# Elastic vibrational approach to giant resonances of deformed nuclei

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Isoscalar giant resonances of arbitrary multipolarity in deformed nuclei are studied with the help of the variational principle within the framework of the nuclear elastic vibration. Particular attention is given to the fragmentation of the giant monopole and quadrupole resonances. It is shown how the nuclear surface deformation contributes to the fragmentation through the coupling between the monopole and quadrupole modes of oscillation. Simple formulas for the giant resonance energies of deformed nuclei are obtained for both axial and nonaxial deformations. Surface tension and Coulomb repulsion are included in the course of formulation but practical evaluations are made without them. Most numerical results are concerned with well-deformed nuclei such as those around $A=150$.

## I. INTRODUCTION

The nuclear elasticity approach to giant resonances has initially been proposed by Bertsch[1] and an extensive study of the isoscalar giant resonances based on elastic vibrations has been made by Wong *et al.*[2,3] Until now, however, only the giant resonances of spherical nuclei have been considered in this approach. This paper describes the isoscalar giant resonances of deformed nuclei of arbitrary multipole degree within the framework of the nuclear elasticity. Particular attention is given to the fragmentation of the giant monopole resonance which has recently been observed[4,5] and which has now been confirmed.

The study of the classical vibrations of an elastic solid body is not new and it has already been worked out in the last century by Lamb[6] for an idealized perfect elastic medium. The conception of free vibrations has then been applied to the oscillations of the earth by many geophysicists.[7] As to nuclear physics, Bertsch[1] has derived macroscopic equations for nuclear vibrations from the random-phase approximation (RPA) equations of motion, and interpreted this mode of collective motion in terms of classical vibrations of an elastic solid body. He has then formulated continuous equations of motion in the framework of the time-dependent Hartree-Fock theory and observed that the nuclear giant resonances can be considered as elastic vibrations of a nucleus. Wong and his collaborators[2,3] have dealt with the dynamics of nuclear fluid starting from the time-dependent Hartree-Fock approximation and have shown that the equation of motion for some collective motions of the nuclear fluid can be approximated by the Lamé equation which governs the classical theory of elasticity. The study of the semiclassical limit of the adiabatic time-dependent Hartree-Fock approximation has become a subject of many investigations.[1,3,8,9]

Since Danos[10] and Okamoto[11] have first investigated the isovector giant dipole resonance of deformed nuclei, the problem of the fragmentation of the giant resonances, especially that of the quadrupole resonance, has been examined by several authors.[12–17] Furthermore, recent experiments[4,5] have indicated that the giant monopole resonance in the $^{154}$Sm nucleus is split into two components. In a previous paper,[17] we have described the fragmentation of the giant resonances of deformed nuclei with the help of the variational principle applied to the scalar Helmholtz-type of equation. However, as is now understood,[18–22] use of arguments based on hydrodynamics for the discussion of the isoscalar-type of giant resonances necessitates considerable effort in justifying such a procedure. Blaizot,[20] and Sagawa and Holzwarth[21] have shown that the energies of isoscalar giant quadrupole resonance can be understood on the basis of a simple macroscopic model involving the Fermi surface distortion. This model has been extended to isoscalar giant resonances of arbitrary multipole degree of spherical nuclei by Nix and Sierk.[22] It seems to us, therefore, worthwhile to describe the isoscalar giant resonances of deformed nuclei in a credible macroscopic model, such as the nuclear elasticity approach. Since the validity of the nuclear elasticity model and its origin have been fully discussed elsewhere,[1–3] we shall focus here our attention on the method of calculation of the isoscalar giant resonances of deformed nuclei starting directly from the Lamé equation having coefficients relevant to nuclear elasticity. One method of dealing with the giant resonances of deformed nuclei is to solve the Lamé equation in the spheroidal coordinates system. However, owing to the difficulty of solving analytically such an equation, it is more convenient to use the variational procedure which has been reviewed[17] in connection with the hydrodynamical approach to the giant resonances of deformed nuclei.

