Summed strength and signature properties of magnetic multipole excitations in a single j shell

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Taking the titanium isotopes as an example, the magnetic multipole excitation strengths are calculated in the single j shell model ($f_{7/2}$). Expressions are derived for the summed strength in this model for the modes with nonvanishing strength—$M1$, $M3$, $M5$, and $M7$. An approximate expression for this sum is obtained in which only the components of the wave function in which neutrons and protons couple to angular momentum $L=0$ and 2 ($v=2$) in the ground state are included. The signature selection rules for the nucleus ^{48}Ti are considered. It is noted that the $M1$ strength is mostly concentrated in the lowest state (at least 80%). The higher multipole modes are more strongly fragmented. For these modes, there is significant strength in the isobaric analog states, especially for $M5$. A closed expression for strength to the isobaric analog strength in ^{48}Ti is obtained.

I. INTRODUCTION

In a previous work,[1] it was noted that an analogy could be made between the behavior of $M1$ excitations in the $f_{7/2}$ region and those recently observed in the deformed nuclei including, and in the neighborhood of, ^{156}Gd. The energies and $M1$ strengths in the two regions were systematically similar, and the theoretical expressions in both cases had the common thread of involving isovector transitions which did not change the overall isospin (except for ^{44}Ti where all 1^+ states have $T=1$). The values of $B(M1)$ in the $f_{7/2}$ region were of the order of one single particle unit and the excitation energy of the lowest 1^+ state was about 4 MeV. These states have been called scissor modes in the deformed region with a picture in which the symmetry axis of the neutrons and that of the protons oscillate with respect to each other. In the context of boson models these states are of interest because they do not occur in IBA-1, only in IBA-2.[2–4]

In this work we extend the work to higher multipoles and we also consider the summed strength. The latter is important to ascertain whether or not the strength for a given multipole is concentrated in the lowest state. The expressions for the summed strength will only involve the ground state wave function and therefore will be less complicated than the corresponding expressions for transitions to individual levels.

Much encouragement for this further work comes from the fact that an Orsay, Darmstadt group, and Michigan State collaboration[5] has found the lowest 1^+ state in ^{46}Ti and 4.3 MeV with a strength $B(M1)=1.0\ \mu_N^2$ (the predicted value was 1.76 μ_N^2 at 4.00 MeV). Not only that but the Darmstadt group found that the 1^+ spin flip state was at 10.2 MeV, quite far away from the "$f_{7/2}$ state." This gives us hope that in zero order at least the low lying $f_{7/2}$ states and the spin-flip states will not be hopelessly intertwined and that the $f_{7/2}$ model will therefore be of considerable use in correlating the data. Of course a residual interaction will mix the basic spin flip and pure $f_{7/2}$ states and there are other states like $f_{7/2}^{-1}\,p_{3/2}$ that enter for higher multipoles. Ultimately larger basis shell model calculations will be done to get a more complete picture. Nevertheless the single j model here will be shown to be quite relevant.

We should remark that although in the original McCullen, Bayman, and Zamick (MBZ) paper[6] it was not possible to list all the energy levels and wave functions [and indeed no states of unnatural parity ($1^+,3^+,5^+,7^+$, etc.) were included], we were wise enough to write a technical report[7] in which all the wave functions were listed. With modern experimental techniques, and especially medium energy probes—pions, protons, and electrons, one can now reach these unnatural parity states. This technical report should therefore be a gold mine of useful information.

II. THEORY

A. General expressions

The wave functions of the titanium isotopes[6–8] in the $f_{7/2}$ model are

$$\psi^J=\sum_{L_pL_n} D^J(L_p,L_n v)[(f_{7/2}^2)^{L_p}(f_{7/2}^n)^{L_n v}]^J ,$$

where $D^J(L_p,L_n v)$ is the probability amplitude that in a state of total angular momentum J the protons couple to angular momentum L_p and the neutrons to angular momentum L_n and seniority v.

We consider a transition from a $J=0$ ground state to a state of angular momentum and parity λ^+ with $\lambda=1$, 3, 5, and 7. In the $f_{7/2}$ model states with angular momentum $J>\lambda$ do occur but they carry no (M^J) strength.

