Triad of three-particle Lippmann-Schwinger equations

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It is shown that usual derivations of the inhomogeneous and homogeneous multiparticle Lippmann-Schwinger equations are lacking from a rigorous mathematical basis. Then, contrary to a more often than not proclaimed assertion, the validity of these equations is still to be settled.

I. INTRODUCTION

In a recent paper,[1] Levin and Sandhas reconsider the validity of the triad of homogeneous and inhomogeneous three-particle Lippmann-Schwinger (LS) equations

$$\psi^+_{\lambda\mathbf{k}}=\delta_{\lambda\lambda'}\varphi_{\lambda\mathbf{k}}+G^+_{\lambda'}(E)V^{\lambda'}\psi^+_{\lambda\mathbf{k}} . \tag{1.1}$$

Here, we are concerned with the collision of three spinless particles interacting via short-range pair potentials $V_1=v_{23}$ and their cycle. We use the notations $V^0=\sum_{\lambda=1}^3 V_\lambda$, $V^\lambda=V^0-V_\lambda$. Denoting by H_λ, $\lambda=0,1,2,3$, the four channel Hamiltonians, the full Hamiltonian (center of mass energy removed) is $H=H_\lambda+V^\lambda$.

We consider a collision process where, initially, the λ pair is in a bound state f_λ of energy $\hat{E}_\lambda$. Then $\varphi_{\lambda\mathbf{k}}$ obeys the equation $(E-H_\lambda)\varphi_{\lambda\mathbf{k}}=0$, $E=\hat{E}_\lambda+(k^2/2M_\lambda)$. In position space it reads

$$\begin{aligned}&\varphi_{\lambda\mathbf{k}}(\mathbf{X})=f_\lambda(\boldsymbol{\rho}_\lambda)e^{i\mathbf{k}_\lambda\cdot\mathbf{r}_\lambda}/(2\pi)^{3/2} ;\\&\mathbf{X}=\{\boldsymbol{\rho}_\lambda,\mathbf{r}_\lambda\}\in\mathbb{R}^6 .\end{aligned} \tag{1.2}$$

In this two-fragment channel state $\boldsymbol{\rho}_\lambda$ is the relative separation in the λ pair and $\mathbf{r}_\lambda$ is the relative position of the third particle.

The stationary collision state which evolves from $\varphi_{\lambda\mathbf{k}}$ is denoted by $\psi^+_{\lambda\mathbf{k}}$ and satisfies $(E-H)\psi^+_{\lambda\mathbf{k}}=0$. The resolvent operators are usually defined as

$$\begin{aligned}&G^+_\lambda(E)=\lim_{\varepsilon\to 0^+}G_\lambda(E+i\varepsilon) ;\\&G_\lambda(E+i\varepsilon)=(E+i\varepsilon-H_\lambda)^{-1} .\end{aligned} \tag{1.3}$$

There exist different methods to present the LS triad (1.1).

The most usual derivation starts from the complex energy domain with the hypothesis

$$\begin{aligned}&\psi^+_{\lambda\mathbf{k}}=\lim_{\varepsilon\to 0^+}\psi_{\lambda\mathbf{k}}(E+i\varepsilon) ,\\&\psi_{\lambda\mathbf{k}}(E+i\varepsilon)=i\varepsilon(E+i\varepsilon-H)^{-1}\varphi_{\lambda\mathbf{k}} .\end{aligned} \tag{1.4}$$

This method of derivation of the LS triad is presented in Sec. II A of Ref. 1.

Levin and Sandhas[1] remark that, on going to the real energy limit, two points are crucial. The first one is the validity of the Lippmann's identity.[2] The second one is to prove the relation

$$\lim_{\varepsilon\to 0^+}G_{\lambda'}(E+i\varepsilon)V^{\lambda'}\psi_{\lambda\mathbf{k}}(E+i\varepsilon)=\lim_{\varepsilon_1\to 0^+}\lim_{\varepsilon_2\to 0^+}G_{\lambda'}(E+i\varepsilon_2)V^{\lambda'}\psi_{\lambda\mathbf{k}}(E+i\varepsilon_1) , \tag{1.5}$$

i.e., that the limits can be performed independently in the resolvents and the state vectors.

