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Optimal approximation to elastic and inelastic scattering on a bound nucleon system

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An approximation to scattering on a bound system which minimizes the correction term is derived. This "optimal" approximation is found to be more precise and more simple for application than the weak binding impulse approximation. The derivations are presented in the framework of a local many-body nuclear interaction for the elastic and inelastic single scattering amplitude for the first-order optical potential and for distorted wave approximations. A simple formula for the correction terms to the optimal approximation is obtained. Pauli antisymmetrization effects and spin-orbit forces are not considered.

I. INTRODUCTION

Any theory describing the scattering of the projectile on a bound nucleon system in terms of scatterings on constituent nucleons has to deal with the problem that the target nucleons are not free but are in bound states. Even in the case of scattering on *one* bound nucleon it involves all the complications of a true many-body problem. Therefore, for a practical treatment of the projectile-nucleus scattering one is forced to make approximations. An approximation which is often used is a weak binding (impulse) approximation,[1–3] where the binding potential is neglected. In that way the scattering is described in terms of the free *off-energy shell* projectile-nucleon amplitudes[4] which are averaged over the nucleon's Fermi motion. Although this approximation is valid only for high energy small angle scattering, it is sometimes used elsewhere such as for low-energy scattering or for processes with large momentum transfer.

Binding corrections have been discussed many times in literature, for example see Refs. 1—3. However, general expressions were not simple and it was quite obvious that inclusion of the binding potential (even in an approximate way) leads to considerable complications. Fortunately, a more precise analysis shows that this may not be the case. The reason is that the target nucleon is in a stationary bound state, where the binding potential and the kinetic energy of the nucleon are mutually cancelled. Therefore, it could be possible to effectively incorporate the binding potential by a proper renormalization of the two-body energy and other kinematical variables in the projectile-nucleon amplitude (see also Ref. 5).

The first work in this direction has been done by Maillet, Dedonder, and Schmit.[6] By analyzing the second Born term they have explicitly shown that the binding potential effects in the first-order optical potential can be taken into account by a proper choice of the kinematical variables in a free projecticle nucleon amplitude.

In Refs. 7 and 8 we further developed this idea by analyzing the full elastic scattering amplitude on the bound nucleon, i.e., we went beyond the second Born term as was done in Ref. 6. Unlike the standard approaches to the problem of binding effects, we considered[7,8] a different starting point in the expansion of the exact scattering amplitude in terms of the approximate one. It is usual to start with the weak binding impulse approximation, where the higher order terms of the expansion are the binding potential corrections. However, our expansion is designed in such a way that the first-order correction term is exactly zero.[9] Therefore a major part of the binding effects are already in the first term of the expansion. This term is our optimal approximation for the scattering amplitude.

Using a model for the elastic scattering on a nucleon bound by an infinitely heavy core[7,8] we demonstrated that such expansion can indeed by found for the case of *local* potentials. In this optimal approximation the first term coincides with the static approximation in Ref. 6, and in fact has been used earlier as a phenomenological prescription in different calculations (see Refs. 6 and 7 and references therein). The amplitude in the optimal approximation takes the form of a product of the free projectile-nucleon amplitude and the nuclear form factor (no Fermi average is necessary). Therefore the optimal approximation is much easier for application than the weak binding impulse approximation. It is also a more exact one since it minimizes the first-order correction term.

In this paper we derive the optimal approximation for the general case of many-body nuclear interactions, and also consider some higher order correction terms.[7,10] We show that these terms can be partially resummed into a simple formula which is a sort of a Fermi average of the off-shell projectile-nucleon amplitudes. This correction term may be important if we apply the optimal approximation in the region of resonance scattering.

Special attention is paid to the nontrivial generalization of the optimal approximation for inelastic scattering. These results are extremely important for the analysis of inclusive scatterings involving large momentum transfers.[11]

We start our discussion with the usual weak binding (impulse) approximation (Sec. II), and demonstrate the inconsistency of this approximation for defining the effective two-body kinematics (Sec. III). In Sec. IV we derive our optimal approximation for elastic and inelastic scattering on a bound nucleon and also extend the final result to the case of relativistic kinematics. In Sec. V we consider corrections to the optimal approximation. In Sec. VI we derive the optimal approximation for the first-order optical potential, and in Sec. VII we derive the distorted wave optimal approximation. Section VIII is a summary.

II. WEAK BINDING (IMPULSE) APPROXIMATION FOR THE SCATTERING ON A BOUND NUCLEON

We start our discussion with a simplified problem, where we consider the nucleus as a system of a nucleon (N) and a core, bound together by a nuclear potential $\tilde{V}$. The masses of the nucleon and the core are m and $M_A - m$, respectively. The projectile (X) of mass μ scatters inelastically (or elastically) on this system (Fig. 1) leading to a final-state nuclear wave function Φ_n (or Φ_0), where the initial nuclear wave function is Φ_0. Both Φ_0 and Φ_n are the eigenstates of the nuclear Hamiltonian $H_A = K_N^r + \tilde{V}$, i.e.,

$$(K_N^r + \tilde{V})\Phi_{0(n)} = \epsilon_{0(n)}\Phi_{0(n)} , \tag{2.1}$$

where K_N^r is the kinetic energy of relative nucleon core motion, ϵ_0 is the ground state energy ($\epsilon_0 < 0$), and ϵ_n is the energy of the final nuclear state. The latter may be a discrete state ($\epsilon_n < 0$) or a continuum state ($\epsilon_n > 0$). The initial and final projectile momenta are $\mathbf{k}$ and $\mathbf{k}' = \mathbf{k} - \mathbf{q}$. The total projectile-nucleus energy and momentum are E_{XA} and $\mathbf{P}_{XA}$.

Now we formulate the problem: consider the case where the projectile interacts only with the nucleon (through the potential V) and does not interact with the core. The question is how to find the transition amplitude for such a process. The formal exact solution of this problem can be written straightforwardly. The transition amplitude $F_{0n}(E_{XA}, \mathbf{P}_{XA}, \mathbf{k}, \mathbf{k}')$ is a matrix element

$$F_{0n}(E_{XA}, \mathbf{P}_{XA}, \mathbf{k}, \mathbf{k}') = \langle \Phi_n, \mathbf{k}' \,|\, \tau \,|\, \Phi_0, \mathbf{k} \rangle , \tag{2.2}$$

where the scattering operator τ satisfies the Lippmann-Schwinger equation

$$\tau = V + V \frac{1}{E_{XA} - K_X - K_N - K_C + \tilde{V}} \tau . \tag{2.3}$$

Here K_X, K_N, and K_C are the kinetic energies for projectile, nucleon, and core. The initial and final states, $|\Phi_0, \mathbf{k}\rangle$ and $|\Phi_n, \mathbf{k}'\rangle$ in Eq. (2.2), correspond to the product of the nuclear wave functions and the plane waves for the projectile (X) and nucleus (A) asymptotic motion, i.e.,

$$|\Phi, \mathbf{k}\rangle = \Phi(r) e^{i(\mathbf{P}_{XA} - \mathbf{k})\mathbf{R}_A} e^{i\mathbf{k}\mathbf{R}} ,$$

or in the momentum space

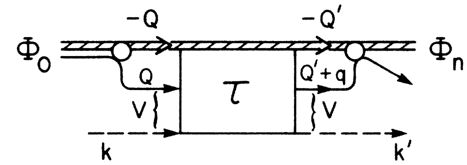

FIG. 1. Graph for the scattering of projectile (dashed line) on the nucleon (solid line) bound by core.

$$\begin{aligned} |\Phi_0, \mathbf{k}\rangle &= \Phi_0(\mathbf{Q}_r)\delta(\mathbf{k} - \mathbf{p})\delta(\mathbf{P}_{XA} - \mathbf{k} - \mathbf{p}_A) , \\ |\Phi_n, \mathbf{k}'\rangle &= \Phi_n(\mathbf{Q}_r')\delta(\mathbf{k}' - \mathbf{p}')\delta(\mathbf{P}_{XA} - \mathbf{k}' - \mathbf{p}_A') , \end{aligned} \tag{2.4}$$

where $\mathbf{Q}_r$ ($\mathbf{Q}_r'$) is the relative nucleon-core momentum in the initial (final) state. The projectile momenta $\mathbf{p}, \mathbf{p}'$ (as well as the nucleus momenta $\mathbf{p}_A, \mathbf{p}'_A$) are variables [as $\mathbf{R}$ ($\mathbf{R}_A$) in the coordinate space] and should not be mixed up with the momenta $\mathbf{k}, \mathbf{k}'$ which are the *external parameters* defining the initial and final projectile states.

The scattering operator τ depends on the external parameters E_{XA} and $\mathbf{P}_{XA}$. In fact, the total projectile nucleus momentum $\mathbf{P}_{XA}$ defines the reference system. For example the choice $\mathbf{P}_{XA} = 0$ corresponds to the center of mass frame, and $\mathbf{P}_{XA} = \mathbf{k}$ corresponds to the laboratory frame. For the physical (on-shell) scattering the total projectile-nucleus energy is

$$\begin{aligned} E_{XA} = E_{XA}^{\text{on shell}} &= \frac{\mathbf{k}^2}{2\mu} + \frac{(\mathbf{P}_{XA} - \mathbf{k})^2}{2M_A} + \epsilon_0 \\ &= \frac{\mathbf{k}'^2}{2\mu} + \frac{(\mathbf{P}_{XA} - \mathbf{k}')^2}{2M_A} + \epsilon_n . \end{aligned} \tag{2.5}$$

The main difficulty in the treatment of Eq. (2.3) comes from the binding potential $\tilde{V}$, which is an integral operator (in momentum representation) in the denominator of the Green's function G. Our goal is not an exact solution of this problem, such as rewriting Eq. (2.3) through the system of Faddeev-type equations (which can be practically solved only for limited cases). Rather we concentrate on the design of the "best" approximation for this problem. Before proceeding with our approach we describe an approximation which presently is in common use. This is the weak binding approximation,[1-3] which is often implied by the term "impulse approximation."[12]

The weak binding (impulse) approximation is the straightforward approximation for Eq. (2.3) in which the binding potential $\tilde{V}$ is neglected. In this case the exact Green's function G is replaced by a free Green's function G_0

$$G_0 = (E_{XA} - K_X - K_N - K_C)^{-1} . \tag{2.6}$$

The operator τ is thus approximated by t_0 which satisfies the equation $t_0 = V + VG_0t_0$. In the form of matrix elements this equation reads

$$\langle \mathbf{p}, \mathbf{Q} \,|\, t_0 \,|\, \mathbf{p}', \mathbf{Q}' \rangle = V(\mathbf{p} - \mathbf{p}')\delta(\mathbf{p} + \mathbf{Q} - \mathbf{p}' - \mathbf{Q}') + \int \frac{V(\mathbf{p} - \mathbf{p}'')\delta(\mathbf{p} + \mathbf{Q} - \mathbf{p}'' - \mathbf{Q}'')\langle \mathbf{p}'', \mathbf{Q}'' \,|\, t_0 \,|\, \mathbf{p}', \mathbf{Q}' \rangle}{E_{XA} - \dfrac{\mathbf{p}''^2}{2\mu} - \dfrac{\mathbf{Q}''^2}{2m} - \dfrac{(\mathbf{P}_{XA} - \mathbf{p}'' - \mathbf{Q}'')^2}{2(M_A - m)}} d^3p''\, d^3Q'' . \tag{2.7}$$

Here p and Q are momenta of the projectile and nucleon, and $\mathbf{P}_{XA}-\mathbf{p}''-\mathbf{Q}''$ is the momentum of the core in the intermediate state. In the following we consider the projectile-nucleus laboratory frame, i.e., we choose $\mathbf{P}_{XA}\equiv\mathbf{k}$. The projectile-nucleon interaction V is taken to be local. To avoid confusion we remind the reader that the momenta p, p′, and p″ are variables in the integral equation (2.7) and should not be mixed up with the *external parameters* $\mathbf{k},\mathbf{k}'$ in Eqs. (2.2), (2.4), and (2.5). Such confusion can produce serious problems in the understanding of our derivations.