In Sec. II, we start with a brief review of the equation of motion of a perfect elastic nuclear medium and then discuss the method of derivation of the eigenvalue equation which is relevant to the subsequent formulation of the theory. The eigenvalue equation we have derived is formally different from the corresponding expression in Ref. 2 but its contents are much the same. In the present study we have not considered the so-called toroidal oscillations in which there is no dilatation. In Sec. III, we discuss the variational principle suitable for the nuclear elasticity approach and then show how to solve the variational equation for the problem of the giant resonances of deformed nuclei. In this section we have obtained explicit formulas for the resonance energies of deformed nuclei as a function of nuclear deformation parameters $\gamma$ and $\delta$. Section IV is devoted to the discussion of the fragmentation of the giant monopole resonance by coupling the monopole mode of oscillation with that of the quadrupole mode with the help of the nuclear deformation parameters. Finally, we give a summary and conclusion in Sec. V.

## II. EQUATION OF MOTION

Since the ingredients contained in the classical theory of elasticity play a prominent part in the present study of the giant resonances of deformed nuclei, we start with the equation of motion which governs a perfect elastic medium:[6,7]

$$\frac{\partial}{\partial x_j} T_{ij} + F_i = \rho \frac{\partial^2 u_i}{\partial t^2} , \tag{2.1}$$

where $T_{ij}$ is the stress-strain tensor component, $F_i$ the body force, $\rho$ the density, and $u_i$ the displacement. Here the usual summation convention of tensor analysis is assumed. The stress-strain tensor $T_{ij}$ is composed of the dilatation $\Delta = \partial u_i / \partial x_i$ and the strain tensor $e_{ij}$:

$$T_{ij} = \lambda \Delta \delta_{ij} + 2\mu e_{ij} , \tag{2.2}$$

where $\lambda$ and $\mu$ are the Lamé coefficients which are assumed to be constant. The strain tensor is explicitly given by

$$e_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] . \tag{2.3}$$

In terms of the strain tensor the dilatation can be expressed simply by $\Delta = e_{ii} = \mathrm{div}\mathbf{u}$, where $\mathbf{u}$ is the displacement vector. If the elastic medium is free from the body force, Eq. (2.1) reduces to

$$(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \times \nabla \times \mathbf{u} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} . \tag{2.4}$$

For the nuclear elasticity the Lamé coefficients are shown to be[1,2]

$$\lambda = \left[ \frac{K}{9} - \frac{2}{15} \frac{\hbar^2}{m^*} k_f^2 \right] \rho , \tag{2.5}$$

$$\mu = \frac{1}{5} \frac{\hbar^2}{m^*} k_f^2 \rho , \tag{2.6}$$

where $K$ is the nuclear compressibility, $m^*$ the effective nucleon mass, and $k_f$ the Fermi momentum. An alternative way of defining the Lamé constant $\lambda$ is to introduce the Landau parameter $F_0$ in place of $K$, namely

$$\lambda = \frac{1}{5} \frac{\hbar^2}{m^*} k_f^2 \rho (1 + \tfrac{5}{3} F_0) .$$

In this case the nuclear compressibility becomes

$$K = 3 \frac{\hbar^2}{m^*} k_f^2 (1 + F_0) .$$

It is noted that the bulk modulus $\kappa = K\rho/9$ is equal to $\lambda + 2\mu/3$. We further assume that the dependence of the displacement vector $\mathbf{u}$ on time is given by $\mathbf{u}(\mathbf{r},t) = \mathbf{u}(\mathbf{r})\exp(i\omega t)$. A general solution of Eq. (2.4) is then given by

$$\begin{aligned} \mathbf{u}(\mathbf{r}) &= \mathbf{u}_1(\mathbf{r}) + \mathbf{u}_2(\mathbf{r}) + \mathbf{u}_3(\mathbf{r}) \\ &= \nabla \chi_1 + \nabla \times \mathbf{r} \chi_2 + \nabla \times \nabla \times \mathbf{r} \chi_3 . \end{aligned} \tag{2.7}$$

Here the function $\chi_1$ is the solution of the scalar Helmholtz equation $\nabla^2 \chi_1 + h^2 \chi_1 = 0$, where $h^2 = (\rho\omega^2)/(\lambda + 2\mu)$, and the functions $\chi_2$ and $\chi_3$ also satisfy the scalar Helmholtz equations $\nabla^2 \chi_{2(3)} + k^2 \chi_{2(3)} = 0$, where $k^2 = \rho\omega^2/\mu$. Of three components of $\mathbf{u}$, $\mathbf{u}_2$ has no radial component and the displacements at a point are always orthogonal to the radius of the point. This is the so-called toroidal motion which is not directly concerned with the electric isoscalar mode of collective vibrations and therefore we discard the solution $\mathbf{u}_2$.