In regard to this relation, Levin and Sandhas admit that it is unproven. This implies that the derivation of LS equations along the line of Sec. II A in Ref. 1 is still heuristic and not founded on a rigorous mathematical basis. Nevertheless, the authors of Ref. 1 do not care about this problem since they claim that the LS equations have been derived by Sandhas[3,4] in two alternative methods. One of these methods is the Møller operator approach.[3] The other method[3,4] establishes the equivalence of the LS triad and the Faddeev equation.

One purpose of the present paper is to show that Sandhas's derivation of the LS triad in the Møller operator approach is incomplete, with the most delicate analysis, which is the transition from vectors in the Hilbert space to non-normalizable stationary scattering states, being omitted. The same remark applies to Ref. 5. On the other hand, we show in Sec. III that there is no proof of the equivalence with the Faddeev equations.

In 1958, Gerjuoy[6] derived the LS triad by a different method based on the existence of Green's functions associated with the partition Hamiltonians of the three-body problem. This method is often referred to as an alternative way to obtain the LS equations. However Gerjuoy himself writes,[6] "We are not prepared to prove them rigorously," in reference to relations (4.9) of the present paper which condition the validity of the LS triad.

Then we conclude in Sec. V that, contrary to an often reiterated affirmation, the multiparticle LS equations are not yet mathematically founded. Moreover, we display the ambiguities of their usual form stemming from the lack of a definition of some relevant mathematical objects, in particular, of the resolvent operators acting on non-normalizable vectors.

Actually, some people might think that the present paper is aimless since they already know that there is no rigorous proof of the multiparticle LS equations. However, since some, among the most involved in the LS formalism, claim that this proof does exist, we think it is worthwhile to clear up this misunderstanding.

II. THE MØLLER OPERATOR APPROACH

The Møller operator approach proposed by Sandhas[3] is summarized in Sec. II C of Ref. 1. However, Sandhas's derivation is open to criticism since it deals with non-normalizable vectors when the Møller operators are defined on the Hilbert space $\mathscr{H}$. Then we shall first reproduce Sandhas's results, remaining in the point of view of the Hilbert space.

Let $\Phi_\lambda(\mathbf{X})$ be a normalizable wave packet in the form

$$\Phi_\lambda(\mathbf{X})=f_\lambda(\rho_\lambda)\chi(\mathbf{r}_\lambda),\quad \chi(\mathbf{r}_\lambda)\in L^2(\mathbb{R}^3)\ , \tag{2.1}$$

where $f_\lambda(\rho_\lambda)$ is the λ pair bound state introduced in Eq. (1.2). Let $\Psi(t)=e^{-iHt}\Psi_\lambda$ be the corresponding scattering state, where

$$\Psi_\lambda=\Omega_\lambda\Phi_\lambda=\underset{t\to-\infty}{s\text{-lim}}\ e^{iHt}e^{-iH_\lambda t}\Phi_\lambda\ . \tag{2.2}$$

Following Sandhas's method, one obtains

$$\Phi_\lambda=\underset{t\to-\infty}{s\text{-lim}}\ e^{iH_\lambda t}e^{-iHt}\Psi_\lambda\ , \tag{2.3}$$

$$0=\underset{t\to-\infty}{w\text{-lim}}\, e^{iH_{\lambda'}t}e^{-iHt}\Psi_\lambda,\quad \lambda\neq\lambda'\ . \tag{2.4}$$