It follows immediately from Eq. (2.7) that the total projectile-nucleon momentum is conserved. Therefore, one can write

$$\langle \mathbf{p},\mathbf{Q}\,|\,t_0\,|\,\mathbf{p}',\mathbf{Q}'\rangle=\langle \mathbf{p},\mathbf{P}_{XN}\,|\,\hat{\mathbf{t}}_0\,|\,\mathbf{p}',\mathbf{P}_{XN}\rangle\delta(\mathbf{p}+\mathbf{Q}-\mathbf{p}'-\mathbf{Q}'), \tag{2.8}$$

where $\mathbf{P}_{XN}$ is the total projectile-nucleon momentum given by

$$\mathbf{P}_{XN}=\mathbf{p}+\mathbf{Q}=\mathbf{p}'+\mathbf{Q}' ,$$

and $\hat{\mathbf{t}}_0$ satisfies the equation

$$\langle \mathbf{p},\mathbf{P}_{XN}\,|\,\hat{\mathbf{t}}_0\,|\,\mathbf{p}',\mathbf{P}_{XN}\rangle=V(\mathbf{p}-\mathbf{p}')+\int \frac{V(\mathbf{p}-\mathbf{p}'')\langle \mathbf{p}'',\mathbf{P}_{XN}\,|\,\hat{\mathbf{t}}_0\,|\,\mathbf{p}',\mathbf{P}_{XN}\rangle}{E_{XA}-\dfrac{(\mathbf{k}-\mathbf{P}_{XN})^2}{2(M_A-m)}-\dfrac{\mathbf{p}''^2}{2\mu}-\dfrac{(\mathbf{P}_{XN}-\mathbf{p}'')^2}{2m}}d^3p'' . \tag{2.9}$$

Now we can see that Eq. (2.9) is the usual Lippmann-Schwinger equation for the free two-body amplitude

$$t(E_{XN},\mathbf{P}_{XN},\mathbf{p},\mathbf{p}')\equiv\langle \mathbf{p},\mathbf{P}_{XN}\,|\,t_0\,|\,\mathbf{p}',\mathbf{P}_{XN}\rangle ,$$

where $\mathbf{P}_{XN}$ is the total projectile-nucleon momentum, $\mathbf{p},\mathbf{p}'$ are the momenta of the projectile, and E_{XN} is the projectile-nucleon energy

$$E_{XN}=E_{XA}-\frac{(\mathbf{P}_{XA}-\mathbf{P}_{XN})^2}{2(M_A-m)} . \tag{2.10}$$

Now we can find the projectile-nucleus transition amplitude F_{0n} in the impulse approximation by taking the matrix element Eqs. (2.2) and (2.4) from the operator $t_0\approx\tau$, given by Eq. (2.8). Notice that the relative nucleon-core momenta are $\mathbf{Q}_r=\mathbf{Q}$ and $\mathbf{Q}_r'=\mathbf{Q}'+[(A-1)/A]\mathbf{q}$, Fig. 1. (The mass of the nucleus is taken to be $M_A=Am$.) Finally we obtain

$$F_{0n}(E_{XA},\mathbf{P}_{XA},\mathbf{k},\mathbf{k}')\equiv F_{0n}(E_{XA},\mathbf{k},\mathbf{k}')$$
$$\cong\int \Phi_n^*\left[\mathbf{Q}+\frac{A-1}{A}\mathbf{q}\right]$$
$$\times t(E_{XN},\mathbf{P}_{XN},\mathbf{k},\mathbf{k}')\Phi_0(Q)d^3Q , \tag{2.11}$$

where the two-body energy E_{XN}, Eq. (2.10),

$$E_{XN}=\frac{k^2}{2\mu}+\epsilon_0-\frac{Q^2}{2m(A-1)} \tag{2.11a}$$

and the total projectile-nucleon momentum

$$\mathbf{P}_{XN}=\mathbf{k}+\mathbf{Q} . \tag{2.11b}$$

Here and elsewhere we denote the projectile-nucleon amplitude as $t(E_{XN},\mathbf{P},\mathbf{p},\mathbf{p}')$ where E_{XN} and $\mathbf{P}$ are the total projectile-nucleon energy and momentum, and $\mathbf{p},\mathbf{p}'$ are the projectile initial and final momenta. In fact using Galilean invariance one finds that the projectile-nucleon amplitude is a function of center of mass energy and relative momenta only.

$$t(E_{XN},\mathbf{P},\mathbf{p},\mathbf{p}')\equiv t^{\text{c.m.}}\left[E_{XN}-\frac{\mathbf{P}^2}{2(m+\mu)},\mathbf{p}-\frac{\mu\mathbf{P}}{m+\mu},\mathbf{p}'-\frac{\mu\mathbf{P}}{m+\mu}\right] , \tag{2.12}$$

where $t^{\text{c.m.}}$ is the solution of the Lippmann-Schwinger equation in the projectile-nucleon c.m. frame:

$$t^{\text{c.m.}}(E_{XN}^{\text{c.m.}},\mathbf{p}_r,\mathbf{p}_r')=V(\mathbf{p}_r-\mathbf{p}_r')+\int\frac{V(\mathbf{p}_r-\mathbf{p}_r'')t^{\text{c.m.}}(E_{XN}^{\text{c.m.}},\mathbf{p}_r'',\mathbf{p}_r')}{E_{XN}^{\text{c.m.}}-\dfrac{\mathbf{p}_r''^2}{2m\mu/(m+\mu)}}d^3p_r'' . \tag{2.13}$$

Equation (2.11) is the standard impulse approximation in the weak binding limit for scattering on a bound nucleon. It corresponds to averaging the free projectile-nucleon amplitude over the nucleon Fermi motion. However, the energy E_{XN} in the two-body amplitude t is off the energy shell. Indeed the on-shell value of E_{XN} would correspond to the sum of the kinetic energies of the projectile and nucleon in the initial or final states:

$$E_{XN}^{\text{on shell}}=\frac{\mathbf{k}^2}{2\mu}+\frac{\mathbf{Q}^2}{2m}=\frac{\mathbf{k}'^2}{2\mu}+\frac{(\mathbf{Q}+\mathbf{k}-\mathbf{k}')^2}{2m} , \tag{2.14}$$

whereas the energy E_{XN} in the two-body amplitude t in Eq. (2.11) is less than that quantity. Indeed we find that the energy shift

$$\Delta E = E_{XN} - E_{XN}^{\text{on shell}} = -|\epsilon_0| - \frac{\mathbf{Q}^2}{2m(A-1)} - \frac{\mathbf{Q}^2}{2m} < 0 . \tag{2.15}$$

Here we should mention that the impulse approximation, Eq. (2.11), is often employed with the two-body amplitude t evaluated at the projectile energy, $E_{XN}=k^2/2\mu$, and $\mathbf{P}_{XN}=\mathbf{k}$. This prescription contradicts Eqs. (2.11a) and (2.11b). However, it is often misinterpreted as the "weak binding limit" which can lead to confusion and ill-conceived "corrections."

A relativistic generalization of the weak impulse approximation, Eq. (2.11), is the triangle Feynman diagram, Fig. 2. It is interesting to note that in order to obtain Eq. (2.11a) for the two-body energy E_{XN} as a nonrelativistic limit one should use a common prescription by taking the spectator core *on the mass shell* in the triangle Feynman diagram. Indeed in this case the energy of the struck nucleon is $M_A-(M_{A-1}^2+\mathbf{Q}^2)^{1/2}$. Therefore the total projectile-nucleon kinetic energy in the nonrelativistic limit is

$$\begin{aligned} E_{XN} &= M_A-(M_{A-1}^2+\mathbf{Q}^2)^{1/2}-m+(\mu^2+\mathbf{k}^2)^{1/2}-\mu \\ &\approx \frac{\mathbf{k}^2}{2\mu}+M_A-M_{A-1}-m-\frac{\mathbf{Q}^2}{2m(A-1)} , \end{aligned} \tag{2.16}$$

which is exactly the same two-body energy E_{XN} as appears in Eq. (2.11a).

We also can see that the weak binding (impulse) approximation, Eq. (2.11), coincides with the first term of the Faddeev three-body equations, which describes the scattering on the bound nucleon. For this reason the choice of two-body energy as in Eq. (2.11) has been called the "three-body choice of energy."[13]

Since Eq. (2.11) involves an explicit three-dimensional integration over the Fermi momenta $\mathbf{Q}$ one tries to simplify this result by an approximate factorization of Eq. (2.11) into the projectile-nucleon amplitude and the transition form factor. The projectile-nucleon amplitude t is evaluated at the value of the struck nucleon's momentum $\mathbf{Q}=\langle \mathbf{Q} \rangle$ which gives the main contribution to the integral (2.11). One thus obtains

$$F_{0n}(E_{XA},\mathbf{k},\mathbf{k}')=t(E_{XN},\mathbf{P}_{XN},\mathbf{k},\mathbf{k}')S_{0n}(\mathbf{q}) , \tag{2.17}$$

where

$$E_{XN}=\frac{\mathbf{k}^2}{2\mu}+\epsilon_0-\frac{\langle \mathbf{Q}\rangle^2}{2m(A-1)} ,$$

$$\mathbf{P}_{XN}=\mathbf{k}+\langle \mathbf{Q}\rangle ,$$

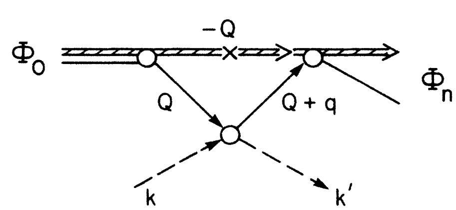

FIG. 2. Graph for the standard impulse approximation. The spectator core is on the mass shell.

and $S_{0n}(\mathbf{q})$ is a transition form factor

$$S_{0n}(\mathbf{q})=\int \Phi_0(\mathbf{Q})\Phi_n^*\left[\mathbf{Q}+\frac{A-1}{A}\mathbf{q}\right]d^3Q . \tag{2.18}$$

An optimal value of $\langle \mathbf{Q}\rangle$

$$\langle \mathbf{Q}\rangle=-\frac{A-1}{2A}\mathbf{q} \tag{2.19}$$

has been obtained in Ref. 14 using some symmetry arguments. (Notice that $\langle \mathbf{Q}\rangle$ here is given in the nuclear laboratory frame, and not in the projectile-nucleus c.m. frame as in Ref. 14.)

In general, the approximate factorization of Eq. (2.11) can be done only if the amplitude t, Eq. (2.12), has a weak energy dependence. Otherwise Eq. (2.17) cannot be a good approximation to Eq. (2.11).[15]

The validity of the factorized form of the impulse approximation, Eq. (2.17), and the energy shift ΔE, Eq. (2.15), in the two-body amplitude, and their implications to data analysis have been often discussed (see for example Refs. 13 and 16). However, one should keep in mind that Eq. (2.11) is itself only an approximation to the exact result, Eq. (2.2), so one should not neglect the important question of the corrections to the impulse approximation, Eq. (2.11), itself. It could appear, for instance, that the higher-order terms compensate the effects of the three-body choice of energy or may influence the factorization approximation, Eq. (2.17).

In fact the "three-body" choice of energy is a specific example of proposals which have been made on the basis of connected-kernel scattering approaches. Analogous N-body extensions of the three-body case, for example, appear in Ref. 17. Serious problems of these approaches have been clearly indicated by Picklesimer, Tandy, and Thaler in Ref. 5. In particular, they demonstrated that for large N the off-shell "N-body energy shift" in two-body amplitudes is a completely unrealistic one. It makes this approach unapplicable. As a possibility to overcome the problem with the energy shift it was proposed in Ref. 5 to take into consideration higher corrections. In that sense our paper provides a desirable correction mechanism which goes beyond the discussion of Ref. 5. In the next section we consider the first-order correction term to Eq. (2.11), and in particular examine the connection with the off-shell energy shift in the weak binding impulse approximation.

III. CORRECTION TO THE WEAK BINDING (IMPULSE) APPROXIMATION

In order to find corrections to the impulse approximation we use a general relation between the exact scattering operator τ and an approximation to it, t_a,

$$\tau=t_a+t_a(G-G_a)\tau , \tag{3.1}$$

where τ and t_a satisfy the Lippmann-Schwinger equations

$$\tau=V+VG\tau , \tag{3.2}$$

$$t_a=V+VG_at_a , \tag{3.3}$$

and where G and G_a are exact and approximate Green's functions. Equation (3.1) can be written in the form of an expansion

$$\tau = t_a + t_a(G-G_a)t_a + t_a(G-G_a)t_a(G-G_a)t_a + \cdots \ . \qquad (3.4)$$

Because approximations are usually applied to the nuclear Hamiltonian (which is in the denominator of the exact Green's function), the "natural parameter" for expanding τ in terms of t_a would be $h \equiv G_a^{-1} - G^{-1}$. In terms of h the expansion (3.4) can be rewritten as

$$\tau = t_a + t_a G_a h G_a t_a + t_a G_a h (G_a + G_a t_a G_a) h G_a t_a + \cdots \ . \qquad (3.5)$$

The weak binding impulse approximation which has been described in the previous section corresponds to $G_a \equiv G_0$, Eq. (2.6), $h \equiv \tilde{V}$ and $t_a \equiv t_0$, Eq. (2.7). In this case we can see that the expansion (3.5) for the transition amplitude F_{0n}, Eq. (2.2), corresponds to the sum of the Feynman diagrams, Fig. 3, where the spectator core is always on the mass shell.

The first diagram, Fig. 3(a), is the weak binding impulse approximation, Eq. (2.11). The second diagram, Fig. 3(b), corresponds to the contribution of the first-order term in h in the expansion (3.5). It is usually considered as a binding correction to the impulse approximation. We thus obtain

$$\langle \Phi_n, \mathbf{k}' \,|\, t_0 G_0 \tilde{V} G_0 t_0 \,|\, \Phi_0, \mathbf{k} \rangle = \int \frac{\Phi_0(\mathbf{Q}) t(E'_{XN}, \mathbf{P}', \mathbf{k}, \mathbf{k}'') \tilde{V}(\mathbf{Q}-\mathbf{Q}') t(E''_{XN}, \mathbf{P}'', \mathbf{k}'', \mathbf{k}')}{\left[E'_{XN} - \dfrac{(\mathbf{P}'-\mathbf{k}'')^2}{2m} - \dfrac{\mathbf{k}''^2}{2\mu} \right] \left[E''_{XN} - \dfrac{(\mathbf{P}''-\mathbf{k}'')^2}{2m} - \dfrac{\mathbf{k}''^2}{2\mu} \right]} \times \Phi_n^* \left[\mathbf{Q}' + \frac{A-1}{A} \mathbf{q} \right] d^3Q \, d^3Q' \, d^3k'' \ , \qquad (3.6)$$

where $\mathbf{P}' = \mathbf{k} + \mathbf{Q}$, $\mathbf{P}'' = \mathbf{k}' + \mathbf{Q}'$ are the total projectile-nucleon momenta and E'_{XN}, E''_{XN} are the total projectile-nucleon energies in the corresponding amplitudes:

$$E'_{XN} = \frac{\mathbf{k}^2}{2\mu} + \epsilon_0 - \frac{\mathbf{Q}^2}{2(M_A - m)} \ , \qquad E''_{XN} = \frac{\mathbf{k}^2}{2\mu} + \epsilon_0 - \frac{\mathbf{Q}'^2}{2(M_A - m)} \ ; \qquad (3.7)$$

$\mathbf{q} = \mathbf{k} - \mathbf{k}'$ is the momentum transfer to the nucleus.