A general solution $\mathbf{u}$ can now be expressed in polar coordinates as $\mathbf{u} = u_r \mathbf{n}_r + u_\theta \mathbf{n}_\theta + u_\phi \mathbf{n}_\phi$, where

$$\begin{aligned} u_r &= U_l(r) Y_{lm}(\theta,\phi) , \\ u_\theta &= V_l(r) \frac{\partial Y_{lm}}{\partial \theta} , \\ u_\phi &= V_l(r) \frac{1}{\sin\theta} \frac{\partial Y_{lm}}{\partial \phi} , \end{aligned} \tag{2.8}$$

and $\mathbf{n}_r$, $\mathbf{n}_\theta$, and $\mathbf{n}_\phi$ denote, respectively, unit vectors in the directions of $r$, $\theta$, and $\phi$. The radial functions $U_l(r)$ and $V_l(r)$ are given by

$$U_l(r) = A_l \frac{1}{h^2} \frac{\partial}{\partial r} j_l(hr) + C_l \frac{1}{k^2} \frac{l(l+1)}{r} j_l(kr) , \tag{2.9}$$

$$V_l(r) = A_l \frac{1}{h^2} \frac{1}{r} j_l(hr) + C_l \frac{1}{k^2} \frac{1}{r} \frac{\partial}{\partial r} [r j_l(kr)] . \tag{2.10}$$

In the above expressions, $Y_{lm}(\theta,\phi)$ is the spherical harmonics and $j_l(x)$ is the spherical Bessel function. The constant multipliers $A_l$ and $C_l$ are to be determined from the boundary conditions which state that the stress-strain tensor components $T_{ij}$ vanish on the nuclear surface. The equations satisfied by the boundary conditions are then

$$\begin{aligned} &T_{rr} = 0 \quad \text{at } r = R_0 , \\ &T_{r\theta} = T_{r\phi} = 0 \quad \text{at } r = R_0 , \end{aligned} \tag{2.11}$$

where $R_0$ is the spherical nuclear radius. In order to obtain explicit expressions corresponding to Eq. (2.11), we now describe the strain tensors in spherical polar coordinates. For example, the symmetric strain tensors are expressed as

$$\begin{aligned} e_{11} &= e_{rr} = \frac{\partial u_r}{\partial r} , \\ e_{22} &= e_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r} u_r , \\ e_{33} &= e_{\phi\phi} = \frac{1}{r\sin\theta} \frac{\partial u_\phi}{\partial \phi} + \frac{1}{r} u_r + \frac{\cot\theta}{r} u_\theta . \end{aligned} \tag{2.12}$$

The antisymmetric strain tensors $e_{ij}$ $(i \neq j)$ are shown in Appendix A. When the explicit expressions of the strain tensors are introduced into Eq. (2.2), we see that the boundary conditions (2.11) take the forms

$$\left\{ (\lambda+2\mu)\frac{\partial U_l}{\partial r}+\frac{\lambda}{r}[2U_l-l(l+1)V_l] \right\}_{r=R_0}=0 , \quad (2.13)$$

$$\left[ \frac{\partial V_l}{\partial r}+\frac{1}{r}(U_l-V_l) \right]_{r=R_0}=0 . \quad (2.14)$$

Upon introducing the explicit forms of $U_l$ and $V_l$, Eqs. (2.13) and (2.14) become the functions of $j_l(hR_0)$, $j_l(kR_0)$ and the constant multipliers $A_l$ and $C_l$. By equating the ratio $C_l/A_l$ obtained from Eq. (2.13) with that obtained from Eq. (2.14) (see Appendix A), we get

$$\left\{ 1+\frac{(l-1)(l+2)}{\eta}\left[ S_l(\eta)-\frac{l+1}{\eta} \right] \right\} \frac{S_l(\xi)}{\eta}+\frac{\eta}{4\xi}\left\{ -1+\frac{2(l-1)(2l+1)}{\eta^2}+\frac{2}{\eta}\left[ 1-\frac{2l(l-1)(l+2)}{\eta^2} \right] S_l(\eta) \right\}=0 , \quad (2.15)$$