To obtain the LS equations in Hilbert space one must turn to time independent formalism. It is shown in Ref. 7 that Eqs. (2.3) and (2.4) are equivalent, respectively, to

$$\Phi_\lambda=\underset{\varepsilon\to0^+}{s\text{-lim}}\int_{-\infty}^{0}dt\, e^{\varepsilon t}e^{iH_\lambda t}e^{-iHt}\Psi_\lambda\ , \tag{2.5}$$

$$0=\underset{\varepsilon\to0^+}{w\text{-lim}}\int_{-\infty}^{0}dt\, e^{\varepsilon t}e^{iH_{\lambda'}t}e^{-iHt}\Psi_\lambda,\quad \lambda\neq\lambda'\ . \tag{2.6}$$

Using the spectral decomposition of the full Hamiltonian $H=\int_{R_1}\mu\, dP_\mu^H$, one obtains[7–9]

$$\Psi_\lambda=\Phi_\lambda+\underset{\varepsilon\to0^+}{s\text{-lim}}\int_{R_1}G_\lambda(\mu+i\varepsilon)(\mu-H_\lambda)dP_\mu^H\Psi_\lambda\ , \tag{2.7}$$

$$\Psi_\lambda=\underset{\varepsilon\to0^+}{w\text{-lim}}\int_{R_1}G_{\lambda'}(\mu+i\varepsilon)(\mu-H_{\lambda'})dP_\mu^H\psi_\lambda,\quad \lambda\neq\lambda'\ . \tag{2.8}$$

Equations (2.7) and (2.8) are called the inhomogeneous and homogeneous LS equations in Hilbert space.

We recalled in Sec. I that LS equations at a fixed energy are usually deduced from Eq. (1.4) and we noticed that a relevant step of this derivation is the proof of Eq. (1.5). This step is avoided in the derivation of the LS equations (2.7) and (2.8) in Hilbert space. Notice that the inhomogeneous LS Eq. (2.7) can be proven to hold in the strong limit sense, when in Ref. 1, it is considered to be valid only with the weak limit. However, the main drawback[7] of the Hilbert space version of the LS equations, viewed as equations for finding Ψ_λ when Φ_λ is given, is that they require the knowledge of the spectral function P_μ^H of the total Hamiltonian H. Now, in practice P_μ^H is an unknown quantity. In fact, if P_μ^H were known, then one could easily compute the time evolution operator and thus solve the dynamics of the problem without even having to resort to scattering theory techniques.

This failure of the LS equations in Hilbert space to be of practical use owing to the occurrence of the spectral function P_μ^H seems to be circumvented in Sandhas's papers since he assumes that relations (2.7) and (2.8) still hold if Φ_λ is not a Hilbert vector by the improper vector $\varphi_{\lambda\mathbf{k}}$ of Eq. (1.2). Actually, this assumption need not be trivially true. In Ref. 7, Prugovecki, after having derived the LS equation in Hilbert space for the two-body problem, proceeds (pp. 503–505) to the derivation of the ordinary LS equation at a fixed energy

$$\psi_{\mathbf{k}}^+(\rho)=e^{i\mathbf{k}\cdot\rho}+\int_{\mathbf{R}^3}G_0^{(3)}(E^+;|\rho-\rho'|)v(\rho')\psi_{\mathbf{k}}^+(\rho')d\rho'\ .$$

This is achieved under conditions on the two-body interaction which normalizes $v\psi_{\mathbf{k}}^+$, and then defines $G_0^+(E)v\psi_{\mathbf{k}}^+$.

We did not succeed in reproducing, in the three-body problem, the sets of arguments used by Prugovecki in the two-body case. One difficulty is that the Green's functions $G_\lambda^{(6)}(E^+,\mathbf{X},\mathbf{X}')$, $\lambda\neq0$, are not known in closed form. But the main ambiguity comes from the fact that $V^{\lambda'}\psi_{\lambda\mathbf{k}}^+$ is not in the Hilbert space, no matter what the two-body interactions may be.

