenta. It is, therefore, of interest to also investigate this quantity in this calculation. In the three-body system the probability of finding a particle with momentum $\mathbf{p}$ is

$$n(\mathbf{p})=\frac{\left\langle\Phi\left|\sum_{i=1}^{3}\delta(\mathbf{p}-\mathbf{p}_i)\right|\Phi\right\rangle}{3\langle\Phi|\Phi\rangle} . \quad (3.31)$$

By using the Faddeev decomposition Eq. (3.9) we get

$$n(\mathbf{p})=\frac{\left\langle\phi_1\left|\sum_{i=1}^{3}\delta(\mathbf{p}-\mathbf{p}_i)\right|\phi_1+\phi_2+\phi_3\right\rangle}{\langle\Phi|\Phi\rangle} . \quad (3.32)$$

TABLE III. Three-body energies. (See the text for notations.)

	E_B	E_{kin}	E_{pot}
Nonrelativistic	-7.5	28.9	-36.4
Relativistic	-7.3	27.9	-35.2
$\langle E_R\rangle_{\mathrm{NR}}$	-7.1	28.0	-35.1
ΔE	0.2	-1.0	1.2
$\Delta E'$	0.7	-1.3	2.0
$\Delta E'$ (RSC)[a]	-0.4	-2.4	2.0

[a]Calculation of Ref. 15 with the correct scale factor $f=1.049$ for the Reid-soft-core potential.

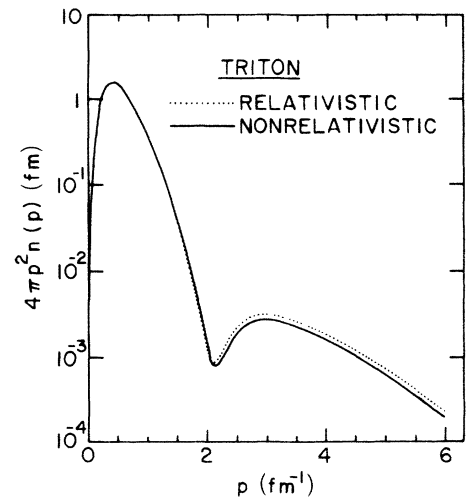

FIG. 3. Weighted triton momentum distribution $4\pi p^2 n(p)$ from nonrelativistic (solid curve) and relativistic (dashed curve) calculations.

Since we have only s-wave interactions of identical particles Eq. (3.32) reduces to

$$n(\mathbf{p})=\tfrac{1}{3}[\langle\phi_1|\delta(\mathbf{p}-\mathbf{p}_1)|\phi_1\rangle+2\langle\phi_1|\delta(\mathbf{p}-\mathbf{p}_2)|\phi_1\rangle+2\langle\phi_1|\delta(\mathbf{p}-\mathbf{p}_2)|\phi_3\rangle+4\langle\phi_1|\delta(\mathbf{p}-\mathbf{p}_1)|\phi_2\rangle]/[\langle\phi_1|\phi_1\rangle+2\langle\phi_1|\phi_2\rangle] . \quad (3.33)$$

By appropriate changes of variables, each term of the above equation can be expressed in terms of an integration over the solution $\phi(k,p)$ of Eq. (4.4). The numerical procedure[17] to solve Eqs. (3.28) and (3.30) is discussed in the Appendix. We use a spline representation[18] of the amplitude $\phi(k,p)$ to carry out the integrations required by Eq. (3.33). The relativistic and nonrelativistic momentum distributions obtained from the three-body wave function are shown in Fig. 3. Figure 2 shows the difference between the relativistic and nonrelativistic momentum distributions for the triton as well as the deuteron.

In Table III, we summarize the results for the relativistic and nonrelativistic energies. The quantity $\langle E_R\rangle_{\mathrm{NR}}$ is the expectation value of the relativistic Hamiltonian with the nonrelativistic wave function. This procedure produces a fair approximation to the relativistic effects in the binding energy. In Table III, we also compare the relativistic correction ΔE with the correction $\Delta E'$ calculated in the approximations of Ref. 15,

$$\Delta E'_{\mathrm{kin}}=-\left\langle\frac{3p^4}{8m^3}\right\rangle_{\mathrm{NR}} , \quad (3.34)$$

$$\Delta E'_{\mathrm{pot}}=(f-1)\langle V_{\mathrm{NR}}\rangle_{\mathrm{NR}}+\left\langle\tfrac{1}{2}\sum_{ij}[(p_i^2+p_j^2),(V_{ij})_{\mathrm{NR}}]\right\rangle_{\mathrm{NR}}/8m^2 . \quad (3.35)$$

The above approximation substantially overestimates the effects of both the kinetic and the potential energy. It appears that expansion in powers of the momenta are not sufficiently accurate to predict the size of the relativistic correction to the binding energy. The relatively large increase (1.7 MeV) in binding found in Ref. 15 is mainly due to a crude estimate in f. In the last line of Table III ($\Delta E'_{\rm RSC}$), we show the result of Ref. 15 calculated with the correct value of the scale factor f. The corresponding estimate of relativistic effect in the binding energy is reduced to 0.4 MeV. As noted in Sec. II our relativistic and nonrelativistic two-body potentials are not exactly phase-shift equivalent. The question arises to what extent the decrease in binding, which we find, is associated with the slightly more repulsive phase shifts of the relativistic two-body potential. We have, therefore, considered a relativistic potential in which the strength of the attractive part is the same as in the nonrelativistic potential, while the repulsive part is reduced by a factor 0.985. The factor is designed to give the same two-body binding energy as the nonrelativistic potential, and phase shifts that are slightly more attractive ($<0.2\%$) than the nonrelativistic ones. The decrease of the three-body binding energy is then reduced to about 2%.