The magnetic operators for protons and neutrons are written as $M_m^\lambda(\pi)$ and $M_m^\lambda(\nu)$ where m is the magnetic quantum number. These are tensor operators of odd rank. The operators are defined such that $B(M\lambda)=\sum_{m=-\lambda}^{\lambda}|X_m^\lambda|^2$, where X_m^λ is the matrix element

$$X_m^\lambda = \langle \psi_m^\lambda \mid M_m^\lambda(\pi) + M_m^\lambda(\nu) \mid 0 \rangle \ .$$

In fact, because the initial state has $J=0$, X_m^λ is independent of m and we can write

$$B(M\lambda) = (2\lambda+1) \mid X_0^\lambda \mid^2 \ .$$

We will use the same convention as originally used by MBZ.[6] The Wigner-Eckart theorem is written as

$$(\psi_{m_B}^{J_B} 0_\mu^\lambda \psi_{m_A}^{J_A}) = (\lambda J_A \mu m_A \mid J_B m_B)(\psi^{J_B}[0^\lambda \psi^{J_A}]^{J_B}) \ ,$$

where the first factor on the right is a Wigner (or Clebsch-Gordan) coefficient, and the second factor is the reduced matrix element in this convention. It is related to the more common "double bar" reduced matrix element of Edmonds[9] (also de Shalit and Talmi,[10] and Bohr and Mottelson[11]) by

$$(\psi^{J_B}[0^\lambda \psi^{J_A}]^{J_B}) = 1/(2J_B+1)^{1/2}(-1)^{\lambda+J_A-J_B} \times (\psi^{J_B} \| 0^\lambda \| \psi^{J_A}) \ .$$

We also use the unitary Racah coefficient U which is related to the six-j symbol as follows:

$$U(abcd;ef) = (-1)^{a+b+c+d}[(2e+1)(2f+1)]^{1/2} \times \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} \ .$$

We now evaluate the matrix element X_0^λ. For convenience we define the single particle matrix elements $\mathcal{M}_\mathrm{p}^\lambda$ and $\mathcal{M}_\mathrm{n}^\lambda$ with

$$\mathcal{M}_\mathrm{p}^\lambda = \langle \psi^j [M^\lambda(\pi)\psi^j]^j \rangle \ ,$$

etc.

Using the Racah algebra we find

$$X_0^\lambda = \sum_{LL_\mathrm{p}L_\mathrm{n}} D^0(LL) D^\lambda(L_\mathrm{p}L_\mathrm{n}) \{ U(\lambda L \lambda L; L_\mathrm{p} 0) \langle (j^2)^{L_\mathrm{p}} [M^\lambda(\pi)(j^2)^L]^{L_\mathrm{p}} \rangle \delta_{L_\mathrm{n},L} + (-1)^{L_\mathrm{p}+L_\mathrm{n}+\lambda} U(\lambda L \lambda L; L_\mathrm{n} 0) \langle (j^n)^{L_\mathrm{n}} [M^\lambda(\nu)(j^n)^L]^{L_\mathrm{n}} \rangle \delta_{L_\mathrm{p},L} \} \ .$$

We can simplify the above by noting the following relation

$$U(\lambda L \lambda L; L_\mathrm{p} 0) = (-1)^{\lambda+L-L_\mathrm{p}} \left[\frac{(2L_\mathrm{p}+1)}{(2\lambda+1)(2L+1)} \right]^{1/2} \ .$$

We further note, following de Shalit and Talmi,[10] that the matrix element of an odd tensor interaction with nucleons of one kind have the following seniority properties

$$\langle (j^n)^{L_\mathrm{n} v_\mathrm{n}} [M^\lambda (j^n)^{Lv}]^{L_\mathrm{n}} \rangle = \delta_{v_\mathrm{n},v} \langle [(j)^v]^{L_\mathrm{n} v} [M^\lambda (j^v)^{Lv}]^{L_\mathrm{n}} \rangle \ .$$

In other words, for particles of one kind the odd tensor interaction is diagonal in seniority. Furthermore, for states of seniority $v<n$ the n particle matrix element is equal to a v particle matrix element. In the $f_{7/2}$ calculation this has the practical consequence that the $n=4$, $v=2$ matrix elements are equal to the $n=2$, $v=2$ matrix elements.

The matrix element X_0^λ can be written as follows

$$X_0^\lambda = X_0^\lambda(v=2) + X_0^\lambda(v=4) \ ,$$

$$X_0^\lambda(v=2) = 2/(2\lambda+1)^{1/2} \sum_{L_0 L} \left[\frac{(2L_0+1)}{(2L+1)} \right]^{1/2} U(\lambda j L_0 j; jL) \times D^0(LL)[D^\lambda(L_0, Lv)(-1)^{\lambda+L-L_0} \mathcal{M}_\mathrm{p}^\lambda + (-1)^{L+L_0} D^\lambda(L, L_0 v_0) \mathcal{M}_\mathrm{n}^\lambda] \delta_{v,2} \delta_{v_0,2} \ .$$

The last ($v=4$) matrix element applies only to the nucleus ^{46}Ti (or the cross conjugate ^{50}Cr). The expression is

$$X_0^\lambda(v=4) = \delta_{4,n} n/(2\lambda+1)^{1/2} \left\{ \sum_{LL_0} D^0(L, Lv) D^\lambda(L, L_0 v_0) \delta_{v,4} \delta_{v_0,4} [(2L_0+1)/(2L+1)]^{1/2} \times \sum_{J_0} (j^{(n-1)} J_0 j \mid \} j^n L v)(j^{(n-1)} J_0 j \mid \} j^n L_0 v_0) U(\lambda j L_0 J_0; jL) \mathcal{M}_\mathrm{n}^\lambda \right\} \ .$$

In the last expression above we have introduced the coefficients of fractional parentage.