Thus we conclude that the derivation of the LS triad (1.1) in the Møller operator approach is incomplete, with the most delicate analysis, which is the transition from LS equations in Hilbert space to LS equations at a fixed energy, being lacking. Indeed, Sandhas[3] is aware of this difficulty. But Ref. 1, where this difficulty is not mentioned, leads people to the erroneous belief that the Møller operator approach is successful in deriving the LS triad (1.1).

III. FADDEEV EQUATION AND LS TRIAD

The three-body Faddeev equation[10] for finding $\psi_{\lambda\mathbf{k}}^+$ reads

$$\chi_{\lambda'}=\delta_{\lambda\lambda'}\varphi_{\lambda\mathbf{k}}+G_{\lambda'}^+(E)V_{\lambda'}\sum_{\lambda''\neq\lambda'}\chi_{\lambda''}\ ,\qquad \psi_{\lambda\mathbf{k}}^+=\sum_{\lambda'=1}^{3}\chi_{\lambda'}\ . \tag{3.1}$$

In Ref. 4 Sandhas, considering the LS triad (1.1) as valid, attempts to prove that Eq. (3.1) can be deduced from it. However, if one follows Sandhas's reasoning one would remark that it relies on the relations

$$G_{\lambda'}^+(E)=G_0^+(E)+G_0^+(E)V_{\lambda'}G_{\lambda'}^+(E)$$
$$=G_0^+(E)+G_{\lambda'}^+(E)V_{\lambda'}G_0^+(E)\ .$$

When these relations are easily verified with pure algebraic manipulations for complex E, their validity on the real energy axis, when they operate on the non-normalizable vector $V^\lambda\psi_{\lambda\mathbf{k}}^+$, has never been proven. It is on the same level of difficulty that one proves Eq. (1.5). We then conclude that there is no proof that the Faddeev equation can be deduced from the LS triad.

In Ref. 3 Sandhas makes a remark which might suggest that the validity of the Faddeev equation would imply the validity of the LS triad. Writing the LS triad in the form

$$\psi_{\lambda'}=\delta_{\lambda\lambda'}\varphi_{\lambda\mathbf{k}}+G_{\lambda'}^+(E)\sum_{\lambda''\neq\lambda'}V_{\lambda''}\psi_{\lambda''}\ ,\qquad \psi_{\lambda'}=\psi_{\lambda\mathbf{k}}^+,\lambda'=1,2,3\ , \tag{3.2}$$

Sandhas calls this equation Faddeev-like because its kernel is connected after one iteration. Then he writes: "It is well known that, appropriately restricting the two-body potentials, a rather sophisticated direct proof of uniqueness has been given by Faddeev for equations of the structure (3.2)." This assertion is obviously wrong. When the Faddeev equation and the LS triad are easily proven to be equivalent for complex energy, the study of their behavior in the limit of real energy entails two completely different problems. The key point of Faddeev's method is the decomposition of the scattering state $\psi^+_{\lambda\mathbf{k}}$ in three pieces $\chi_{\lambda'}$, with specific boundary conditions. After the sound work of Faddeev, its success may be understood if one remarks that $V_{\lambda'}\chi_{\lambda''}$, $\lambda'\neq\lambda''$ is in the Hilbert space. Then $G^+_{\lambda'}(E)V_{\lambda'}\chi_{\lambda''}$ is defined. The main difference in Eq. (3.2) is that $V_{\lambda''}\psi^+_{\lambda\mathbf{k}}$ is never normalizable no matter what the two-body interactions may be.

IV. LS EQUATIONS IN THE GREEN'S FUNCTIONS APPROACH

Another method to obtain the three-particle LS equations at a fixed energy was presented by Gerjuoy[6] in 1958 to explain the origins and the consequences of the simultaneous existence of inhomogeneous and homogeneous LS equations, outlined by Foldy and Tobocman[11] in 1957 and derived along the heuristic method mentioned in Sec. I.