On first sight the binding potential corrections are not connected with the impulse approximation term, Eq. (2.11) [Fig. 3(a)]. However, the following analysis shows that this is not true. Consider for instance the case where the Green's functions and the two-body amplitudes in Eq. (3.6) have a weak dependence on the arguments $\mathbf{Q}, \mathbf{Q}'$ compared to the strong $\mathbf{Q}, \mathbf{Q}'$ dependence in $\tilde{V}$ and Φ_0, Φ_n. Then assuming that $\tilde{V}$ commutes with t and G_0, i.e., no spin or isospin dependence, the $\mathbf{Q}$ (or $\mathbf{Q}'$) integration in (3.6) can be carried out using the Schrödinger equation

$$\int \tilde{V}(\mathbf{Q}-\mathbf{Q}') \Phi_0(\mathbf{Q}) d^3Q = \left[\epsilon_0 - \frac{\mathbf{Q}'^2}{2\bar{m}} \right] \Phi_0(\mathbf{Q}') \ , \qquad (3.8)$$

where $\bar{m} = m(A-1)/A$ is the nucleon-core reduced mass. If the main contribution to the integral is coming from the region $\mathbf{Q}' \approx \mathbf{Q}$ (as is definitely the case for small q^2), then we can replace $\mathbf{P}'' \approx \mathbf{P}' = \mathbf{k} + \mathbf{Q} = \mathbf{P}$, and $E'_{XN} \approx E''_{XN} = E_{XN}$ in (3.6). Afterwards the integration over the $\mathbf{k}''$ variable can be carried out in (3.6) using the following relation for the energy derivative of the two-body amplitude:[18]

$$\frac{dt(E_{XN}, \mathbf{P}, \mathbf{k}, \mathbf{k}')}{dE_{XN}} = -\int \frac{t(E_{XN}, \mathbf{P}, \mathbf{k}, \mathbf{k}'') t(E_{XN}, \mathbf{P}, \mathbf{k}'', \mathbf{k}')}{\left[E_{XN} - \dfrac{(\mathbf{P}-\mathbf{k}'')^2}{2m} - \dfrac{\mathbf{k}''^2}{2\mu} \right]^2} d^3k'' \ . \qquad (3.9)$$

Using Eqs. (3.8) and (3.9) to evaluate the binding correction term of Eq. (3.6), we obtain

$$\langle \Phi_n, \mathbf{k}' \,|\, t_0 G_0 \tilde{V} G_0 t_0 \,|\, \Phi_0, \mathbf{k} \rangle \approx -\int \left[\epsilon_0 - \frac{\mathbf{Q}^2}{2\bar{m}} \right] \Phi_0(\mathbf{Q}) \frac{dt(E_{XN}, \mathbf{P}, \mathbf{k}, \mathbf{k}')}{dE_{XN}} \Phi_n^* \left[\mathbf{Q} + \frac{A-1}{A} \mathbf{q} \right] d^3Q \ . \qquad (3.10)$$

Now it is clear that the binding potential correction term (3.10) and the impulse approximation term (2.11) are related. Indeed

$$t(E_{XN}, \mathbf{P}, \mathbf{k}, \mathbf{k}') - \left[\epsilon_0 - \frac{\mathbf{Q}^2}{2\bar{m}} \right] \frac{dt(E_{XN}, \mathbf{P}, \mathbf{k}, \mathbf{k}')}{dE_{XN}} \approx t \left[E_{XN} + \frac{\mathbf{Q}^2}{2\bar{m}} - \epsilon_0, \mathbf{P}, \mathbf{k}, \mathbf{k}' \right] \qquad (3.11)$$

and therefore

$$\langle \Phi_n, \mathbf{k}' \,|\, t_0 + t_0 G_0 \tilde{V} G_0 t_0 \,|\, \Phi_0, \mathbf{k} \rangle \approx \int \Phi_n^* \left[\mathbf{Q} + \frac{A-1}{A} \mathbf{q} \right] t \left[E_{XN} = \frac{\mathbf{k}^2}{2\mu} + \frac{\mathbf{Q}^2}{2m}, \mathbf{P} = \mathbf{k} + \mathbf{Q}, \mathbf{k}, \mathbf{k}' \right] \Phi_0(\mathbf{Q}) d^3Q \ . \qquad (3.12)$$

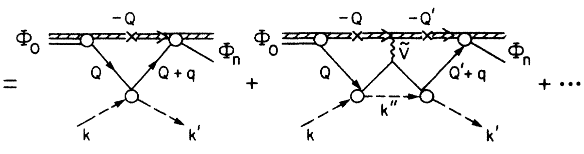

FIG. 3. Graphs for the impulse approximation (a) and for the binding correction term (b). The spectator core is on the mass shell.

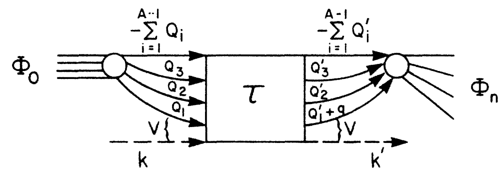

FIG. 4. Graph for projectile-bound nucleon scattering in the case of the many-body nuclear Hamiltonian. The projectile interacts only with the nucleon "1."

Comparing Eq. (3.12) with the impulse approximation term, Eq. (2.11), we find that the binding corrections generate an upward energy shift in the projectile-nucleon amplitude, which cancels the off-energy-shell shift Δ, Eq. (2.15), in the weak binding approximation.

Although our evaluation of the correction term is not precise, it clearly shows the inconsistency of defining the two-body kinematics in the impulse approximation term without regard to the higher order correction term. Therefore the large effects which result from the three-body choice of energy[13] (spectator core on the mass shell) can be completely artificial and are compensated by higher order terms.

We have thus demonstrated that the determination of the effective two-body kinematics for the scattering on a bound nucleon cannot be separated from the simultaneous considerations of binding effects. In the next section we deal with this problem in more detail.

IV. OPTIMAL APPROXIMATION FOR THE SCATTERING ON BOUND NUCLEON

Consider again the problem of a projectile-nucleus scattering where the projectile (X) interacts with only one of the target nucleus "1" (through the potential V), but it does not interact with the others. Here we consider all nucleons as distinguishable particles. Now we do not take the nucleus as a system of nucleon and core, as has been done in the previous sections. We rather consider the nucleus as a true many-body system, Fig. 4, where the nuclear wave functions Φ_0 and Φ_n for the initial and final states are the eigenstates of the exact nuclear Hamiltonian H_A:

$$H_A\Phi_{0(n)}=\left[\sum_i K_i^r+\sum_{i>j}\tilde{V}_{ij}\right]\Phi_{0(n)}=\epsilon_{0(n)}\Phi_{0(n)} . \quad (4.1)$$

Here K_i^r is the kinetic energy of the relative motion of nucleon i relative to the nucleus center of mass, and $\tilde{V}_{ij}$ is the two-nucleon potential. All derivations are presented in the nucleus laboratory frame. We denote the momenta of target nucleons in the initial state as $\mathbf{Q}_1,\mathbf{Q}_2,\mathbf{Q}_3,\ldots,\mathbf{Q}_A$ and in the final state $\mathbf{Q}'_1+\mathbf{q},\mathbf{Q}'_2,\mathbf{Q}'_3,\ldots,\mathbf{Q}'$, Fig. 4, so that

$$\sum_{i=1}^{A}\mathbf{Q}_i=\sum_{i=1}^{A}\mathbf{Q}'_i=0 .$$

$\mathbf{q}=\mathbf{k}-\mathbf{k}'$ is the momentum transfer to the nucleus.

As in Eqs. (2.2) and (2.3), the transition amplitude F_{0n} is a matrix element of the operator τ

$$F_{0n}(E_{XA},\mathbf{k},\mathbf{k}')=\langle\Phi_n,\mathbf{k}'\,|\,\tau\,|\,\Phi_0,\mathbf{k}\rangle , \quad (4.2)$$

which describes the scattering of the projectile on the bound nucleon "1," and satisfies the Lippmann-Schwinger equation

$$\tau=V+V\frac{1}{E_{XA}-K_X-K_A-H_A}\tau , \quad (4.3)$$

where K_X and K_A are the kinetic energies of projectile and nucleus center of mass motion.

Now we are ready to formulate the problem we are going to solve. Consider the expansion (3.5) for the operator τ in terms of approximate operators G_a, t_a, and $h=G_a^{-1}-G^{-1}$. We look for the approximate Green's function G_a [which defines the approximate operator t_a by means of Eq. (3.3)], so that the contribution to the scattering amplitude F_{0n} from the first-order correction term in expansion (3.5) vanishes. This means that G_a satisfies the equation

$$\langle\Phi_n,\mathbf{k}'\,|\,t_aG_a(G_a^{-1}-G^{-1})G_at_a\,|\,\Phi_0,\mathbf{k}\rangle=0 , \quad (4.4)$$

where t_a is given by Eq. (3.3). Such an optimal choice for t_a corresponds to the inclusion of some of the binding potential effects in the approximate operator t_a.

Equation (4.4) is highly nonlinear and its straightforward treatment would be very complicated. Therefore, we first have to guess a general class for the Green's functions G_a in which to look for a solution to Eq. (4.4). We take G_a to have the form

$$G_a=(E_{XA}-K_X-K_A-\bar{\epsilon})^{-1} ,$$

where $\bar{\epsilon}$ is an operator which acts only on the projectile variables and may depend on external parameters like E_{XA}, $\mathbf{k}$, $\mathbf{q}$, ϵ_0, and ϵ_n, but it *does not depend* on any target nucleon variables. The matrix element of G_a^{-1} in momentum space is

$$\langle\mathbf{p},\mathbf{Q}_1,\mathbf{Q}_2,\ldots\,|\,G_a^{-1}\,|\,\mathbf{p}',\mathbf{Q}'_1,\mathbf{Q}'_2,\ldots\rangle=\left[E_{XA}-\frac{\mathbf{p}^2}{2\mu}-\frac{(\mathbf{k}-\mathbf{p})^2}{2mA}-\bar{\epsilon}(\mathbf{p},\mathbf{k},\mathbf{q},\epsilon_0,\epsilon_n)\right]\delta(\mathbf{p}-\mathbf{p}')\prod_i\delta(\mathbf{Q}_i-\mathbf{Q}'_i) , \quad (4.5)$$

where $\mathbf{p}$ ($\mathbf{p}'$) is the momentum of the projectile, $\mathbf{Q}_i$ ($\mathbf{Q}'_i$) are the momenta of the target nucleons, and $\mathbf{k}-\mathbf{p}$ is the momentum of the nucleus c.m. motion. (We remind the reader that the total projectile-nucleus momentum is $\mathbf{P}_{XA}\equiv\mathbf{k}$

since all derivations are carried out in the nucleus laboratory frame.)

We are now going to demonstrate that a G_a of the form of Eq. (4.5) can satisfy Eq. (4.4). We consider the case where all potentials are local. [For nonlocal potentials our result can be considered only as an approximate solution of Eq. (4.4) (Ref. 19)]. However, before substituting G_a into Eq. (4.4) we must consider Eq. (3.3) for the operator t_a which involves the same Green's function G_a, Eq. (4.5). In the form of matrix elements Eq. (3.3) reads

$$\begin{aligned}\langle \mathbf{p},\mathbf{Q}_1,\ldots \mid t_a \mid \mathbf{p}',\mathbf{Q}_1',\ldots\rangle &= V(\mathbf{p}-\mathbf{p}')\delta(\mathbf{p}+\mathbf{Q}_1-\mathbf{p}'-\mathbf{Q}_1')\prod_{i\neq 1}\delta(\mathbf{Q}_i-\mathbf{Q}_i')\\ &\quad+\int V(\mathbf{p}-\mathbf{p}'')\delta(\mathbf{p}+\mathbf{Q}_1-\mathbf{p}''-\mathbf{Q}_1'')\\ &\quad\times\prod_{i\neq 1}\delta(\mathbf{Q}_i-\mathbf{Q}_i'')\frac{\langle \mathbf{p}'',\mathbf{Q}_1'',\ldots \mid t_a \mid \mathbf{p}',\mathbf{Q}_1',\ldots\rangle d^3p''d^3Q_1''d^3Q_2''\cdots}{E_{XA}-\dfrac{\mathbf{p}''^2}{2\mu}-\dfrac{(\mathbf{k}-\mathbf{p}'')^2}{2mA}-\bar{\epsilon}(\mathbf{p}'',\mathbf{k},\mathbf{q},\epsilon_0,\epsilon_n)}\ .\end{aligned} \tag{4.6}$$

We can see immediately from this equation that t_a can be written in the form

$$\langle \mathbf{p},\mathbf{Q}_1,\ldots \mid t_a \mid \mathbf{p}',\mathbf{Q}_1',\ldots\rangle = \langle \mathbf{p} \mid \hat{\mathbf{t}}_a \mid \mathbf{p}'\rangle\delta(\mathbf{p}+\mathbf{Q}_1-\mathbf{p}'-\mathbf{Q}_1')\prod_{i\neq 1}\delta(\mathbf{Q}_i-\mathbf{Q}_i') \tag{4.7}$$

where $\hat{\mathbf{t}}_a$ satisfies the equation

$$\langle \mathbf{p} \mid \hat{\mathbf{t}}_a \mid \mathbf{p}'\rangle = V(\mathbf{p}-\mathbf{p}')+\int\frac{V(\mathbf{p}-\mathbf{p}'')\langle \mathbf{p}'' \mid \hat{\mathbf{t}}_a \mid \mathbf{p}'\rangle d^3p''}{E_{XA}-\dfrac{\mathbf{p}''^2}{2\mu}-\dfrac{(\mathbf{k}-\mathbf{p}'')^2}{2mA}-\bar{\epsilon}(\mathbf{p}'',\mathbf{k},\mathbf{q},\epsilon_0,\epsilon_n)}\ . \tag{4.8}$$

Since $\bar{\epsilon}$ does not depend on the target nucleons momenta $\mathbf{Q}_i,\mathbf{Q}_i'$, neither will the operator $\hat{\mathbf{t}}_a$. Therefore all the dependence on the nucleon momenta $\mathbf{Q}_i,\mathbf{Q}_i'$ in the operator t_a, Eq. (4.7), is coming from the δ functions.