B. Summed strength

For nuclei in which there are no components with seniority higher than 2, such as ^{44}Ti and ^{48}Ti, we can obtain expressions for the summed strength in each multipole, $\sum B(M\lambda)$. The expression is also approximately applicable to ^{46}Ti if we neglect the seniority 4 component of the neutron wave function in the *ground* state (of course the unnatural parity *excited* states can and will have large seniority 4 components).

We make use of the relation

$$\sum_\alpha D^{\lambda\alpha}(L,L_0v_0)D^{\lambda\alpha}(L',L_0'v_0')=\delta_{LL'}\delta_{L_0L_0'}\delta_{v_0v_0'} .$$

We obtain

$$\sum_\alpha B^\alpha(M\lambda)=4\sum_{\substack{L_0,L\\ \text{even}}} |D^0(LL)|^2\frac{(2L_0+1)}{(2L+1)}|U(\lambda jL_0j;jL)|^2(|\mathcal{M}_{\rm p}^\lambda|^2+|\mathcal{M}_{\rm n}^\lambda|^2)$$

$$-8\sum_{L_0,L} D^0(LL)D^0(L_0L_0)U(\lambda jL_0j;jL)U(\lambda jLj;jL_0)\mathcal{M}_{\rm p}^\lambda\mathcal{M}_{\rm n}^\lambda .$$

We can simplify things further by noting

$$\sum_{\substack{L_0\\ \text{even}}}\frac{(2L_0+1)}{(2L+1)}|U(\lambda jL_0j;jL)|^2=\tfrac{1}{2}[1+(-1)^\lambda U(\lambda jjL;jj)] .$$

Hence

$$\sum_\alpha B^\alpha(M\lambda)=4\sum_L |D^0(LL)|^2[1+(-1)^\lambda U(\lambda jjL;jj)]/2(|\mathcal{M}_{\rm p}^\lambda|^2+|\mathcal{M}_{\rm n}^\lambda|^2)$$

$$-8\sum_{LL_0} D^0(LL)D^0(L_0L_0)U(\lambda jL_0j;jL)U(\lambda jLj;jL_0)\mathcal{M}_{\rm p}^\lambda\mathcal{M}_{\rm n}^\lambda .$$

This is our main result in this section.

C. The magnetic dipole case

For the magnetic dipole case only terms with $L=L_0$ contribute. The Racah coefficients have the value

$$|U(1jLj;jL)|^2=\frac{L(L+1)}{4(j)(j+1)} .$$

Hence,

$$\sum_\alpha B^2(M1)=\frac{|\mathcal{M}_{\rm p}^\lambda-\mathcal{M}_{\rm n}^\lambda|^2}{j(j+1)}\left[\sum_L |D^0(LL)|^2L(L+1)\right] .$$

Note that only the isovector combination enters. We could have obtained this result quicker by using the results of the first paper in this series[1] where it was shown

$$B(M1)=\tfrac{3}{4}\pi\mu_{\rm N}^2(g_{\rm p}-g_{\rm n})^2\left|\sum_L D^0(LL)D^1(LL)\sqrt{L(L+1)}\right|^2 .$$

Using

$$\sum_\alpha D^{1\alpha}(LL)D^{1\alpha}(L'L')=\delta_{LL'}$$

we obtain

$$\sum_\alpha B^\alpha(M1)=\tfrac{3}{4}\pi\mu_{\rm N}^2(g_{\rm p}-g_{\rm n})^2\left[\sum_L |D^0(LL)|^2L(L+1)\right] .$$

One finds that the lowest 1^+ state gets most of the strength. The percentages are 93.7 for ^{44}Ti, 86.0 for ^{46}Ti, and 80.5 for ^{48}Ti.