where $\xi=hR_0$, $\eta=kR_0=\xi(\lambda/\mu)^{1/2}+2$, and $S_l(x)=j_{l+1}(x)/j_l(x)$. Equation (2.15) is the desired eigenvalue equation for free oscillations of a uniform, perfectly elastic sphere. This equation is formally different from the corresponding equation obtained by Wong and Azziz[2] but its contents are the same. It is remarked that the present constant multipliers $A_l$ and $C_l$ are not identical to those of Ref. 2. For $l=0$, the expressions in the two curly braces of Eq. (2.15) yield a common factor which can be eliminated from the equation and the eigenvalue equation reduces to $4\xi j_1(\xi)=\eta^2 j_0(\xi)$. The lowest solution of the eigenvalue equation for $l=0$ can be used to calculate the monopole giant resonance energy. Once the eigenvalues for a multipolarity $l$ are evaluated from Eq. (2.15), the giant resonance energies of spherical nuclei can be estimated by introducing the values of $\lambda$ and $\mu$, defined by Eqs. (2.5) and (2.6), into the expression

$$E^2=(\lambda+2\mu)\frac{\hbar^2}{\rho R_0^2}\xi^2=\mu\frac{\hbar^2}{\rho R_0^2}\eta^2 . \quad (2.16)$$

It is remarked that the Lamé coefficients are not a function of $m$ but a function of the effective mass $m^*$ in order to compensate for absence in the formulation of the effects such as the momentum dependence and nonlocality of the nucleon-nucleon interaction and also the coupling of phonon to the single-particle motion. Although we use the value of 0.75 for $m^*/m$ throughout the present study, it is certainly advised to choose carefully the value of the effective mass for further accurate comparisons with experimental results. For example, with the values of $K=220$ MeV, $m^*/m=0.75$, and $k_f=1.25$ fm$^{-1}$ we can realize a very crude estimate of the giant resonance energies, namely $62.4A^{-1/3}$, $84.9A^{-1/3}$, $66.9A^{-1/3}$, and $126.4A^{-1/3}$ MeV for $l=0$, 1, 2, and 4, respectively. The monopole resonance energy in this estimation is too low compared with the phenomenological value $80A^{-1/3}$ MeV. Inclusion of the mass-dependent nuclear compressibility improves much of the resonance energies. Table I lists the lowest eigenvalues of $\xi$ and corresponding energies for $l=0$, 1, 2, 3, and 4, obtained using the same values of $m^*/m$ and $k_f$ as before but with $K=220+K_SA^{-1/3}$ MeV with the surface compressibility $K_S=-550$ MeV. Except for $A=50$, the monopole resonance energies are now correctly reproduced. For more realistic calculations, use of more refined values of the Lamé constants through the effective mass and the Fermi momentum is necessary, as has been done in Ref. 2.

TABLE I. The lowest eigenvalues $\xi=hR_0$ and the corresponding giant resonance energies for $l=0$, 1, 2, 3, and 4. All results are obtained using the parameters $K=K_\infty+K_SA^{-1/3}$ with $K_\infty=220$ MeV and $K_S=-550$ MeV, $m^*/m=0.75$, and $k_f=1.25$ fm$^{-1}$.

| $l$ | $A$ | 50 | 100 | 150 | 200 | 250 |
|---|---|---|---|---|---|---|
| 0 | $\xi$ | 1.849 | 2.068 | 2.150 | 2.195 | 2.225 |
| | $E$ ($A^{-1/3}$ MeV) | 62.75 | 73.99 | 78.74 | 81.56 | 83.49 |
| 1 | $\xi$ | 1.868 | 1.993 | 2.024 | 2.037 | 2.043 |
| | $E$ | 63.39 | 71.30 | 74.14 | 75.68 | 76.68 |
| 2 | $\xi$ | 1.905 | 1.838 | 1.803 | 1.781 | 1.766 |
| | $E$ | 64.65 | 65.75 | 66.05 | 66.19 | 66.28 |
| 3 | $\xi$ | 2.720 | 2.667 | 2.629 | 2.603 | 2.585 |
| | $E$ | 92.32 | 95.40 | 96.30 | 96.75 | 97.03 |
| 4 | $\xi$ | 3.422 | 3.372 | 3.331 | 3.303 | 3.282 |
| | $E$ | 116.14 | 120.63 | 122.02 | 122.73 | 123.17 |