Consider again Eq. (4.4). In the momentum representation it is

$$\begin{aligned}&\int \langle \Phi_n \mid \mathbf{Q}_1'+\mathbf{q},\mathbf{Q}_2',\ldots\rangle\langle \mathbf{k}' \mid \mathbf{p}'\rangle\langle \mathbf{p}',\mathbf{Q}_1'+\mathbf{q},\mathbf{Q}_2',\ldots \mid t_aG_a(G_a^{-1}-G^{-1})G_at_a \mid \mathbf{p},\mathbf{Q}_1,\mathbf{Q}_2,\ldots\rangle\\ &\quad\times\langle \mathbf{p} \mid \mathbf{k}\rangle\langle \mathbf{Q}_1,\mathbf{Q}_2,\ldots \mid \Phi_0\rangle\delta\left[\sum_{i=1}^{A}\mathbf{Q}_i'\right]\delta\left[\sum_{i=1}^{A}\mathbf{Q}_i\right]\prod_{i=1}^{A}d^3Q_i'\,d^3Q_id^3p\,d^3p'=0\ .\end{aligned} \tag{4.9}$$

Here

$$\langle \Phi_n \mid \mathbf{Q}_1'+\mathbf{q},\mathbf{Q}_2',\ldots\rangle = \Phi_n^*\left[\mathbf{Q}_1'+\frac{A-1}{A}\mathbf{q},\mathbf{Q}_2'-\frac{\mathbf{q}}{A},\ldots\right]$$

and $\langle \mathbf{Q}_1,\mathbf{Q}_2,\ldots \mid \Phi_0\rangle = \Phi_0(\mathbf{Q}_1,\mathbf{Q}_2,\ldots)$ are the nuclear wave functions for the initial and the final states, Fig. 4, depending on the nucleons's momenta relative to the nucleus center of mass motion; and $\langle \mathbf{p} \mid \mathbf{k}\rangle = \delta(\mathbf{p}-\mathbf{k})$, $\langle \mathbf{k}' \mid \mathbf{p}'\rangle = \delta(\mathbf{p}'-\mathbf{k}')$ are the wave functions for the projectile-nucleus asymptotic motion. The inverse Green's function $G^{-1}=E_{XA}-K_X-K_A-H_A$ in momentum space is

$$\begin{aligned}\langle \mathbf{p},\mathbf{Q}_1,\mathbf{Q}_2,\ldots \mid G^{-1} \mid \mathbf{p}',\mathbf{Q}_1',\mathbf{Q}_2',\ldots\rangle &= \left\{\left[E_{XA}-\frac{\mathbf{p}^2}{2\mu}-\frac{(\mathbf{k}-\mathbf{p})^2}{2mA}-\sum_{i=1}^{A}\frac{\left[\mathbf{Q}_i-\dfrac{\mathbf{k}-\mathbf{p}}{A}\right]^2}{2m}\right]\prod_i\delta(\mathbf{Q}_i-\mathbf{Q}_i')\right.\\ &\quad\left.-\sum_{i>j}\tilde{V}_{ij}(\mathbf{Q}_i-\mathbf{Q}_i')\delta(\mathbf{Q}_i+\mathbf{Q}_j-\mathbf{Q}_i'-\mathbf{Q}_j')\prod_{l\neq i,j}\delta(\mathbf{Q}_l-\mathbf{Q}_l')\right\}\delta(\mathbf{p}-\mathbf{p}')\ .\end{aligned} \tag{4.10}$$

Here the kinetic energy operator for the A-body target includes an overall kinetic energy term $(\mathbf{k}-\mathbf{p})^2/2mA$ and a sum over the individual kinetic energies of a *relative* nucleon-nucleus center of mass motion. The interaction term corresponds to the local many-body potential $\sum_{i>j}\tilde{V}_{ij}(\mathbf{r}_i-\mathbf{r}_j)$ in the momentum representation. It is important to note that due to the locality of interactions the potential $\tilde{V}$ is not changed when momentum $\mathbf{k}-\mathbf{p}$ is transferred to the nucleus.

Inserting Eqs. (4.5), (4.7), and (4.10) into Eq. (4.9), we obtain

$$\int\frac{\langle \mathbf{k}' \mid \hat{\mathbf{t}}_a \mid \mathbf{p}''\rangle\langle \mathbf{p}'' \mid \hat{\mathbf{t}}_a \mid \mathbf{k}\rangle I(\mathbf{p}'')d^3p''}{\left[E_{XA}-\dfrac{\mathbf{p}''^2}{2\mu}-\dfrac{(\mathbf{k}-\mathbf{p}'')^2}{2mA}-\bar{\epsilon}(\mathbf{p}'',\mathbf{k},\mathbf{q},\epsilon_0,\epsilon_n)\right]^2}=0\ , \tag{4.11}$$

where

$$I(\mathbf{p}'')=\int \prod_{i=1}^{A-1} d^3Q_i d^3Q_i' \,\Phi_n^* \left[\mathbf{Q}_1'+\frac{A-1}{A}\mathbf{q},\mathbf{Q}_2'-\frac{\mathbf{q}}{A},\ldots\right]$$
$$\times\left\{\left[\frac{\left[\mathbf{Q}_1+\frac{A-1}{A}\mathbf{q}_1\right]^2}{2m}+\sum_{i=2}^{A}\frac{\left[\mathbf{Q}_i-\frac{\mathbf{q}_1}{A}\right]^2}{2m}-\bar{\epsilon}\right]\prod_i \delta(\mathbf{Q}_i-\mathbf{Q}_i')\right.$$
$$\left.+\sum_{i>j}\tilde{V}_{ij}(\mathbf{Q}_i-\mathbf{Q}_i')\delta(\mathbf{Q}_i+\mathbf{Q}_j-\mathbf{Q}_i'-\mathbf{Q}_j')\prod_{l\neq i,j}\delta(\mathbf{Q}_l-\mathbf{Q}_l')\right\}\Phi_0(\mathbf{Q}_1,\mathbf{Q}_2,\ldots)\ . \tag{4.12}$$

Here $\mathbf{q}=\mathbf{k}-\mathbf{k}'$, $\mathbf{q}_1=\mathbf{p}-\mathbf{p}''$, and $\mathbf{Q}_A=-\sum_{j=1}^{A-1}\mathbf{Q}_j$. We assumed commutativity $\hat{\mathbf{t}}_a$ and $\tilde{V}$ since all the spin-isospin dependence of interactions has been neglected.

We have to find $\bar{\epsilon}$ as a function of $\mathbf{p}''$, which is independent of $\mathbf{Q}_i$ and satisfies Eq. (4.11) (and does not involve $\hat{\mathbf{t}}_a$). This is the case where $\bar{\epsilon}$ satisfies the equation $I(\mathbf{p}'')=0$ for any values of $\mathbf{p}''$. Consider thus Eq. (4.12) and $I(p'')$. Eliminating the nuclear potential $\tilde{V}_{ij}$ by using the Schrödinger equation, $\sum_{i>j}\tilde{V}_{ij}\Phi_0=(\epsilon_0-\sum_i K_i)\Phi_0$, we obtain

$$I(\mathbf{p}'')=\int \Phi_n^*\left[\mathbf{Q}_1+\frac{A-1}{A}\mathbf{q},\mathbf{Q}_2-\frac{\mathbf{q}}{A},\ldots\right]$$
$$\times\left[\epsilon_0-\sum_{i=1}^{A}\frac{\mathbf{Q}_i^2}{2m}+\frac{\left[\mathbf{Q}_1+\frac{A-1}{A}\mathbf{q}_1\right]^2}{2m}+\sum_{i=2}^{A}\frac{\left[\mathbf{Q}_i-\frac{\mathbf{q}_1}{A}\right]^2}{2m}-\bar{\epsilon}\right]\Phi_0(\mathbf{Q}_1,\mathbf{Q}_2,\ldots)\prod_{i=1}^{A-1}d^3Q_i\ . \tag{4.13}$$

After simple algebra we get

$$I(\mathbf{p}'')=\int \Phi_n^*\left[\mathbf{Q}_1+\frac{A-1}{A}\mathbf{q},\mathbf{Q}_2-\frac{\mathbf{q}}{A},\ldots\right]\left[\epsilon_0+\frac{(A-1)\mathbf{q}_1^2}{2mA}+\frac{\mathbf{Q}_1\mathbf{q}_1}{m}-\bar{\epsilon}\right]\Phi_0(\mathbf{Q}_1,\mathbf{Q}_2,\ldots)\prod_i d^3Q_i\ . \tag{4.14}$$

We see that all quadratic terms in $\mathbf{Q}_i^2$ are cancelled in the integrand (4.14). However, the term linear in $\mathbf{Q}_1$, $\mathbf{Q}_1\mathbf{q}/m$, remains. On first sight we cannot find a $\bar{\epsilon}$ which makes $I(\mathbf{p}'')\equiv 0$ and is also independent of the nucleus variables $(\mathbf{Q}_i,\mathbf{Q}_i')$. However, this problem can be overcome and the desirable function $\bar{\epsilon}$ can be eventually found. We discuss separately the cases of elastic and inelastic scattering.

A. Elastic projectile-nucleus scattering

In this case the final nucleus is in the ground state ($\epsilon_n=\epsilon_0$), and we can use the symmetry properties of the ground state nuclear wave function with a given parity

$$\Phi_0(\mathbf{Q}_1,\mathbf{Q}_2,\ldots)\Phi_0(\mathbf{Q}_1',\mathbf{Q}_2',\ldots)=\Phi_0(-\mathbf{Q}_1,-\mathbf{Q}_2,\ldots)\Phi_0(-\mathbf{Q}_1',-\mathbf{Q}_2',\ldots)\ . \tag{4.15}$$

If we replace $\mathbf{Q}_1=\tilde{\mathbf{Q}}_1-(A-1/2A)\mathbf{q}$ and $\mathbf{Q}_i=\tilde{\mathbf{Q}}_i+(1/2A)\mathbf{q}$ (for $i\neq 1$) in Eq. (4.14) and use (4.15) we can see that the term $(1/m)\tilde{\mathbf{Q}}_1\mathbf{q}$ does not contribute in the integral (4.14). Returning to the variables $\mathbf{Q}_i$, we thus obtain

$$I(\mathbf{p}'')=\int \Phi_0\left[\mathbf{Q}_1+\frac{A-1}{A}\mathbf{q},\mathbf{Q}_2-\frac{\mathbf{q}}{A},\ldots\right]\left[\epsilon_0+\frac{(A-1)\mathbf{q}_1^2}{2mA}-\frac{(A-1)\mathbf{q}\mathbf{q}_1}{2mA}-\bar{\epsilon}\right]\Phi_0(\mathbf{Q}_1,\mathbf{Q}_2,\ldots)\prod_i d^3Q_i\ . \tag{4.16}$$

Therefore the quantity $\bar{\epsilon}$ which corresponds to $I(\mathbf{p}'')\equiv 0$ is

$$\bar{\epsilon}(\mathbf{p}'',\mathbf{k},\mathbf{q},\epsilon_0)=\epsilon_0+\frac{A-1}{2mA}\mathbf{q}_1^2-\frac{(A-1)}{2mA}\mathbf{q}_1\mathbf{q}\ . \tag{4.17}$$