D. The single particle matrix elements

The general expression for the M_λ matrix element in the $q\rightarrow 0$ limit is[11]

$$\mathcal{M}^\lambda = \langle j_2[M^\lambda j_1]j_2\rangle$$
$$= -\left[\frac{(2j_1+1)(2\lambda+1)}{(2j_2+1)4\pi}\right]^{1/2} \mu_N \langle j_2 r^{\lambda-1} j_1\rangle$$
$$\times\left\{\left[g_s - \frac{2}{\lambda+1}g_l\right]\frac{\lambda}{2}(j_1\lambda\tfrac{1}{2}0\,|\,j_2\tfrac{1}{2})[1+1/\lambda(-1)^{l_1+(1/2)-j_1}(j_1+\tfrac{1}{2})+(-1)^{j_1+j_2-\lambda}(j_2+\tfrac{1}{2})]\right\}$$
$$+(-1)^{j_1+j_2+\lambda}2g_l/(\lambda+1)[\lambda(2\lambda-1)(2\lambda+1)j_1(j_1+1)(2j_1+1)]^{1/2}$$
$$\times[j_1(\lambda-1)\tfrac{1}{2}0\,|\,j_2\tfrac{1}{2}]\begin{Bmatrix} j_1 & 1 & j_1 \\ \lambda-1 & j_2 & \lambda \end{Bmatrix}.$$

The free particle values of g_l and g_s are

	g_l	g_s
Proton	1	5.5855
Neutron	0	−3.8256

The values of the single particle reduced matrix elements for $f_{7/2}$ shell orbits are

	$\mathcal{M}^\lambda$	
$\lambda=1$	$-0.276\,67g_s-1.660\,03g_l$	
$\lambda=3$	$(0.661\,57g_s+1.694\,85g_l)$	$\langle r^2\rangle$
$\lambda=5$	$(-1.664\,63g_s-1.109\,76g_l)$	$\langle r^4\rangle$
$\lambda=7$	$3.575\,17g_s$	$\langle r^6\rangle$

Note that $\lambda=7$ is a pure spin mode. The increasing importance of the spin part relative to the orbital part with increasing λ has been emphasized by Heyde and Sau.[12] We evaluate $r^{\lambda-1}$ with harmonic oscillators using $b=2$ fm. The expressions are

$$\langle r^2\rangle = \tfrac{9}{2}b^2,\quad \langle r^4\rangle = \tfrac{11}{2}\times\tfrac{9}{2}b^4,$$
$$\langle r^6\rangle = \tfrac{13}{2}\times\tfrac{11}{2}\times\tfrac{9}{2}b^6.$$

We obtain

	$\mathcal{M}^\lambda$		
$\lambda=1$	$-0.275\,67g_s$	$-1.660\,03g_l$	μ_N
$\lambda=3$	$11.908\,31g_s$	$+30.507\,30g_l$	$\mu_N\text{fm}^2$
$\lambda=5$	$-659.1938g_s$	$-439.4635g_l$	$\mu_N\text{fm}^4$
$\lambda=7$	$36\,809.91g_s$		$\mu_N\text{fm}^6$

With the unquenched values of g_l and g_s we obtain the following values (in appropriate units)

$\mathcal{M}_p^\lambda$	$\mathcal{M}_n^\lambda$	$\mathcal{M}_p^\lambda-\mathcal{M}_n^\lambda$	$\mathcal{M}_p^\lambda+\mathcal{M}_n^\lambda$	$\left[\dfrac{\mathcal{M}_p+\mathcal{M}_n}{\mathcal{M}_p-\mathcal{M}_n}\right]^2$
−3.2054	1.0584	−4.2638	−2.1469	0.253
97.0212	−45.5564	142.5776	51.4647	0.130
−4121.39	2521.812	−6643.20	1599.58	0.058
206 160.3	−140 820	346 980.3	65 340.3	0.035

E. Strength distribution in ^{48}Ti: Signature

The wave function in the self-cross conjugate nucleus has the properties $D^\lambda(L_pL_n)=(-1)^{s+\lambda}D^\lambda(L_n,L_p)$ where s represents the signature. The value $s=0$ corresponds to even signature and $s=1$ to odd signature.[13–16] The expression for X_0^λ,

$$X_0^\lambda = -2/(2\lambda+1)^{1/2}$$
$$\times\sum_{L_0L}\left[\frac{(2L_0+1)}{(2L+1)}\right]^{1/2} U(\lambda jL_0j;jL)D^0(LL)$$
$$\times D^\lambda(L_0,Lv)[\mathcal{M}_p^\lambda-(-1)^{s+\lambda}\mathcal{M}_n^\lambda].$$

For odd signature the matrix element is proportional to $\mathcal{M}_p - \mathcal{M}_n$, while for even signature to $(\mathcal{M}_p + \mathcal{M}_n)$. Since $\mathcal{M}_p^\lambda$ and $\mathcal{M}_n^\lambda$ have opposite signs for electron probes, the states of odd signature should, other things being equal, be more strongly excited. For all intents and purposes we can limit the sum over L to one value $L=2$ (we must however sum over L_0). This is in fact done here.

We now list in Table I the ratio

$$B(M\lambda)/(\mathcal{M}_p \mp \mathcal{M}_n)^2 ,$$

where the minus sign is for odd signature and the plus for even. The list includes all odd multiples and all calculated energies.