This method deals with $\mathbb{R}^6$ position space where the initial state (1.2) obeys the equation

$$[E-H_\lambda(\mathbf{X})]\varphi_{\lambda\mathbf{k}}(\mathbf{X})=0\ . \tag{4.1}$$

The corresponding stationary scattering state is the solution of

$$[E-H(\mathbf{X})]\Psi^+_{\lambda\mathbf{k}}(\mathbf{X})=0\ , \tag{4.2}$$

defined by the condition that

$$D_{\lambda\mathbf{k}}(\mathbf{X})=\psi^+_{\lambda\mathbf{k}}(\mathbf{X})-\varphi_{\lambda\mathbf{k}}(\mathbf{X}) \tag{4.3}$$

be purely outgoing.

One assumes that there exists Green's functions which are purely outgoing solutions of the equations

$$\begin{aligned}[E-\overrightarrow{H_{\lambda'}(\mathbf{X}')}]&G^{(6)}_{\lambda'}(E^+;\mathbf{X}',\mathbf{X})\\ &=G^{(6)}_{\lambda'}(E^+;\mathbf{X}',\mathbf{X})[E-\overleftarrow{H_{\lambda'}(\mathbf{X})}]\\ &=\delta(\mathbf{X}-\mathbf{X}')\ .\end{aligned} \tag{4.4}$$

Following Gerjuoy, by combining Eqs. (4.2) and (4.4) and integrating into a finite volume $v\in\mathbb{R}^6$ which contains $\mathbf{X}'$, one obtains

$$\begin{aligned}\psi^+_{\lambda\mathbf{k}}(\mathbf{X}')=&I_v[G^+_{\lambda'}/\varphi_{\lambda\mathbf{k}}]+I_v[G^+_{\lambda'}/D_{\lambda\mathbf{k}}]\\ &+\int_v G^{(6)}_{\lambda'}(\mathbf{X}',\mathbf{X})\langle\mathbf{X}\,|\,V^{\lambda'}\psi^+_{\lambda\mathbf{k}}\rangle d\mathbf{X}\ ,\end{aligned} \tag{4.5}$$

$$\begin{aligned}I_v[G^+_{\lambda'}/f]=&\int_v G^{(6)}_{\lambda'}(\mathbf{X}'',\mathbf{X})\\ &\times[\overrightarrow{H_0(\mathbf{X})}-\overleftarrow{H_0(\mathbf{X})}]f(\mathbf{X})d\mathbf{X}\ .\end{aligned} \tag{4.6}$$

By making the same operations with Eqs. (4.1) and (4.4), one obtains

$$\begin{aligned}I_v(G^+_{\lambda'}/\varphi_{\lambda\mathbf{k}})=&\varphi_{\lambda\mathbf{k}}(\mathbf{X}')\\ &+\int_v G^{(6)}_{\lambda'}(\mathbf{X}',\mathbf{X})\langle\mathbf{X}\,|\,V_{\lambda'}-V_\lambda\,|\,\varphi_{\lambda\mathbf{k}}\rangle d\mathbf{X}\ .\end{aligned} \tag{4.7}$$

Now, Gerjuoy remarks that the inhomogeneous and homogeneous LS equations in position space

$$\begin{aligned}\psi^+_{\lambda\mathbf{k}}(\mathbf{X}')=&\delta_{\lambda\lambda'}\varphi_{\lambda\mathbf{k}}(\mathbf{X}')\\ &+\int_{\mathbb{R}^6}G^{(6)}_{\lambda'}(\mathbf{X}',\mathbf{X})\langle\mathbf{X}\,|\,V^{\lambda'}\psi^+_{\lambda\mathbf{k}}\rangle d\mathbf{X}\end{aligned} \tag{4.8}$$

are obtained from Eq. (4.5) by letting v go to infinity and by granting the conditions

$$\begin{aligned}&\lim_{v\to\infty}I_v(G^+_{\lambda'}/\varphi_{\lambda\mathbf{k}})=\delta_{\lambda\lambda'}\varphi_{\lambda\mathbf{k}}(\mathbf{X}')\ ;\\ &\lim_{v\to\infty}I_v(G^+_{\lambda'}/D_{\lambda\mathbf{k}})=0\ .\end{aligned} \tag{4.9}$$

Thus the proof of the validity of the LS equations (4.8) relies, in the Green's function method, on two conditions: (a) the proof of Eq. (4.9); (b) the existence of the integral in Eq. (4.8). However Gerjuoy[6] admits, "The assertions of this paragraph are crucial and can be made plausible, but we are not prepared to prove them rigorously."