The target nucleon variables do not appear in Eq. (4.17) and therefore $\bar{\epsilon}$ has the desirable form. Substituting $\bar{\epsilon}$ from Eq. (4.17) into Eq. (4.5) for G_a we find after simple algebra

$$\langle \mathbf{p},\mathbf{Q}_1,\mathbf{Q}_2,\ldots|G_a^{-1}|\mathbf{p}',\mathbf{Q}_1',\mathbf{Q}_2',\ldots\rangle=\left[E_{XA}-\frac{\mathbf{p}^2}{2\mu}-\epsilon_0-\frac{(\mathbf{k}+\tilde{\mathbf{Q}}-\mathbf{p})^2}{2m}+\frac{\tilde{\mathbf{Q}}^2}{2m}\right]\delta(\mathbf{p}-\mathbf{p}')\prod_i\delta(\mathbf{Q}_i-\mathbf{Q}_i')\ , \tag{4.18}$$

where

$$\tilde{\mathbf{Q}} = -\frac{A-1}{2A}\mathbf{q} \,. \tag{4.19}$$

Now we can find the scattering operator t_a, Eq. (4.7). Inserting Eq. (4.17) into Eq. (4.8) and rearranging the terms in the same way as in Eq. (4.18) we get

$$\langle \mathbf{p} | \hat{\mathbf{t}}_a | \mathbf{p}' \rangle = V(\mathbf{p}-\mathbf{p}') + \int \frac{V(\mathbf{p}-\mathbf{p}'')\langle \mathbf{p}'' | \hat{\mathbf{t}}_a | \mathbf{p}' \rangle d^3p''}{E_{XA} - \epsilon_0 + \frac{\tilde{\mathbf{Q}}^2}{2m} - \frac{\mathbf{p}''^2}{2\mu} - \frac{(\mathbf{k}+\tilde{\mathbf{Q}}-\mathbf{p}'')^2}{2m}} \,. \tag{4.20}$$

Comparing Eq. (4.20) with the Lippmann-Schwinger equation for free two-body scattering amplitude $t(E_{X\mathrm{N}}, \mathbf{P}_{X\mathrm{N}}, \mathbf{p}, \mathbf{p}')$

$$t(E_{X\mathrm{N}}, \mathbf{P}_{X\mathrm{N}}, \mathbf{p}, \mathbf{p}') = V(\mathbf{p}-\mathbf{p}') + \int \frac{V(\mathbf{p}-\mathbf{p}'')t(E_{X\mathrm{N}}, \mathbf{P}_{X\mathrm{N}}, \mathbf{p}'', \mathbf{p}')}{E_{X\mathrm{N}} - \frac{(\mathbf{P}_{X\mathrm{N}}-\mathbf{p}'')^2}{2m} - \frac{\mathbf{p}''^2}{2\mu}} d^3p'' \,, \tag{4.21}$$

we realize that

$$\langle \mathbf{p} | \hat{\mathbf{t}}_a | \mathbf{p}' \rangle \equiv t(E_{X\mathrm{N}}, \mathbf{P}_{X\mathrm{N}}, \mathbf{p}, \mathbf{p}') \,, \tag{4.22}$$

with

$$\begin{aligned} E_{X\mathrm{N}} &= E_{XA} - \epsilon_0 + \frac{\tilde{\mathbf{Q}}^2}{2m} = \frac{\mathbf{k}^2}{2\mu} + \frac{\tilde{\mathbf{Q}}^2}{2m} \,, \\ \mathbf{P}_{X\mathrm{N}} &= \mathbf{k} + \tilde{\mathbf{Q}} \,. \end{aligned} \tag{4.22a}$$

It corresponds to the kinematics shown in Fig. 5. [In order to avoid confusion we remind the reader that the momenta **p** and **p**″ are the running variables in the integral equation (4.20), whereas **k** and **q** are the external momenta.]

Using Eqs. (4.2), (4.7), and (4.22) we can find an expression for the elastic scattering amplitude in the optimal approximation

$$\begin{aligned} F_{00}(E_{XA}, \mathbf{k}, \mathbf{k}') &= \int \Phi_0 \left[\mathbf{Q}_1' + \frac{(A-1)}{A}\mathbf{q}, \mathbf{Q}_2' - \frac{\mathbf{q}}{A}, \dots \right] \delta(\mathbf{k}'-\mathbf{p}') \\ &\quad \times \langle \mathbf{p}', \mathbf{Q}_1' + \mathbf{q}, \mathbf{Q}_2', \dots | \tau | \mathbf{p}, \mathbf{Q}_1, \mathbf{Q}_2, \dots \rangle \delta(\mathbf{p}-\mathbf{k}) \Phi_0(\mathbf{Q}_1, \mathbf{Q}_2, \dots) \prod_{i=1}^{A-1} d^3Q_i d^3Q_i' \\ &\approx t \left[E_{X\mathrm{N}} = \frac{\mathbf{k}^2}{2\mu} + \frac{\tilde{\mathbf{Q}}^2}{2m}, \mathbf{P} = \mathbf{k} + \tilde{\mathbf{Q}}, \mathbf{k}, \mathbf{k}' \right] S_{00}(\mathbf{q}) \,, \end{aligned} \tag{4.23}$$

where $S_{00}(\mathbf{q})$ is the nuclear elastic form factor [cf. with Eq. (2.18)]

$$S_{00}(\mathbf{q}) = \int \Phi_0 \left[\mathbf{Q}_1 + \frac{A-1}{A}\mathbf{q}, \mathbf{Q}_2 - \frac{\mathbf{q}}{A}, \dots \right] \Phi_0(\mathbf{Q}_1, \mathbf{Q}_2, \dots) \prod_{i=1}^{A-1} d^3Q_i \,, \tag{4.24}$$

and the two-body scattering amplitude t corresponds to the kinematics shown in Fig. 5, where $\mathbf{p} = \mathbf{k}$ and $\mathbf{p}' = \mathbf{k}' = \mathbf{k} - \mathbf{q}$. This amplitude can be identically rewritten in the corresponding projectile-nucleon c.m. frame [cf. Eq. (2.12)]

$$t \left[E_{X\mathrm{N}} = \frac{\mathbf{k}^2}{2\mu} + \frac{\tilde{\mathbf{Q}}^2}{2m}, \mathbf{P} = \mathbf{k} + \tilde{\mathbf{Q}}, \mathbf{k}, \mathbf{k}' \right] \equiv t(E_{X\mathrm{N}}^{\mathrm{c.m.}}, \mathbf{q}_r, \mathbf{q}_r') \,, \tag{4.25}$$

where

$$E_{X\mathrm{N}}^{\mathrm{c.m.}} = \frac{\mathbf{k}^2}{2\mu} + \frac{\tilde{\mathbf{Q}}^2}{2m} - \frac{(\mathbf{k}+\tilde{\mathbf{Q}})^2}{2(m+\mu)} \tag{4.26}$$

and the relative momenta

$$\begin{aligned} \mathbf{q}_r &= \mathbf{k} - \frac{\mu}{m+\mu}(k + \tilde{\mathbf{Q}}) \,, \\ \mathbf{q}_r' &= \mathbf{k}' - \frac{\mu}{m+\mu}(\mathbf{k} + \tilde{\mathbf{Q}}) \,. \end{aligned} \tag{4.27}$$

For the case $A \to \infty$, Eq. (4.23) coincides with the result of Ref. 7, where we derived the optimal approximation for the scattering on a nucleon bound by an infinitely heavy core.

B. Inelastic projectile-nucleus scattering

We are going back to Eq. (4.14). Unfortunately we cannot use the symmetry properties of the ground state nuclear wave function, Eq. (4.15), to help evaluate the $(\mathbf{Q}_1\mathbf{q}_1)/m$ term in Eq. (4.14), therefore we use a different way. Consider an expression for $I(\mathbf{p}'')$ which is equivalent to Eq. (4.14), but is obtained from Eq. (4.12) by eliminating the nuclear potential $\tilde{V}$ through the trick of applying it to the final state nuclear wave function, $\sum_{i>j} \tilde{V}_{ij}\Phi_n = (\epsilon_n - \sum_i K_i)\Phi_n$:

$$I(\mathbf{p}'') = \int \Phi_n^* \left[\mathbf{Q}_1 + \frac{A-1}{A}\mathbf{q}, \mathbf{Q}_2 - \frac{\mathbf{q}}{A}, \ldots\right] \left[\epsilon_n + \frac{(A-1)(\mathbf{q}_1^2 - \mathbf{q}^2)}{2mA} + \frac{\mathbf{Q}_1(\mathbf{q}_1 - \mathbf{q})}{m} - \bar{\epsilon}\right] \Phi_0(Q_1, Q_2, \ldots) \prod_i d^3Q_i \ . \tag{4.28}$$

From Eqs. (4.14) and (4.28) we obtain that

$$\int \Phi_n^* \left[\mathbf{Q}_1 + \frac{A-1}{A}\mathbf{q}, \mathbf{Q}_2 - \frac{\mathbf{q}}{A}, \ldots\right] \frac{\mathbf{Q}_1\mathbf{q}}{m} \Phi_0(\mathbf{Q}_1, \mathbf{Q}_2, \ldots) \prod_i d^3Q_i = \int \Phi_n^* \left[\mathbf{Q}_1 + \frac{A-1}{A}\mathbf{q}, \mathbf{Q}_2 - \frac{\mathbf{q}}{A}, \ldots\right]$$
$$\times \left[\epsilon_n - \epsilon_0 - \frac{(A-1)\mathbf{q}^2}{2mA}\right] \Phi_0(\mathbf{Q}_1, \mathbf{Q}_2, \ldots) \prod_i d^3Q_i \ . \tag{4.29}$$

In order to evaluate the contribution of the $(\mathbf{Q}_1\mathbf{q}_1)/m$ term in integral (4.14) we rewrite

$$\mathbf{Q}_1\mathbf{q}_1 \equiv \frac{(\mathbf{Q}_1\mathbf{q})(\mathbf{q}_1\mathbf{q})}{\mathbf{q}^2} + \mathbf{Q}_{1\perp}\mathbf{q}_{1\perp} \ , \tag{4.30}$$

where $\mathbf{Q}_{1\perp}$ and $\mathbf{q}_{1\perp}$ are the projections of the vectors $\mathbf{Q}_1$ and $\mathbf{q}_1 = \mathbf{k} - \mathbf{p}''$ onto the plane perpendicular to the momentum transfer $\mathbf{q}$.

Substituting (4.30) into Eq. (4.14) [or into Eq. (4.28)] we can see that the term $Q_{1\perp}q_{1\perp}$ does not contribute to the integral if the product $\Phi_n^*\Phi_0$ is symmetric under $\mathbf{Q}_{1\perp} \to -\mathbf{Q}_{1\perp}$. This will be so if the wave function of the final nuclear state

$$\Phi_n \left[\mathbf{Q}_1 + \frac{A-1}{A}\mathbf{q}, \mathbf{Q}_2 - \frac{\mathbf{q}}{A}, \ldots\right]$$

has a symmetry axis along the momentum transfer to the nucleus $\mathbf{q}$. In this case the contribution from the $(\mathbf{Q}_1\mathbf{q}_1)/m$ term in (4.14) can be easily found using Eqs. (4.29) and (4.30), and the quantity $\bar{\epsilon}$ which corresponds to $I(\mathbf{p}'') \equiv 0$ is obtained

$$\bar{\epsilon}(\mathbf{p}'', \mathbf{k}, \mathbf{q}, \epsilon_0, \epsilon_n) = \epsilon_0 + \frac{(A-1)\mathbf{q}_1^2}{2mA} - \frac{(A-1)}{2mA}\mathbf{q}_1\mathbf{q} + (\epsilon_n - \epsilon_0)\frac{(\mathbf{q}_1\mathbf{q})}{\mathbf{q}^2} \ , \tag{4.31}$$

where $\mathbf{q}_1 = \mathbf{k} - \mathbf{p}''$. We see that $\bar{\epsilon}$ does not depend on the target nucleon variables in the case of inelastic scattering as well. [For the elastic scattering ($\epsilon_n = \epsilon_0$) Eq. (4.31) goes over Eq. (4.17).]

Another case where the contribution from the $(\mathbf{Q}_{1\perp}\mathbf{q}_{1\perp})$ term vanishes is backward scattering. There the vector $\mathbf{q}$ is parallel to $\mathbf{k}$ and the integral (4.11) is symmetric under $\mathbf{q}_{1\perp} \to -\mathbf{q}_{1\perp}$. In general, the contribution from the $(\mathbf{Q}_{1\perp}\mathbf{q}_{1\perp})$ term does not vanish, but it still remains small. The reason is that the values of $|\mathbf{Q}_{1\perp}|$ which contribute in the integral (4.14) [or (4.28)] are of the order of the Fermi momentum (compare to Q_{1z} which is of the order of q). Thus the choice of $\bar{\epsilon}$ as in Eq. (4.31) would lead to the minimization of the first-order correction also in a general case of the inelastic scattering.

Now we substitute $\bar{\epsilon}$ from Eq. (4.31) into Eq. (4.5) for the Green's function G_a, obtaining

$$\langle \mathbf{p}, \mathbf{Q}_1, \mathbf{Q}_2, \ldots | G_a^{-1} | \mathbf{p}', \mathbf{Q}_1', \mathbf{Q}_2', \ldots \rangle = \left[E_{XA} - \frac{\mathbf{p}^2}{2\mu} - \epsilon_0 - \frac{(\mathbf{k} + \tilde{\mathbf{Q}} - \mathbf{p})^2}{2m} + \frac{\tilde{\mathbf{Q}}^2}{2m}\right] \times \delta(\mathbf{p} - \mathbf{p}') \prod_i \delta(\mathbf{Q}_i - \mathbf{Q}_i') \ , \tag{4.32}$$

where

$$\tilde{\mathbf{Q}} = -\frac{A-1}{2A}\mathbf{q} + (\epsilon_n - \epsilon_0)\frac{m\mathbf{q}}{\mathbf{q}^2} \ . \tag{4.33}$$

We thus obtain for G_a the same expression as in elastic scattering, Eq. (4.18), where the only difference is that the vector $\tilde{\mathbf{Q}}$ is given by Eq. (4.33) instead of Eq. (4.19). [Note that Eq. (4.33) goes over Eq. (4.19) for the case of elastic scattering, $\epsilon_n = \epsilon_0$.] Therefore for the scattering operator $\hat{\mathbf{t}}_a$ we also obtain the same expressions as for elastic scattering, Eqs. (4.20) and (4.22), with $\tilde{\mathbf{Q}}$ defined by Eq. (4.33).

$$\langle \vec{p} | \hat{t}_a | \vec{p}' \rangle \equiv t(E_{XN}, \vec{P} = \vec{k} + \vec{Q}, \vec{p}, \vec{p}') =$$

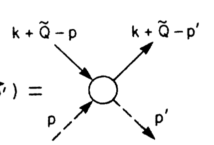

FIG. 5. Schematic representation of the projectile nucleon kinematics obtained in the optimal approximation.