Concerning the fragmentation, we see that for the dipole case most of the strength is in the lowest state at 3.83 MeV. For the higher multipoles there is much more fragmentation. However, because the summed strength for the dipole case is so small, there are individual transitions for higher multipoles for which the strength ratios are even larger. For example, the $M5$ strength ratio to the $T=3$ isobaric analog state at 10.99 MeV is 15.93, whereas it is only 5.47 for the strongest $M1$ state.

We note that many of the states of even signature have large strength ratios, e.g., for $\lambda=3$ the state at 5.29 MeV, for $\lambda=5$ the state at 7.76 MeV, and for $\lambda=7$ the state at 5.92 MeV.

However, when we go to the next column on the percentage distribution of Table I, we see that the states of even signature carry relatively little strength. This is of course because for these states $B(M\lambda)$ is proportional to $(\mathcal{M}_p^\lambda + \mathcal{M}_n^\lambda)^2$ and $\mathcal{M}_p^\lambda$ and $\mathcal{M}_n^\lambda$ are of opposite sign. Indeed, the percentage distribution in this column is very close to the pure isovector case $\mathcal{M}_p^\lambda = 1$, $\mathcal{M}_n^\lambda = -1$. For the isovector case the states of even signature carry no strength.

We list the percentage of total strength for various choices of $\mathcal{M}_p^\lambda$ and $\mathcal{M}_n^\lambda$. First we use the values as calculated in subsection D using the unquenched values of g_l and g_s. Next we choose $\mathcal{M}_p^\lambda = 1$, $\mathcal{M}_n^\lambda = 0$ ($\mathcal{M}_p^\lambda = 0$, $\mathcal{M}_n^\lambda = 1$ will give the same answer). Then we consider the isovector extreme $\mathcal{M}_p^\lambda = -\mathcal{M}_n^\lambda$ and finally the isoscalar extreme $\mathcal{M}_p^\lambda = \mathcal{M}_n^\lambda$.

Concerning the table's third column of the strength ratio

$$B(M\lambda)/(\mathcal{M}_p^\lambda \mp \mathcal{M}_n^\lambda)^2$$

it is of interest to compare the sums for the different mul-

TABLE I. Calculated $M\lambda$ strength in ^{48}Ti.

λ	E (MeV)	$B(M\lambda)/(\mathcal{M}_p \mp \mathcal{M}_n)^2$ $\times 10^2$	Percent of total strength[c] $\mathcal{M}_p,\mathcal{M}_n$ as calculated	$\mathcal{M}_p=1$ $\mathcal{M}_n=0$	$\mathcal{M}_p=1$ $\mathcal{M}_n=-1$	$\mathcal{M}_p=1$ $\mathcal{M}_n=1$
1	3.83	5.471	87.53	87.53	87.53	0
	7.67	0.555	8.88	8.88	8.88	0
	13.82[b]	0.228	3.66	3.66	3.66	0
	Sum	6.25				
3	3.01	3.698	16.78	11.78	17.91	0
	5.29[a]	9.327	5.51	29.70	0	86.72
	6.04	9.737	44.15	31.00	47.14	0
	7.03	0.503	2.28	1.60	2.43	0
	9.04[a]	1.428	0.84	4.55	0	13.28
	9.95	0.487	2.21	1.55	2.36	0
	10.99[b]	6.225	28.23	19.82	30.14	0
	Sum	31.4				
5	4.58[a]	1.98	0.43	3.96	0	7.92
	5.10	0.541	2.04	1.08	2.16	0
	6.61[a]	13.56	2.97	27.01	0	54.02
	6.95	4.160	15.65	8.29	16.58	0
	7.76[a]	9.551	2.08	19.03	0	38.06
	7.81	4.316	16.25	8.60	17.2	0
	9.37	0.144	0.55	0.29	0.58	0
	10.53[b]	15.93	59.99	31.74	63.48	0
	Sum	50.2				
7	5.56	4.594	50.51	26.15	52.30	0
	5.92[a]	6.595	2.57	37.54	0	75.08
	6.97	0.023	0.25	50.13	0.26	0
	8.29[a]	2.187	0.85	12.45	0	24.90
	8.39	2.215	24.35	12.61	25.22	0
	11.71[b]	1.952	21.45	11.11	22.22	0
	Sum	17.6				

[a]State of even signature.

[b]$T=3$ state.

[c]Only $L=0$ and $L=2$ components of the *ground* state wave function were included in the calculation.

tipoles. Such a comparison is meaningful because in this ratio we have divided out the single particle reduced matrix elements. The values of the sums for $\lambda=1$, 3, 5, and 7 are, respectively, 6.25, 31.4, 50.2, and 17.6. We see that the $M5$ sum is, in the sense, a factor of 8 larger than the $M1$ sum.