Let us now explain why neither condition (a) nor condition (b) are satisfied. Our criticism relies on one definition and two theorems of the integration theory. Let

$$J_v=\int_v f(\mathbf{X})d\mathbf{X},\quad \mathbf{X}\in\mathbb{R}^n,\quad n\geq 2\ . \tag{4.10}$$

Definition: The integral $J=\lim_{v\to\infty}J_v$ is said to exist if and only if

$$\lim_{v\to\infty}|J-J_v|=0\ ,$$

whatever the way v is going to infinity. In other words, it means that $\lim_{v\to\infty}J_v$ must not depend on the choice of the integration variables.

Theorem A (Ref. 12). A Lebesque integral exists if and only if $|f(\mathbf{X})|$ is integrable.

Theorem B (Ref. 13). A multiple Riemann integral exists if and only if $\lim_{v\to\infty}\int_v|f(\mathbf{X})|d\mathbf{X}$ exists.

Following the definition, theorem B implies that if $|f(\mathbf{X})|$ is not integrable, J_v does not go to a unique limit J when v is going to infinity in different ways.

Let us now consider the integral

$$\begin{aligned}J_v(\mathbf{X}')=&\int_v G^{(6)}_{\lambda'}(\mathbf{X}',\mathbf{X})\langle\mathbf{X}\,|\,V_\lambda\varphi_{\lambda\mathbf{k}}\rangle d\mathbf{X}\\ =&\int_v G^{(6)}_{\lambda'}(\mathbf{X}',\mathbf{X})V_\lambda(\rho_\lambda)f_\lambda(\rho_\lambda)\\ &\times e^{i\mathbf{k}\cdot\mathbf{r}_\lambda}d\mathbf{X}/(2\pi)^{3/2}\ ,\end{aligned} \tag{4.11}$$

which occurs, when $\lambda\neq\lambda'$, in Eq. (4.7) and as a part of the integral (4.5).

From theorems A and B,

$$\lim_{v\to\infty}J_v(\mathbf{X}')$$

exists if and only if the integral

$$K(\mathbf{X}')=\int_{\mathbb{R}^6}|G^{(6)}_{\lambda'}(\mathbf{X}',\mathbf{X})|\,|V_\lambda(\rho_\lambda)f(\rho_\lambda)|\,d\mathbf{X} \tag{4.12}$$

exists.

Now, in the asymptotic region of the $\mathbb{R}^6=\{\mathbf{X}\}$ hyperspace where the relative distance ρ_λ remains bounded, the relative distance $\rho_{\lambda'}$ goes to infinity; for short-range two-body interactions $V_{\lambda'}$ becomes negligible. Then it follows from Eq. (4.4) that $G_{\lambda'}^{(6)}(E^+;\mathbf{X}',\mathbf{X})$ behaves as $G_0^{(6)}(E^+;|\mathbf{X}'-\mathbf{X}|)$ for fixed $\mathbf{X}'$ in this asymptotic region.[14] The Green's function $G_0^{(6)}(E;|\mathbf{X}'-\mathbf{X}|)$ is known in closed form[15] and $|\mathbf{X}'-\mathbf{X}|^{5/2} \times |G_0^{(6)}(E;|\mathbf{X}'-\mathbf{X}|)|$ remains bounded as $|\mathbf{X}|$ goes to infinity. Then $K(\mathbf{X}')$ is not converging and $\lim_{v\to\infty} J_v(\mathbf{X}')$ [Eq. (4.11)] does not exist.