With these results we can find the inelastic transition amplitude F_{0n}, Eq. (4.2), in the optimal approximation, $\tau \cong t_a$. Similarly to the elastic scattering case, Eq. (4.23), we obtain

$$F_{0n}(E_{XA},\mathbf{k},\mathbf{k}') \approx t\left[E_{XN}=\frac{\mathbf{k}^2}{2\mu}+\frac{\tilde{\mathbf{Q}}^2}{2m},\mathbf{P}=\mathbf{k}+\tilde{\mathbf{Q}},\mathbf{k},\mathbf{k}'\right]S_{0n}(\mathbf{q}) , \quad (4.34)$$

where $S_{0n}(\mathbf{q})$ is the nuclear transition form factor

$$S_{0n}(\mathbf{q})=\int \Phi_n^*\left[\mathbf{Q}_1+\frac{A-1}{A}\mathbf{q},\mathbf{Q}_2-\frac{\mathbf{q}}{A},\ldots\right] \times\Phi_0(\mathbf{Q}_1,\mathbf{Q}_2,\ldots)\prod_{i=1}^{A-1}d^3Q_i , \quad (4.35)$$

and $\tilde{\mathbf{Q}}$ is given by Eq. (4.33). The projectile-nucleon amplitude t can be given also in the corresponding projectile-nucleon c.m. frame using Eqs. (4.25)–(4.27).

Consider Eq. (4.33) which defines the momentum $\tilde{\mathbf{Q}}$. It can be rewritten in terms of the energy transfer ν to the nucleus

$$\nu=\epsilon_n+\frac{\mathbf{q}^2}{2Am}-\epsilon_0 \quad (4.36)$$

as

$$\tilde{\mathbf{Q}}=-\frac{\mathbf{q}}{2}\left[1-\frac{2m\nu}{\mathbf{q}^2}\right] . \quad (4.37)$$

We can see that $\tilde{\mathbf{Q}}$ is the *minimal* momentum of the struck nucleon which can provide the *on-shell* projectile-nucleon scattering with given momentum and energy transfer ($\mathbf{q}$ and ν) to the nucleon. We note that the same quantity appears in the Fermi gas model for electron-nucleus scattering.[20]

Let us compare the optimal approximation for the scattering on a bound nucleon, Eq. (4.34), with the weak binding approximation, Eq. (2.11). It is quite surprising that in spite of binding effects, which should complicate the final answer, our result looks much simpler than the weak binding impulse approximation formula. Firstly, it is because we obtain the scattering amplitude in a factorized form, whereas Eq. (2.11) involves three-dimensional integration over the Fermi motion. Secondly, we find that the projectile-nucleon amplitude is on the energy shell, whereas it is the off-shell amplitude in Eq. (2.11). Indeed, we can see from Eqs. (4.33) and (4.34) that the two-body energy in the optimal approximation

$$E_{XN}=\frac{\mathbf{k}^2}{2\mu}+\frac{\tilde{\mathbf{Q}}^2}{2m}=\frac{(\mathbf{k}-\mathbf{q})^2}{2\mu}+\frac{(\tilde{\mathbf{Q}}+\mathbf{q})^2}{2m}$$

is the sum of the kinetic energies for incoming and also for outcoming particles. This is precisely the on-shell condition for the scattering amplitude.

We can also see that the optimal approximation, Eq. (4.34), looks like the approximate factorized form of the impulse approximation, Eq. (2.17), if we take the average momentum of Fermi motion $\langle\mathbf{Q}\rangle=\tilde{\mathbf{Q}}$. [In fact for the elastic scattering $\tilde{\mathbf{Q}}$ coincides with the optimal choice of $\langle\mathbf{Q}\rangle$, Eq. (2.19), proposed in Ref. 14 for the factorized weak binding impulse approximation.] However, if we compare the two-body energies, E_{XN} with the total energy of the system $E_{XA}=k^2/2\mu-|\epsilon_0|$, we find that E_{XN} in Eq. (2.17) is shifted down to value $\langle\mathbf{Q}\rangle^2/2m(A-1)$; whereas E_{XN} in Eq. (4.34) is shifted up to value $\tilde{\mathbf{Q}}^2/2m+|\epsilon_0|$. It is in accordance with estimation of the binding effects done in the previous section.

C. Relativistic kinematics

Although we made all derivations in the framework of nonrelativistic potential theory, the final result can be easily extended for the case of relativistic kinematics. Indeed, the momentum $\tilde{\mathbf{Q}}$, Eq. (4.37), is the minimal momentum of the struck nucleon N in the *on-shell* scattering $X+\mathrm{N}\to X'+\mathrm{N}'$, where the momenta of the projectile X in the initial and final states are $\mathbf{k}$ and $\mathbf{k}'$. In the relativistic case we obtain

$$\tilde{\mathbf{Q}}=-\frac{\mathbf{q}}{2}\left[1-\frac{\nu}{|\mathbf{q}|}\left[1+\frac{4m^2}{\mathbf{q}^2-\nu^2}\right]^{1/2}\right] , \quad (4.38)$$

where $\mathbf{q}=\mathbf{k}-\mathbf{k}'$ and

$$\nu=(\mathbf{k}^2+\mu^2)^{1/2}-(\mathbf{k}'^2+\mu^2)^{1/2}$$

are the momentum and energy transfers to the nucleus. The two-body kinetic energy E_{XN}, Eq. (4.38), is also replaced by its relativistic equivalent expression

$$E_{XN}^{\rm kin}=(\mathbf{k}^2+\mu^2)^{1/2}+(\tilde{\mathbf{Q}}^2+m^2)^{1/2}-\mu-m , \quad (4.39)$$

and the relativistic kinematics of the projectile-nucleon amplitude in Eq. (4.34) is uniquely defined.

We should only point out that this relativistic extension of the optimal approximation is in some extent *ad hoc*. In fact, the projectile can be treated relativistically in the derivation above. However, the relativistic treatment of the target nucleon has to involve the relativistic nuclear wave functions, which considerably complicates the problem.

V. CORRECTIONS TO THE OPTIMAL APPROXIMATION

In this section we consider the nonvanishing correction terms, which are of order h^2 and higher ($h\equiv G_a^{-1}-G^{-1}$), Eq. (3.5). In fact, the h^2 correction term has been analyzed earlier in Refs. 7 and 10 for the case of elastic scattering on a nucleon bound by an infinitely heavy core. These results can be generalized for the case of the many-body nuclear interaction but this extension is not significant at this point. We rather consider the same model as in Refs. 7 and 10. However, we obtain a new expression for the correction to the optimal approximation, which is much more useful than that derived earlier in Refs. 7 and 10.

Consider the expansion (3.5) for the scattering operator τ in terms of the approximate operators t_a and G_a, Eqs. (4.18) and (4.22), for the case of scattering on a nucleon bound by an infinitely heavy core. For the elastic scattering amplitude we obtain[7,10] [cf. Eq. (4.23) for $A\rightarrow\infty$]

$$F_{00}\left[E_{XA}=\frac{k^2}{2\mu}+\epsilon_0,\mathbf{k},\mathbf{k}'\right]=\langle\Phi_0,\mathbf{k}'\,|\,\tau\,|\,\Phi_0,\mathbf{k}\rangle$$

$$=\int\left\{t(E_{XN},\mathbf{P},\mathbf{k},\mathbf{k}')+\int\frac{t(E_{XN},\mathbf{P},\mathbf{k},\mathbf{p}_1)t(E_{XN},\mathbf{P},\mathbf{p}_1,\mathbf{k}')}{\left[E_{XN}-\frac{\mathbf{p}_1^2}{2\mu}-\frac{(\mathbf{P}-\mathbf{p}_1)^2}{2m}\right]^2}\frac{\mathbf{Q}(\mathbf{k}-\mathbf{p}_1)}{m}d^3p_1\right.$$

$$+\int\frac{t(E_{XN},\mathbf{P},\mathbf{k},\mathbf{p}_1)t(E_{XN},\mathbf{P},\mathbf{p}_1,\mathbf{k}')}{\left[E_{XN}-\frac{\mathbf{p}_1^2}{2\mu}-\frac{(\mathbf{P}-\mathbf{p}_1)^2}{2m}\right]^3}\frac{\mathbf{Q}(\mathbf{k}-\mathbf{p}_1)\mathbf{Q}(\mathbf{k}'-\mathbf{p}_1)}{m^2}d^3p_1$$

$$+\int\frac{t(E_{XN},\mathbf{P},\mathbf{k},\mathbf{p}_1)t(E_{XN},\mathbf{P},\mathbf{p}_1,\mathbf{p}_2)t(E_{XN},\mathbf{P},\mathbf{p}_2,\mathbf{k}')}{\left[E_{XN}-\frac{\mathbf{p}_1^2}{2\mu}-\frac{(\mathbf{P}-\mathbf{p}_1)^2}{2m}\right]^2\left[E_{XN}-\frac{\mathbf{p}_2^2}{2\mu}-\frac{(\mathbf{P}-\mathbf{p}_2)^2}{2m}\right]^2}$$

$$\left.\times\frac{[\mathbf{Q}(\mathbf{k}-\mathbf{p}_1)][\mathbf{Q}(\mathbf{k}'-\mathbf{p}_2)]}{m^2}d^3p_1d^3p_2\right\}\Phi_0\left[\mathbf{Q}+\frac{\mathbf{q}}{2}\right]\Phi_0\left[\mathbf{Q}-\frac{\mathbf{q}}{2}\right]d^3Q+\cdots\,, \tag{5.1}$$

where the projectile-nucleon energy E_{XN} and the total projectile-nucleon momentum $\mathbf{P}$ are given by Eq. (4.22). In our case where $A\rightarrow\infty$ they are

$$E_{XN}=\frac{\mathbf{k}^2}{2\mu}+\frac{\mathbf{q}^2}{8m}\,,$$
$$\mathbf{P}=\mathbf{k}-\frac{\mathbf{q}}{2}=\frac{\mathbf{k}+\mathbf{k}'}{2}\,. \tag{5.2}$$

The first term in the expansion (5.1) is the optimal approximation for the elastic amplitude, Eqs. (4.23) and (4.24), where we keep explicitly the elastic form factor as an integral over the overlap of the nuclear wave functions. The second term in (5.1) is the first-order correction to the optimal approximation. After d^3Q integration this term is zero. However, it would be useful to retain it in the expansion (5.1). The third and the fourth terms in (5.1) represent the contribution from the h^2 term [Eq. (3.5)] to the elastic amplitude.[7,10]

We now try to resum the expansion (5.1) into an integral over projectile-nucleon amplitude $t(E'_{XN},\mathbf{P}',\mathbf{k},\mathbf{k}')$, where E'_{XN} and $\mathbf{P}'$ are different from those given by Eq. (5.2). We again use a general relation (3.4), where τ is replaced by the two-body operator describing the projectile-nucleon scattering with total energy E'_{XN} and total momentum $\mathbf{P}$. We obtain

$$t(E'_{XN},\mathbf{P}',\mathbf{k},\mathbf{k}')=t(E_{XN},\mathbf{P},\mathbf{k},\mathbf{k}')+\int\frac{t(E_{XN},\mathbf{P},\mathbf{k},\mathbf{p}_1)t(E_{XN},\mathbf{P},\mathbf{p}_1,\mathbf{k}')}{\left[E_{XN}-\frac{\mathbf{p}_1^2}{2\mu}-\frac{(\mathbf{P}-\mathbf{p}_2)^2}{2m}\right]^2}\left[\delta E-\frac{\mathbf{\Delta}(\mathbf{P}-\mathbf{p}_1-\mathbf{\Delta}/2)}{m}\right]d^3p_1$$

$$+\int\frac{t(E_{XN},\mathbf{P},\mathbf{k},\mathbf{p}_1)t(E_{XN},\mathbf{P},\mathbf{p}_1,\mathbf{k}')}{\left[E_{XN}-\frac{\mathbf{p}_1^2}{2\mu}-\frac{(\mathbf{P}-\mathbf{p}_1)^2}{2m}\right]^3}\left[\delta E-\frac{\mathbf{\Delta}(\mathbf{P}-\mathbf{p}_1-\mathbf{\Delta}/2)}{m}\right]^2d^3p_1$$

$$+\int\frac{t(E_{XN},\mathbf{P},\mathbf{k},\mathbf{p}_1)t(E_{XN},\mathbf{P},\mathbf{p}_1,\mathbf{p}_2)t(E_{XN},\mathbf{P},\mathbf{p}_2,\mathbf{k}')}{\left[E_{XN}-\frac{\mathbf{p}_1^2}{2\mu}-\frac{(\mathbf{P}-\mathbf{p}_1)^2}{2m}\right]^2\left[E_{XN}-\frac{\mathbf{p}_2^2}{2\mu}-\frac{(\mathbf{P}-\mathbf{p}_1)^2}{2m}\right]^2}$$

$$\times\left[\delta E-\frac{\mathbf{\Delta}(\mathbf{P}-\mathbf{p}_1-\mathbf{\Delta}/2)}{m}\right]\left[\delta E-\frac{\mathbf{\Delta}(\mathbf{P}-\mathbf{p}_2-\mathbf{\Delta}/2)}{m}\right]d_1^3d^3p_2+\cdots\ , \tag{5.3}$$

where

$$\delta E=E_{XN}-E'_{XN}\ ,\qquad \mathbf{\Delta}=\mathbf{P}-\mathbf{P}'\ . \tag{5.4}$$