In the last column of Table I we consider the isoscalar case $\mathcal{M}_{\mathrm{p}}^{\lambda}=1$, $\mathcal{M}_{\mathrm{n}}^{\lambda}=1$. For this case all $M1$ transitions vanish. All transitions to states of odd signature vanish. Since there are relatively few states of even signature some of them carry considerable strength.

Clearly then different probes, e.g., π^+, π^-, electrons, and protons should excite these states quite differently. It would be of great interest to try to excite as many of these odd multipole states as possible. To see which of these look promising we return to the percentage distribution for $\mathcal{M}_{\mathrm{p}}$ and $\mathcal{M}_{\mathrm{n}}$ as calculated with the electromagnetic operator.

Clearly the lowest 1^+ state in ^{48}Ti should be seen easily by electron scattering. Richter's group has seen the corresponding state in ^{46}Ti.[5] The 1^+ state has been identified by Rasmussen from Bartol using resonance fluorescence.[17]

For $\lambda=3$ the lowest state, calculated to be at 3 MeV though not predicted to be the strongest, is at a sufficiently low energy to be seen. In passing, it should be remarked that the presence of an unusually low lying 3^+ state in ^{48}Ti has been cited as evidence of triaxiality of this nucleus. Such a low lying state appears quite naturally in a shell model calculation. Another state at a calculated energy of 6.04 MeV which is predicted to carry 44% of the strength certainly looks like a promising candidate. Certainly, the 3^+ isobaric analog state at about 11 MeV excitation should be looked for. The calculation gives it 28% of the strength.

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An interesting behavior for the $M5$ and $M7$ modes is worth mentioning. As seen in Table I the sum of the strength ratio $B(M\lambda)/(\mathcal{M}_{\mathrm{p}}^{\lambda}\pm\mathcal{M}_{\mathrm{n}}^{\lambda})^2$ is the same for states of even signature as it is for states of odd signature. This comes from our approximation of keeping only the $D^0(2,2)$ term. By examining the expression for the summed strength the reason for this behavior becomes clear. The second term, proportional to $\mathcal{M}_{\mathrm{p}}^{\lambda}\mathcal{M}_{\mathrm{n}}^{\lambda}$, will vanish because $2+2$ cannot add up to 5 or 7. The first term is then the only finite term. It is proportional to $\mathcal{M}_{\mathrm{p}}^2+\mathcal{M}_{\mathrm{n}}^2$ which can be written as $\frac{1}{2}(\mathcal{M}_{\mathrm{p}}-\mathcal{M}_{\mathrm{n}})^2+\frac{1}{2}(\mathcal{M}_{\mathrm{p}}+\mathcal{M}_{\mathrm{n}})^2$, i.e., half isovector and half isoscalar.

We thus see that in the single j shell mode the different multipoles behave in rather different ways—from the very high concentration of strength in the lowest 1^+ state for the dipole case (80%), generally strong fragmentation for the octupole mode, high concentration in the analog state for the $M5$ mode (60%), back to substantial concentration (50%) in the lowest 7^+ state for the $M7$ mode.

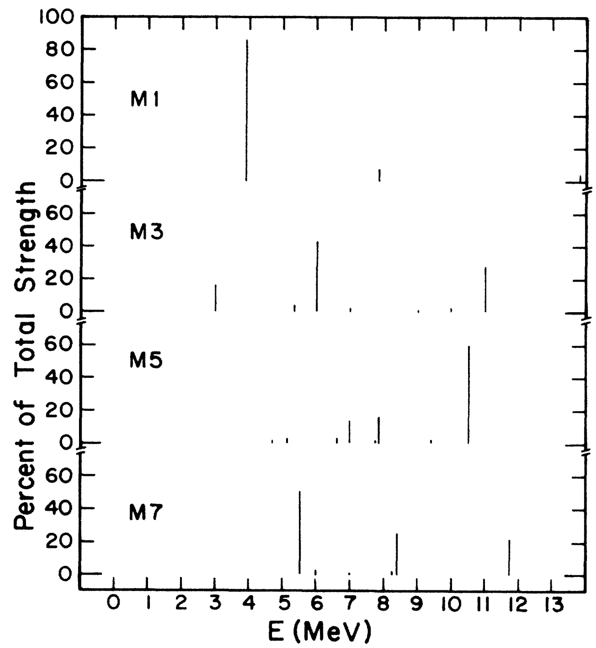

FIG. 1. The percent, to a given state, of the total strength for each multipole, $\lambda=1$, 3, 5, and 7 is shown. The single particle matrix elements of the $M\lambda$ operators are calculated with base g factors. The results are very close to what one obtains in the isovector case $\mathcal{M}_{\mathrm{p}}=-\mathcal{M}_{\mathrm{n}}$.