Therefore, conditions (a) and (b) are not satisfied when $\lambda\neq\lambda'$, which invalidates the proof of the two homogeneous LS equations. On the same line, the proof of relations (4.9) published by Adhikari and Glöckle[16] is questionable since it contradicts theorems A and B.

One notes that the undefined integral $\lim_{v\to\infty} J_v(\mathbf{X}')$ [Eq. (4.11)] can be eliminated if Eq. (4.5), with the help of Eqs. (4.3) and (4.7), is written

$$D_{\lambda\mathbf{k}}(\mathbf{X}')=I_v[G_{\lambda'}^{+}/D_{\lambda\mathbf{k}}]+\int_v G_{\lambda'}^{(6)}(\mathbf{X}',\mathbf{X})\langle\mathbf{X}\,|\,V^{\lambda}\varphi_{\lambda\mathbf{k}}\rangle d\mathbf{X} + \int_v G_{\lambda'}^{(6)}(\mathbf{X}',\mathbf{X})\langle\mathbf{X}\,|\,V^{\lambda'}D_{\lambda\mathbf{k}}\rangle d\mathbf{X}\,. \tag{4.13}$$

This equation is considered by Gerjuoy[14] as "a rigorous starting point." However, [if $\lim_{v\to\infty} I_v(G_{\lambda'}^{+}/D_{\lambda\mathbf{k}})=0$], the three corresponding integral equations that calculate $D_{\lambda\mathbf{k}}$ are inhomogeneous.

Indeed, theorem B need not prevent relations (4.9) from holding for some particular choice of the integration variables, which might explain the results of Ref. 16. This means that the validity of the homogeneous LS equations in position space might be restored if (and only if) they are accompanied by a prescription of the way the integration must be performed.

V. CONCLUSIONS

We have shown that four different methods used to derive the three-body LS equations (1.1) are lacking from a rigorous mathematical basis. The method[1] mentioned in Sec. I lacks the proof of Eq. (15); the Møller operator approach[1,3] lacks the analysis of the transition from LS equations (2.7) and (2.8) in the Hilbert space to LS equations (1.1) at a fixed energy; there is no proof of the equivalence between the LS triad and the Faddeev equation (3.1); and the Green's function method[6] lacks the proof of the relevant relations (4.9). As far as we know, every other so-called proof of the LS triad is mathematically failing.

The main point is that, as a consequence of these incomplete derivations, the usual form (1.1) of the LS equations is meaningless. Indeed, let us recall that time independent collision theory originates in time dependent theory devised for normalized initial wave packets. Then the resolvent operators $G_\lambda(z)=(z-H_\lambda)^{-1}$ enter into the theory as bounded operators on the Hilbert space $\mathscr{H}$ when $\mathrm{Im}z\neq 0$. When E belongs to the continuum spectrum of H_λ, for all $f,g\in\mathscr{H}$, $\langle g\,|\,G_\lambda(z)\,|\,f\rangle$ goes to a limit as ε goes to 0^+, which is symbolically written as

$$\lim_{\varepsilon\to 0^{\pm}}\langle g\,|\,G_\lambda(z)\,|\,f\rangle=\langle g\,|\,G_\lambda^{\pm}(E)\,|\,f\rangle\,, \tag{5.1}$$

which defines the meaning of

$$\lim_{\varepsilon\to 0^{\pm}} G_\lambda(E+i\varepsilon)\,|\,f\rangle=G_\lambda^{\pm}(E)\,|\,f\rangle,\quad f\in\mathscr{H}\,. \tag{5.2}$$

Now the crucial problem, with regard to the LS triad (1.1), is that $V^{\lambda'}\psi_{\lambda\mathbf{k}}^{+}$ is never normalizable, no matter what the two-body interactions may be. Then $G_{\lambda'}^{+}(E)V^{\lambda'}\psi_{\lambda\mathbf{k}}^{+}$ need to be first defined.