First consider the forward scattering $\mathbf{k}=\mathbf{k}'$ in Eq. (5.1). [In this case $E_{XN}=\mathbf{k}^2/2\mu$ and $\mathbf{P}=\mathbf{k}$, Eq. (5.2).] Comparing expansions (5.1) and (5.3) we find that the first four terms of expansion (5.1) coincide with the corresponding terms of expansion (5.3) if

$$E'_{XN}=\frac{\mathbf{k}^2}{2\mu}+\frac{\mathbf{Q}^2}{2m}\ ,\qquad \mathbf{P}'=\mathbf{k}+\mathbf{Q}\ , \tag{5.5}$$

and therefore

$$F_{00}(E_{XA},\mathbf{k},\mathbf{k})\cong\int t\left[E'_{XN}=\frac{k^2}{2\mu}+\frac{Q^2}{2m},\mathbf{P}'=\mathbf{k}+\mathbf{Q},\mathbf{k},\mathbf{k}\right]\times|\Phi_0(Q)|^2d^3Q\ . \tag{5.6}$$

Unfortunately the contribution from h^3 and the higher-order terms in Eq. (5.1) are not fully taken into account by the corresponding terms in expansion (5.3). It is due to the commutators of nuclear potential with the kinetic energy term, which appear in h^3 and higher-order terms in Eq. (5.1), but not in Eq. (5.3).[21]

We thus found that the elastic amplitude F_{00} for *forward scattering* can be approximated as a Fermi-averaged elementary projectile-nucleon amplitude, Eq. (5.6). It is similar to the result of the weak binding impulse approximation, Eq. (2.11), for the elastic forward scattering. However, the projectile-nucleon amplitude in Eq. (5.6) is the *on-shell* amplitude. The optimal approximation corresponds to the factorization of the integral (5.6) when the two-body amplitude t is taken at $\mathbf{Q}=0$. It is clearly the region which mainly contributes in the integral (5.6).

If the scattering is not in the forward direction ($\mathbf{k}\neq\mathbf{k}'$) we cannot resum (5.1) into an integral over the one on-shell projectile-nucleon amplitude. However, we can rewrite (5.1) as an integral over the combination of different on- and off-shell projectile-nucleon amplitudes. Indeed, consider again Eq. (5.1). First, we use relations

$$[\mathbf{Q}(\mathbf{k}-\mathbf{p}_1)][\mathbf{Q}(\mathbf{k}'-\mathbf{p}_1)]=[\mathbf{Q}(\mathbf{P}-\mathbf{p}_1)]^2-\left[\frac{\mathbf{Qq}}{2}\right]^2$$

and

$$[\mathbf{Q}(\mathbf{k}-\mathbf{p}_1)][\mathbf{Q}(\mathbf{k}'-\mathbf{p}_2)]=[\mathbf{Q}(\mathbf{P}-\mathbf{p}_1)][\mathbf{Q}(\mathbf{P}-\mathbf{p}_2)]-\left[\frac{\mathbf{Qq}}{2}\right]^2+\left[\mathbf{Q}\frac{\mathbf{q}}{2}\right][\mathbf{Q}(\mathbf{p}_1-\mathbf{p}_2)] \tag{5.7}$$

in the third and fourth terms of expansion (5.1). Notice that the term

$$\left[\mathbf{Q}\frac{\mathbf{q}}{2}\right][\mathbf{Q}(\mathbf{p}_1-\mathbf{p}_2)]$$

does not contribute in (5.1), since the corresponding part of the integrand changes the sign under the interchange $\mathbf{p}_1\leftrightarrow\mathbf{p}_2$. Then using Eq. (5.3) and choosing in an appropriate way the values of E'_{XN} and $\mathbf{P}'$ we can resum Eq. (5.1) thus obtaining

$$F_{00}(E_{XA},\mathbf{k},\mathbf{k}')\simeq t(E_{XN},\mathbf{P},\mathbf{k},\mathbf{k}')\int \Phi_0\left[\mathbf{Q}+\frac{\mathbf{q}}{2}\right]\Phi_0\left[\mathbf{Q}-\frac{\mathbf{q}}{2}\right]d^3Q$$
$$+\int\left[t\left[E_{XN}+\frac{\mathbf{Q}^2}{2m},\mathbf{P}+\mathbf{Q},\mathbf{k},\mathbf{k}'\right]-t\left[E_{XN}-\frac{\mathbf{qQ}}{2m},\mathbf{P},\mathbf{k},\mathbf{k}'\right]\right]\Phi_0\left[Q+\frac{q}{2}\right]\Phi_0\left[\mathbf{Q}-\frac{\mathbf{q}}{2}\right]d^3Q\ , \tag{5.8}$$

where E_{XN} and $\mathbf{P}$ are given in Eq. (5.2).

The first term in Eq. (5.8) is the optimal approximation. The second term in Eq. (5.8) is the correction to the optimal approximation. Expanding two-body amplitudes t in this term through $t(E_{XN},\mathbf{P},\mathbf{k},\mathbf{k}')$ we exactly reproduce h^2 correction terms in the expansion (5.1). [h^3 and higher-order terms in (5.1) are not fully reproduced by the second term in Eq. (5.8) for the same reason as explained above.[21]] However, Eq. (5.8) is much more simple for the analysis of correction to the optimal approximation than the direct evaluation of the third and fourth terms in Eq. (5.1), as has been done in Refs. 7 and 10. We can also see that Eq. (5.8) goes over Eq. (5.6) for $\mathbf{q}=0$.

VI. ELASTIC SCATTERING IN THE FIRST-ORDER OPTICAL POTENTIAL THEORY

In the previous sections we derived the optimal approximation for elastic and inelastic scattering of projectile on a single bound nucleon. Now we deal with the full scattering amplitude, which includes the rescattering of the projectile on different nucleons. In this section we consider the case of elastic projectile-nucleus scattering. The elastic projectile-nucleus amplitude $T_{00}(E_{XA},\mathbf{k},\mathbf{k}')$ (the kinematics is in the nuclear laboratory frame) is the matrix element of the scattering operator T

$$T_{00}(E_{XA},\mathbf{k},\mathbf{k}')=\langle\Phi_0,\mathbf{k}'\,|\,T\,|\,\Phi_0,\mathbf{k}\rangle\ , \tag{6.1}$$

which satisfies the Lippmann-Schwinger equation

$$T=\sum_i V_i+\sum_i V_iGT\ . \tag{6.2}$$

Here the potential V_i describes the interaction of the projectile with nucleon i, and the Green's function G is the same as in Eq. (4.3). The projectile-nucleus energy E_{XA} is given by Eq. (2.5). The elastic scattering amplitude T_{00} can be written in a form of the two-body Lippmann-Schwinger equation[3]

$$T_{00}=U'_{\rm opt}+U'_{\rm opt}P_0GT_{00}\ , \tag{6.3}$$

where $U'_{\rm opt}$ is the optical potential and the operator P_0 projects the Green's function G into a ground nuclear state. The optical potential $U'_{\rm opt}$ is given by a multiple scattering expansion[3]

$$U'_{\rm opt}=\langle\Phi_0,\mathbf{p}'\,|\,\sum_i\tau'_i+\sum_{i\neq 1}\tau'_iQG\tau'_j+\sum_{\substack{i\neq j\\ j\neq k}}\tau'_iQG\tau'_jQG\tau'_k+\cdots\,|\,\Phi_0,\mathbf{p}\rangle\ , \tag{6.4}$$

where the projection operator Q excludes the ground state, $Q=1-P_0$, and the scattering operators τ' satisfy the equation

$$\tau'_i=V_i+V_iQG\tau'_i\ . \tag{6.5}$$

It differs from a similar equation for τ, Eq. (4.3), only by the projection operator Q in the Green's function.

The first term in expansion (6.4)

$$U'^{(1)}_{\rm opt}=\langle\Phi_0,\mathbf{p}'\,|\,\sum_i\tau'_i\,|\,\Phi_0,\mathbf{p}\rangle$$
$$=A\langle\Phi_0,\mathbf{p}'\,|\,\tau'\,|\,\Phi_0,\mathbf{p}\rangle\equiv A\tau'_{00} \tag{6.6}$$

is the first-order optical potential. The higher order terms in (6.4) are proportional to nucleon correlation functions or include reflections (local field corrections). In this paper we concentrate only on the first-order optical potential, Eq. (6.6).

The optimal approximation for the first-order optical potential can be found in the same way as in the previous section. The complications arise only from the projection operator Q in the Green's function of Eq. (6.5). However, this operator can be eliminated easily if we express τ' through τ, defined by Eq. (4.3), which does not contain the projection operator Q. Using Eq. (3.1) we obtain

$$\tau'=\tau-\tau P_0G\tau'\ . \tag{6.7}$$

Substituting Eq. (6.7) into Eq. (6.6), and Eq. (6.6) into Eq. (6.3) for the first-order optical potential, we find

$$\begin{aligned}T_{00}&=A\tau'_{00}+A\tau'_{00}P_0GT_{00}\\&=A\tau_{00}-\tau_{00}P_0GA\tau'_{00}+A\tau_{00}P_0GT_{00}\\&\quad-\tau_{00}P_0GA\tau'_{00}P_0GT_{00}\\&=A\tau_{00}+(A-1)\tau_{00}P_0GT_{00}\ ,\end{aligned} \tag{6.8}$$

where

$$\tau_{00}=\langle\Phi_0,\mathbf{p}'\,|\,\tau\,|\,\Phi_0,\mathbf{p}\rangle\ , \tag{6.9}$$

and the operator τ is defined by Eq. (4.3) where the Green's function G *does not involve* the projection operator Q. Equation (6.8) can be rewritten in a form

$$\frac{A-1}{A}T_{00}=U^{(1)}_{\rm opt}\left[1+P_0G\frac{A-1}{A}T_{00}\right]\ , \tag{6.10}$$

with

$$U_{\rm opt}^{(1)}=(A-1)\langle\Phi_0,\mathbf{p}'\,|\,\tau\,|\,\Phi_0,\mathbf{p}\rangle\ . \tag{6.11}$$

These are the Kerman, McManus, and Thaler (KMT) equations for the first-order optical potential,[2] whereas Eqs. (6.3) and (6.6) are Watson's equations for the first-order optical potential, which both are fully equivalent.[22]

Now we can apply the optimal approximation to the KMT first-order optical potential, Eq. (6.11), using the results of Sec. IV. For the sake of convenience we considered there the nucleus in the laboratory frame, in which the kinematics of the optimal approximation has a simple form both for elastic and inelastic single scattering amplitudes. However, the nucleus laboratory frame is not convenient for a treatment of Eq. (6.10) for the full elastic amplitude T_{00}. A more appropriate frame would be the projectile-nucleus c.m. frame. The transformation from c.m. to the laboratory frame can be done easily by using the Galilean invariance of the scattering amplitude:

$$t(E,\mathbf{K},\mathbf{k},\mathbf{k}')\equiv t\left[E+\frac{\mu+M}{2}\mathbf{v}^2-\mathbf{K}\mathbf{v},\mathbf{K}-(M+\mu)\mathbf{v},\mathbf{k}-\mu\mathbf{v},\mathbf{k}'-\mu\mathbf{v}\right], \tag{6.12}$$

where E and $\mathbf{K}$ are the total energy and momentum of the system; M and μ are the masses of target and projectile; $\mathbf{k}$ and $\mathbf{k}'$ are the initial and final momenta of the projectile; and $\mathbf{v}$ is the velocity of the frame. The transformation to the laboratory frame corresponds to $\mathbf{v}=\mathbf{K}-\mathbf{k}/M$. Therefore the projectile nucleus transition amplitude $F_{0n}(E_{XN},\mathbf{K},\mathbf{k},\mathbf{k}')$ can be rewritten in the nucleus laboratory frame as

$$F_{0n}(E_{XA},\mathbf{K},\mathbf{k},\mathbf{k}')=F_{0n}(E_{XA}^{*},\mathbf{K}^{*}=\mathbf{k}^{*},\mathbf{k}^{*},\mathbf{k}'^{*})\equiv F_{0n}(E_{XA}^{*},\mathbf{k}^{*},\mathbf{k}'^{*})\ , \tag{6.13}$$

where

$$E_{XA}^{*}=E_{XA}+\frac{\mathbf{k}^2-\mathbf{K}^2}{2M_A}+\frac{\mu}{2M_A^2}(\mathbf{K}-\mathbf{k})^2 \tag{6.14}$$

and

$$\mathbf{k}^{*}=\mathbf{k}-\frac{\mu}{M_A}(\mathbf{K}-\mathbf{k})\ ,\qquad \mathbf{k}'^{*}=\mathbf{k}'-\frac{\mu}{M_A}(\mathbf{K}-\mathbf{k})\ . \tag{6.15}$$

Afterwards the optimal approximation can be applied straightforwardly.