F. Closed expressions for the magnetic multipole strengths to isobaric analog states in ^{48}Ti

The ground state of ^{48}Ti has isospin $T=2$. The $T=3$ states of the $f_{7/2}$ configuration for $\lambda=1$, 3, 5, and 7 are predicted to be at excitation energies $E^*=13.8$, 11.0, 10.5, and 11.7 MeV, respectively. These states are analogs of states in ^{48}Sc, which have in the single j shell model, unique configurations

$$\psi^{\lambda}=[f_{7/2\pi}f_{7/2\nu}^{-1}]^{\lambda}\ .$$

Because of the uniqueness of the ^{48}Sc wave functions one expects the coefficients $D^{\lambda}(L,L_0)$ for the $T=3$ states in ^{48}Ti to have simple closed forms. We write the wave function for ^{48}Sc in a form similar to that for ^{48}Ti. We obtain

$$[j_\pi j_\nu^{-1}]^\lambda = \sum_{\substack{L_0 \\ \text{even}}} (j^{-2}L_0 j \mid \} j^{-1} j) \{ j_\pi [j_\nu (j^{-2})_\nu^{L_0}]^j \}^\lambda ,$$

$$= \sum_{\substack{L_0 \\ \text{even}}} (j^{-2}L_0 j \mid \} j^{-1} j) U(jj\lambda L_0;Lj) [(j_\pi j_\nu)^L (j^{-2})_\nu^{L_0}]^\lambda .$$

Using a relation from page 4 of the MBZ technical report[7] one can show that

$$(j^{-2}L_0 j \mid \} j^{-1} j) = [(2L_0+1)/28]^{1/2} .$$

We thus expect $D^\lambda(L,L_0)$ for $T=3$ states in ^{48}Ti to be proportional to

$$\sqrt{(2L_0+1)}\, U(jj\lambda L_0;Lj) ,$$

and indeed by imposing the normalization condition

$$\sum_{\substack{LL_0 \\ \text{even}}} | D^\lambda(LL_0) |^2 = 1$$

we find

$$D^\lambda(LL_0) = \left[\frac{(2L_0+1)}{12} \right]^{1/2} U(jj\lambda L_0;Lj) .$$

This expression is symmetric under the interchange of L and L_0. The expression for X_0^λ for ^{48}Ti involves a sum over L_0. Since all the dependence on L_0 is now explicit this sum can be performed. The expression for $B(M_\lambda)$ becomes

$$B(M\lambda)_{T=3} = (\mathcal{M}_p^\lambda - \mathcal{M}_n^\lambda)^2 \left| \sum_L D^0(LL) \left[\frac{2L+1}{12} \right]^{1/2} [1+(-1)^\lambda U(\lambda jjL;jj)] \right|^2 .$$

This is the main result of this section.

With the further approximation that only the $L=2$ part contributes, we obtain

$$B(M\lambda)_{T=3} \approx (\mathcal{M}_p^\lambda - \mathcal{M}_n^\lambda)^2 | D^0(22) |^2 \tfrac{5}{12} \times [1+(-1)^\lambda U(\lambda jj2;jj)]^2 .$$

From the above expression we can readily understand why the strength ratio for different multipoles is so different. The factor

$$[1+(-1)^\lambda U(\lambda jj2;jj)]$$

has the following values

$$\lambda=1 \ \tfrac{4}{21} , \quad \lambda=5 \ \tfrac{32}{21} ,$$

$$\lambda=3 \ \tfrac{20}{21} , \quad \lambda=7 \ \tfrac{8}{15} .$$

This factor is smallest for $\lambda=1$ and largest for $\lambda=5$ consistent with previous discussion.

Note that for $\lambda=5$ and 7 the summed strength for these modes is proportional to $[1+(-1)^\lambda U(\lambda jj2;jj)]$. Thus the ratio of analog to summed strength for these states is proportional to $[1+(-1)^\lambda U(\lambda jj2;jj)]$. For $\lambda=5$ this factor is $\frac{20}{7}$ times larger than for $\lambda=7$. More generally, we can show in the large j limit that, for odd λ, the above factor is a maximum for $\lambda \approx \sqrt{2}j$.

G. Comparison with Lawson wave functions

The expressions for the transition strength depend very strongly on the coefficient $D^0(22)$ in the ^{48}Ti ground state wave function

$$\psi = \sum D^0(LL)[LL]^0 .$$

Recall that Lawson[18] was able to obtain wave functions in the $f_{7/2}$ region by constructing intrinsic states then projecting out states of good angular momentum. Although he used deformed model ideas, Lawson[18] restricted the intrinsic wave functions to have only $f_{7/2}$ components. A comparison was made between the MBZ (Ref. 6) and Lawson[18] wave functions, and it was emphasized that the overlap was large.

However overlap comparisons can be deceiving. More relevant are the comparisons with measurable quantities. Here we will compare the summed strengths for magnetic multipole excitations in the two models.