It is often assumed that a candidate for this definition is that $G_{\lambda'}^{+}(E)$ is an integral operator, i.e.,

$$\langle\mathbf{X}\,|\,G_{\lambda'}^{+}(E)V^{\lambda'}\psi_{\lambda\mathbf{k}}^{+}\rangle=\int_{\mathbb{R}^6} G_{\lambda'}^{(6)}(E^+;\mathbf{X},\mathbf{X}') \times\langle\mathbf{X}'\,|\,V^{\lambda'}\psi_{\lambda\mathbf{k}}^{+}\rangle d\mathbf{X}'\,.$$

We have seen in Sec. IV that, when $\lambda\neq\lambda'$, this integral does not exist as long as some prescription to perform the integral has not been given.

Another candidate is the integral representation of $G_\lambda^{\pm}(E)$,

$$G_\lambda^{\pm}(E)=\frac{\mathscr{P}}{E-H_\lambda}\mp i\pi\delta(E-H_\lambda)\,,$$

which is valid in Eq. (5.1) when f and $g\in\mathscr{H}$. However, in the same way, it has never been proven rigorously that $G_{\lambda}^{+},V^{\lambda'}\psi_{\lambda\mathbf{k}}^{+}$ exists when this integral representation is used. Then Eq. (1.3) of the present paper, reproduced from a large number of papers, is in the present context meaningless.

Once more, if the LS equations (1.1) could be derived from the Hilbert space relations (2.7) and (2.8) their meaning would be unambiguous. In the absence of such a derivation they are, as they stand, undefined.

In a recent paper,[17] Gerjuoy and Adhikari prove the uniqueness of solutions to the LS equations in a soluble one-dimensional three-body model. They conclude "that any claims that the (inhomogeneous) LS equation (1.1) has unique scattering solutions in three-particle systems first must explain why our results cannot be extrapolated to actual three-dimensional three-particle systems, as well as why proofs that the solutions are unique fail in one dimension but not in three dimensions." However, in their model the integral corresponding to Eq. (4.11) is a simple integral, which does not converge absolutely, and does not exist as a Lebesque integral but which is a converging Riemann integral. Since the key point of our objections against the homogeneous LS equations is that the integral in Eq. (4.8) ($\lambda\neq\lambda'$) in a three-dimensional three-body problem is not defined, we conclude that, even after the elegant one-dimensional proof[17] of the validity of the LS equations, the question of their validity in three dimensions is still open.

We have not proven in the present paper that the LS equations (1.1) are not valid, but we have shown that the often reiterated affirmation that the triad of homogeneous and inhomogeneous LS equations is firmly established is premature.

For more than twenty years, numerous papers[18] on multiparticle scattering theory relied on the LS equations.

Most of these works consisted of formal manipulations of the LS equations aiming at integral equations with a connected kernel. A small number of papers, having in view the interpretation of experimental results, were numerical applications. Purely formal works are open to the same criticism as that of the LS equations: Even if, at a complex energy and inasmuch as one does not go out of the Hilbert space, these manipulations are justified, the transition to the real energy limit is not studied and remains unconfirmed and undefined. As to the numerical applications, their reasonable agreement with the experimental results in some domain, is by no means the proof of the validity of the starting point equations, in particular, because numerous numerical approximations are necessary. We would like to distinguish here between "theory" and "model" on the basis of the following definitions. A theory is a set of equations which follows, through rigorous mathematical steps, from the principles of quantum mechanics with a dynamical input; such are the three-body Faddeev equations. A model relies on equations which are not theoretically founded in the above sense. This does not prevent some models from being powerful tools in the investigation of the nuclear properties. Regarding the LS equations and the following formal developments, they are not yet theories. It is high time that researchers be interested in the study of their foundation. Otherwise, it only remains the possibility of numerical tests to verify if these equations supply a satisfactory model to interpret the experimental results. One notes that in this way, the building of complicated integral equations with a connected kernel is not essential, a sensible ansatz for the resolvents G_χ^+ would do the same.

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