Consider again Eq. (6.10) for the full elastic scattering amplitude. In the projectile-nucleus center of mass frame it is

$$\frac{A-1}{A}T_{00}(E_{XA}^{\rm c.m.},\mathbf{p},\mathbf{p}')=U_{\rm opt}^{(1)}(E_{XA}^{\rm c.m.}\mathbf{p},\mathbf{p}')+\int\frac{U_{\rm opt}^{(1)}(E_{XA}^{\rm c.m.},\mathbf{p},\mathbf{p}'')\frac{A-1}{A}T_{00}(E_{XA}^{\rm c.m.},\mathbf{p}''\mathbf{p}')}{E_{XA}^{\rm c.m.}-\epsilon_0-\frac{\mathbf{p}''^2}{2\overline{M}_A}}d^3p''\ , \tag{6.16}$$

where $\overline{M}_A=\mu M_A/(\mu+M_A)$, and

$$E_{XA}^{\rm c.m.}=\frac{(\mathbf{k}^{\rm c.m.})^2}{2\overline{M}_A}+\epsilon_0=\frac{(\mathbf{k}'^{\rm c.m.})^2}{2\overline{M}_A}+\epsilon_0\ . \tag{6.17}$$

Here $\mathbf{k}^{\rm c.m.}$ ($\mathbf{k}'^{\rm c.m.}$) is the projectile-nucleus center-of-mass (on-shell) momentum. We remind the reader that $\mathbf{p}$ ($\mathbf{p}'$) are the variables in the integral equation (6.16) and therefore can be on or off the energy shell.

Using Eqs. (4.22) and (6.13)–(6.15) for $\mathbf{K}=0$ we find that the optimal approximation for the first-order optical potential $U_{\rm opt}^{(1)}$ is

$$U_{\rm opt}^{(1)}(E_{XA}^{\rm c.m.},\mathbf{p},\mathbf{p}')\cong(A-1)t\left[E_{XN},\mathbf{P},\mathbf{p}+\frac{\mu}{M_A}\mathbf{p},\mathbf{p}'+\frac{\mu}{M_A}\mathbf{p}\right]\times S_{00}(\mathbf{p}-\mathbf{p}')\ , \tag{6.18}$$

where

$$E_{XN}=E_{XA}^{\rm c.m.}+\frac{p^2}{2M_A}\left[1+\frac{\mu}{M_A}\right]-\epsilon_0+\left[\frac{A-1}{2A}\right]^2\frac{(\mathbf{p}-\mathbf{p}')^2}{2m}\ ,\qquad \mathbf{P}=\frac{M_A+\mu}{M_A}\mathbf{p}-\frac{A-1}{2A}(\mathbf{p}-\mathbf{p}')\ . \tag{6.19}$$

The amplitude t can also be rewritten in the corresponding projectile-nucleon c.m. frame, Eqs. (4.25)–(4.27),

$$t\left[E_{XN},\mathbf{P},\mathbf{p}+\frac{\mu}{M_A}\mathbf{p},\mathbf{p}'+\frac{\mu}{M_A}\mathbf{p}\right]\equiv t^{\rm c.m.}\left[E_{XN}^{\rm c.m.},\mathbf{p}+\frac{\mu}{M_A}\mathbf{p}-\frac{\mu}{m+\mu}\mathbf{P},\mathbf{p}'+\frac{\mu}{M_A}\mathbf{p}-\frac{\mu}{m+\mu}\mathbf{P}\right], \tag{6.20}$$

where

$$E_{XN}^{\text{c.m.}}=E_{XN}-\frac{\mathbf{P}^2}{2(m+\mu)} \ . \tag{6.21}$$

We have thus obtained the optimal approximation for the full scattering amplitude in the framework of the first-order optical potential, Eqs. (6.16)–(6.18). As in the case of the single scattering amplitude, Eq. (4.23), the first-order optical potential is found in a factorized form and therefore does not involve the Fermi averaging. The Fermi averaging appears only in the second-order correction to the optimal approximation, Eq. (5.8) (the first-order correction is zero). This correction can be taken into account (if it is necessary) by using Eq. (5.8) in the calculation of the first-order optical potential. But, in general, the correction term is small.[7,10]

VII. DISTORTED WAVE OPTIMAL APPROXIMATION

To calculate the full inelastic amplitude T_{0n}, we use the distorted wave approach. A consistent microscopic KMT-type theory for distorted wave treatment of inelastic scattering has been recently proposed by Picklesimer, Tandy, and Thaler (PTT).[23,24] Finally, they obtained for the first-order transition amplitude for inelastic scattering [Eq. (5.30) in Ref. 24] the following expression (we use here our notations):

$$T_{0n}(E_{XA},\mathbf{k},\mathbf{k}')=A\langle\Phi_n,\widetilde{\Psi}_{\mathbf{k}'}^{(-)}\,|\,\tau\,|\,\Psi_{\mathbf{k}}^{(+)},\Phi_0\rangle \ , \tag{7.1}$$

where $\Psi_{\mathbf{k}}^{(+)}$, $\widetilde{\Psi}_{\mathbf{k}'}^{(-)}$ are KMT-type projectile distorted wave functions for the initial and final projectile-nuclear states. Notice that according to the results of PTT the initial and final distorted wave functions $\Psi_{\mathbf{k}}^{(+)}$ and $\widetilde{\Psi}_{\mathbf{k}'}^{(-)}$ satisfy different equations:

$$|\,\Psi_{\mathbf{k}}^{(+)}\rangle=|\,\mathbf{k}\rangle+(A-1)G_0\tau_{00}\,|\,\Psi_{\mathbf{k}}^{(+)}\rangle \tag{7.2}$$

and

$$\langle\widetilde{\Psi}_{\mathbf{k}'}^{(-)}\,|=\langle\mathbf{k}'\,|+\langle\widetilde{\Psi}_{\mathbf{k}'}^{(-)}\,|\,G_n\widetilde{\tau}_{nn}(A-1) \ , \tag{7.3}$$

where τ_{00} is given by Eq. (6.9), but $\widetilde{\tau}_{nn}$ is given by a system of coupled equations

$$\widetilde{\tau}_{nn}=\tau_{nn}-\tau_{n0}G_0\widetilde{\tau}_{0n} \ ,$$
$$\widetilde{\tau}_{0n}=\tau_{0n}-\tau_{00}G_0\widetilde{\tau}_{0n} \ . \tag{7.4}$$

Here $G_0=(E_{XA}-K_X-K_A-\epsilon_0)^{-1}$ and

$$\tau_{0n}=\langle\Phi_n,\mathbf{p}'\,|\,\tau\,|\,\Phi_0,\mathbf{p}\rangle \ ,$$
$$\tau_{nn}=\langle\Phi_n,\mathbf{p}'\,|\,\tau\,|\,\Phi_n,\mathbf{p}\rangle \ . \tag{7.5}$$

(Notice that in the case of the off diagonal transitions τ_{0n} are weak relative to the diagonal transition, τ_{00} and τ_{nn}; one can approximate $\widetilde{\tau}_{nn}\approx\tau_{nn}$ and therefore $\widetilde{\Psi}_{\mathbf{k}'}^{(-)}\approx\Psi_{\mathbf{k}'}^{(-)}$ which is the standard first-order KMT distortion for scattering from the excited state.[24])

We can see that all the ingredients in the distorted wave approach of PTT are the "elastic," τ_{00}, τ_{nn}, and "inelastic," τ_{0n}, amplitudes for the scattering on the single bound nucleon. Therefore the optimal approximation can be applied for calculation of these amplitudes straightforwardly using Eqs. (4.23) and (4.24) for elastic and Eqs. (4.34) and (4.35) for inelastic cases. The amplitude τ_{nn} is given by the same equations as the amplitude τ_{00}, where only the form factor $S_{00}(\mathbf{q})$ is replaced by

$$S_{nn}(\mathbf{q})=\int\Phi_n^*\left[\mathbf{Q}_1+\frac{A-1}{A}\mathbf{q},\mathbf{Q}_2-\frac{\mathbf{q}}{A},\ldots\right]\times\Phi_n(\mathbf{Q}_1,\mathbf{Q}_2,\ldots)\prod_{i=1}^{A-1}d^3Q_i \ . \tag{7.6}$$

Equation (7.1) can be rewritten in more explicit form as

$$T_{0n}(E_{XA}^{\text{c.m.}},\mathbf{k},\mathbf{k}')=A\int\psi_{\mathbf{k}'}^{(-)*}(\mathbf{p}')\langle\mathbf{p}',\Phi_n\,|\,\tau\,|\,\mathbf{p},\Phi_0\rangle\times\psi_{\mathbf{k}}^{(+)}(\mathbf{p})d^3p\,d^3p' \ . \tag{7.7}$$

As in the previous section we present our result in the projectile-nucleus c.m. frame. Therefore,

$$E_{XA}^{\text{c.m.}}=\frac{\mathbf{k}^2}{2\overline{M}_A}+\epsilon_0=\frac{\mathbf{k}'^2}{2\overline{M}_A}+\epsilon_n \ , \tag{7.8}$$

and $\mathbf{k},\mathbf{k}'$ are the initial and final projectile c.m. momentum. The optimal approximation for the matrix element $\langle\mathbf{p}',\Phi_n\,|\,\tau\,|\,\mathbf{p},\Phi_0\rangle$ can be found in the same way as we did in Sec. IV for the calculation of the single scattering inelastic amplitude, Eq. (4.34). We have only to perform the Galilean transformation to the corresponding laboratory frame in which the optimal approximation, Eq. (4.34), has been derived. Therefore, using Eqs. (6.13)–(6.15) for $\mathbf{K}=0$, and Eqs. (4.33)–(4.35) we find

$$\langle\mathbf{p}',\Phi_n\,|\,\tau\,|\,\mathbf{p},\Phi_0\rangle\cong t\left[E_{XN},\mathbf{P},\mathbf{p}+\frac{\mu}{M_A}\mathbf{p},\mathbf{p}'+\frac{\mu}{M_A}\mathbf{p}\right]S_{0n}(\mathbf{p}-\mathbf{p}') \ , \tag{7.9}$$

where

$$E_{XN}=E_{XA}^{\text{c.m.}}+\frac{p^2}{2M_A}\left[1+\frac{\mu}{M_A}\right]-\epsilon_0+\frac{(\mathbf{p}-\mathbf{p}')^2}{2m}\left[\frac{A-1}{2A}-(\epsilon_n-\epsilon_0)\frac{m}{(\mathbf{p}-\mathbf{p}')^2}\right]^2 \ ,$$
$$\mathbf{P}=\frac{M_A+\mu}{M_A}\mathbf{p}-\left[\frac{A-1}{2A}-(\epsilon_n-\epsilon_0)\frac{m}{(\mathbf{p}-\mathbf{p}')^2}\right](\mathbf{p}-\mathbf{p}') \ . \tag{7.10}$$

Equations (7.7)–(7.10) define optimal approximation for the inelastic transition amplitude in the first-order PTT distorted wave theory. We should mention that the projectile-nucleon amplitude t can be on- or off-energy shell. Therefore, there appears a singular point $\mathbf{p}=\mathbf{p}'$ in Eqs. (7.9) and (7.10), which cannot be reached for the on-shell scattering. Fortunately, this point does not contribute, since the inelastic form factor $S_{0n}(\mathbf{p}-\mathbf{p}')$ in Eq. (7.10) is zero if $\mathbf{p}=\mathbf{p}'$. It is due to the orthogonality of nuclear wave functions of the ground and excited states, Eq. (4.35).

VIII. SUMMARY

In this paper we have concentrated on the binding potential effects in projectile-nucleus scattering. These effects are neglected in the weak binding impulse approximation; however, we have showed that it is impossible to determine consistently the effective two-body kinematics in the projectile-target nucleon scattering if we do not involve the binding potential. One usually assumes that the inclusion of the binding potential results in large complications. Here we have demonstrated that this is not the case. The compensation between the binding potential and the Fermi motion kinetic energy leads to a factorization of the single scattering amplitude into the projectile-nucleon amplitude and nuclear transition form factor. Therefore, our optimal approximation which is based on this compensation is much simpler for applications than the weak binding impulse approximation. In the latter, the Fermi average of the projectile-nucleon amplitude is required. In our case the Fermi average appears only in the second-order correction term to the optimal approximation (the first-order correction is zero) which is much smaller than the correction term to the weak binding impulse approximation. We also found a way of recasting the corrections to the optimal approximation into a simple, practical formula.

Since the binding potential is included, the optimal approximation has no inconsistencies in the effective projectile-nucleon kinematics. We found that for the on-shell *projectile-nucleus* (elastic or inelastic) single scattering the corresponding *projectile-nucleon* amplitude in the optimal approximation is also on the mass shell. This result is very important for the interpretation of proton-nucleus large angle data.[11,25]

Finally we obtained the optimal approximation for the first-order optical potential theory for elastic scattering, and in the framework of the distorted wave approximation for inelastic scattering. The result is very similar to the optimal approximation for the *off-shell* single scattering amplitude, only the nuclear recoil effects make some difference.

We believe that the optimal approximation can be very useful for the treatment of different nuclear reactions. It is especially important for the analysis of inclusive nuclear reactions with large momentum transfer. In this case it naturally leads to the two-body scaling observed in these reactions.[11,26]

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