The appropriate intrinsic state for ^{48}Ti is

$$A(f_{7/2},k=\tfrac{1}{2} f_{7/2},k=-\tfrac{1}{2})_\pi A(f_{7/2},k=\tfrac{7}{2} f_{7/2},k=-\tfrac{7}{2}) .$$

This can be written as

$$\psi = \sum (jj\tfrac{1}{2}-\tfrac{1}{2} \mid L_0)(jjj-j \mid L'0)(LL'00 \mid J0)[LL']^J .$$

From this we can see that

$$D^0(LL)_{\text{Lawson}} \propto (\tfrac{7}{2}\tfrac{7}{2}\tfrac{1}{2}-\tfrac{1}{2} \mid L0)(\tfrac{7}{2}\tfrac{7}{2}\tfrac{7}{2}-\tfrac{7}{2} \mid L0) \times (LL\,00 \mid 00) .$$

Hence,

$$\psi_{\text{Lawson}} = -0.7929[00]+0.5835[22] -0.2135[44]+0.0329[66] ,$$

whereas,

$$\psi_{\text{MBZ}} = -0.9136[00]+0.4058[22] +0.0196[44]-0.0146[66] .$$

The quantity $| D^0(22) |^2$ is equal to 0.165 for MBZ

(Refs. 6 and 7) and 0.341 for Lawson.[18] Thus we expect the summed strength to be about a factor of 2 larger for the Lawson wave functions than for those of MBZ. A more detailed calculation shows that the summed strengths for the $M1$, $M3$, $M5$, and $M7$ modes are, respectively, factors of 3, 2.5, 2.2, and 2.5 larger. The difference is due to the lack of pairing in the Lawson calculation.

III. CLOSING REMARKS

We have considered magnetic multipole transitions in the single j shell model. Although this model is far from complete it should have sufficient validity to give a rough picture of what is going on and to direct experimentalists to energy regions where interesting results should appear. The model should be especially relevant to low lying excitations and for the excitation of isobaric analog states.

We have obtained results not only for the individual transitions but have also obtained formulas for the *summed* strength for the relevant odd multipoles $\lambda=1$, 3, 5, and 7. Whereas one could not possibly describe most of the odd multipole states in terms of only $L=0$ and $L=2$ couplings of neutrons and protons, the summed strength involves only the *ground state* wave functions, and for ^{48}Ti the $L=2$ component $D^0(2,2)$ is the only relevant one [$D^0(0,0)$ is of course large but $L=0$ cannot contribute due to an angular momentum selection rule].

We have noted a varied pattern for the different multipoles. For $\lambda=1$ the lowest state is the one most strongly excited. For $\lambda=5$ the isobaric analog state is the most strongly excited state.

We are able to completely understand the isobaric analog state excitation strength for the various multipoles in ^{48}Ti because the wave functions can be determined in closed form and the sum over L_0 can be performed to give a closed expression. This expression is proportional to

$$[1+(-1)^{\lambda}U(\lambda jjL;jj)]^2 \quad L=2 \text{ dominantly} .$$

This is smallest for $\lambda=1$ and largest for $\lambda=5$ (for odd multipoles).

In a previous work[1] we noted an analogy between the lowest 1^+ states in the titanium isotopes and those occurring in deformed nuclei in the region of ^{156}Gd (such modes were first prediced by LoIudice and Palumbo[19] and Suzuki and Rowe[20]). The energies and excitation strengths of these scissor modes in the two regions are systematically similar. In the $f_{7/2}$ model, though, the excitation involves almost equal mixtures of spin and orbital excitation, whereas for the heavier nuclei the orbital mostly enters. The 1^+ state has been found in ^{46}Ti at 4.3 MeV via electron scattering by the Darmstadt group,[5] and via proton scattering by the Orsay and Michigan State collaboration.[5] The energy and $B(M1)$ strength in electron scattering ($\sim 1\,\mu_N^2$) agrees pretty well with the $f_{7/2}$ model. However a comparison of the proton and electron scattering results suggests that the orbital contribution is larger than the spin contribution, somewhat more than is predicted by the $f_{7/2}$ model, even allowing for a respectable amount of spin quenching.

This experimental result may be a manifestation of the general trend in which the residual interaction causes the wave function to go from the j-j limit towards the L-S limit.

At any rate, we hope that this work will stimulate further experimental work to help locate the odd multipole states, and further theoretical work to help understand the systematics of odd state spectroscopy.

Before closing we should take note of work in other parts of the periodic table on higher multipoles. These include works by Scholten *et al.*,[21–24] who also find that for higher multipoles, e.g., $\lambda=3$ and 5, the lowest state does not necessarily get the most strength. It would appear that even for higher multipoles there is a strong analogy between the behaviors in the deformed and in the single j shell approximations.

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