IV. SUMMARY AND CONCLUSIONS

The requirements of Lorentz invariance necessarily affect the conclusions which can be drawn from conventional nuclear dynamics, which describes few-body and many-body systems in a common framework. Canonical relativistic dynamics[7] is particularly suited for the precise formulation and quantitative investigation of this problem. It should be emphasized that relativistic effects depend in both sign and magnitude on the choice of the dynamics. Different choices do not amount to different estimates of the same effect. In the context of canonical relativistic dynamics the instant form and the front form[19] are equivalent only if appropriate three-body forces are added. Relativistic effects that can be deduced from a covariant quasipotential dynamics[20,21] are a feature of that dynamics and cannot be expected to be comparable to the effects discussed here.

We have solved both relativistic and nonrelativistic Faddeev equations for a simple s-wave potential. From the Faddeev wave functions we have obtained momentum distributions and expectation values of both kinetic and potential energies. We have also evaluated the expectation values of various approximate relativistic corrections. We find that the relativistic effect on the binding energy is a small decrease of about 3%. In general we can expect that the magnitude of both the kinetic and the potential energy will be smaller in the relativistic case. The net effect on the binding energy depends sensitively on the amount of cancellation of the two effects. Our model calculation can therefore not be used to predict the sign of the effect for realistic potentials. Compared to the nonrelativistic momentum distribution the relativistic momentum distribution is enhanced at very low momenta, depleted at moderate momenta, and increased at high momenta. The expectation values of the relativistic kinetic and potential energy operators with the nonrelativistic wave functions are a fair approximation to the exact values. On the other hand approximate kinetic and potential energies obtained with expansion in powers of the momenta overestimate the relativistic effects by substantial factors. These errors preclude a reliable estimate of the relativistic effects in the binding energy by expansion in powers of the momenta.

We have two main conclusions: (1) The quantitative effects of the Lorentz invariance of the dynamics are small. (2) Easy approximations of relativistic effects can be quite misleading.

APPENDIX

We solve Eq. (3.28) in the partial-wave representation defined by

$$|\mathbf{k},\mathbf{p}\rangle = \sum_{\substack{l\lambda L\\ m_l m_\lambda M}} \langle LM\,|\,l\lambda m_l m_\lambda\rangle\,|\,kp(l\lambda)LM\rangle \times Y_{lm_l}(\hat{\mathbf{k}})Y_{\lambda m_\lambda}(\hat{\mathbf{p}})\ . \tag{A1}$$

Partial-wave decomposition of Eq. (3.28) is more complicated than that of Eq. (3.29), because of the angle dependence of $N(\mathbf{p}_2,\mathbf{p}_3)$ and $\mathbf{k}(\mathbf{p}_2,\mathbf{p}_3)$. Our restriction to s-wave two-body interactions without spin and isospin dependence is a source of great simplification. In that case the matrix $t(\mathbf{k},\mathbf{k}')$, and therefore the amplitude $\phi_1(\mathbf{k},\mathbf{p})$, depend only on the magnitudes k,k',p of the vector arguments. Since N is a scalar function of its vector arguments it is a function of p, p', and z which is the cosine of the angle between the two vectors

$$N(\mathbf{p},-\mathbf{p}-\mathbf{p}')\rightarrow N(p,p',z)\ . \tag{A2}$$

The same is true for the magnitude of the vector $\mathbf{k}(\mathbf{p},-\mathbf{p}-\mathbf{p}')$,

$$|\,\mathbf{k}(\mathbf{p},-\mathbf{p}-\mathbf{p}')\,|\rightarrow k(p,p',z)\ . \tag{A3}$$

Then the Faddeev Eq. (3.28) becomes

$$\phi(k,p)=\frac{4\pi}{E_B-E(k,p)} \times \int_0^\infty dp'p'^2\int_{-i}^{+1}dz\,T[k,k(p',p,z);p] \times\frac{1}{N(p,p',z)N(p',p,z)} \times\phi[k(p,p',z),p']\ . \tag{A4}$$

For the computations we must reduce the integral equation to a finite set of linear equations. To this end we introduce grids of points k_n,p_m for the variables k and p. Integrals over p are approximated by quadratures. For any function $f(p)$, we have

$$\int dp\,p^2f(p)\rightarrow\sum_m W_m f(p_m)\ . \tag{A5}$$

We use a spline interpolation when functions of k are needed for values of k which are not on the grid,

$$\phi(k) \simeq \sum_n \phi(k_n) S_n(k) , \tag{A6}$$

where $S_n(k)$ is the spline function.

The approximate Faddeev equation obtained in this manner has the form

$$\phi_{nm} = \sum_{n',m'} \frac{4\pi}{E_B - E_{nm}} K_{nm,n'm'} \phi_{n'm'} , \tag{A7}$$

where

$$\phi_{nm} \equiv \phi(k_n, p_m) W_m^{1/2}, \quad E_{nm} \equiv E(k_n, p_m)$$

and

$$K_{nm,n'm'} \equiv (W_m W_{m'})^{1/2} \times \sum_{n''} \int_{-1}^{+1} dz\, T(k_n, k_{n''}; p_m) C_{n''n'}(m,m') \tag{A8}$$

and

$$C_{n''n'}(m,m') = \int_{-1}^{+1} dz \frac{S_{n''}[k(p_{m'}, p_m, z)] S_{n'}[k(p_m, p_{m'}, z)]}{N(p_{m'}, p_m, z) N(p_m, p_{m'}, z)} . \tag{A9}$$

We employ the efficient Malfliet-Tjon iteration method[12] to solve Eq. (A7).

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