So far we have implicitly assumed a constant nuclear density. The case where the nuclear density is a function of radial variable can be treated by integrating numerically the equation of motion, but this is beyond the scope of the present purpose. A simple method of taking into account the effect of Coulomb force and surface tension is to modify the boundary conditions[2] so as to include these forces in the eigenvalue equation. In fact, the displacement vector field slightly changes the shape of the nucleus. The restoring force maintaining equilibrium can then be assumed to be surface tension as in the case of a liquid drop. The total stress at the nuclear surface is then the sum of the displacement stress $T_{rr}$ and the change in the pressure due to the surface tension. As in Ref. 2, we can also include the additional effect of Coulomb interaction in the pressure change. The boundary value equation corresponding to the first equation of (2.11) becomes now

$$[T_{rr}+u_r(r)F]_{r=R_0}=0\ , \tag{2.17}$$

where

$$F=\frac{\sigma}{R_0^2}(l-1)(l+2)\left[1-\frac{20x}{(2l+1)(l+2)}\right] .$$

Here $\sigma$ is the constant surface tension coefficient and $x$ is the fissibility parameter. Wong and Azziz[2] have shown that in this way one can uncover the low-lying vibrational states but the inclusion of surface tension and Coulomb force does not make many changes in the energies and characteristics of the elastic vibrational states. Because of the modification of the boundary condition, Eq. (2.13) has an additional term $U_l(R_0)F$ and the ratio of the multiplier constants $A_l/C_l$ is modified and so is the eigenvalue equation, but the method of derivation of the modified eigenvalue equation remains unchanged.

## III. VARIATIONAL PROCEDURE

For further formulations it is more convenient to describe the Lame equation (2.4) in terms of the stress-strain tensors $T_{ij}$ expressed in spherical polar coordinates. We have (see Appendix B)

$$\mathbf{T}+\frac{1}{r}\mathbf{S}=-\rho\omega^2\mathbf{u}\ , \tag{3.1}$$

where

$$\mathbf{T}=\left[\frac{\partial}{\partial r}+\frac{2}{r},\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cot\theta}{r},\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right]\times\begin{bmatrix}T_{rr} & T_{\theta r} & T_{\phi r}\\ T_{r\theta} & T_{\theta\theta} & T_{\phi\theta}\\ T_{r\phi} & T_{\theta\phi} & T_{\phi\phi}\end{bmatrix}\begin{bmatrix}\mathbf{n}_r\\ \mathbf{n}_\theta\\ \mathbf{n}_\phi\end{bmatrix}, \tag{3.2}$$

$$\mathbf{S}=-(T_{\theta\theta}+T_{\phi\phi})\mathbf{n}_r+(T_{\theta r}-\cot\theta T_{\phi\phi})\mathbf{n}_\theta+(T_{\phi r}+\cot\theta T_{\phi\theta})\mathbf{n}_\phi\ . \tag{3.3}$$

Multiplying Eq. (3.1) by $\mathbf{u}^*$ and integrating the result over the volume, we can extract the quantity $\omega^2$ from the equation. Having integrated by part all integrands of derivative forms and by taking account of the boundary conditions (2.11) and (2.17), we arrive at (see Appendix B)

$$\omega^2=\frac{\lambda\int|\nabla\cdot\mathbf{u}|^2d\tau+2\mu\int\sum_{i,j}|e_{ij}|^2d\tau+U_l^2(R_0)R_0^2F}{\int\rho|\mathbf{u}|^2d\tau}\ , \tag{3.4}$$

where $e_{ij}$ and $F$ are those defined in Sec. II. This equation, which is the variational expression for the Lamé equation, forms the basis of the present study and of a subsequent paper[23] which deals with the giant resonances of fast rotating nuclei. If we use the boundary conditions without surface tension and Coulomb interaction, the quantity $F$ disappears and Eq. (3.5) reduces to the corresponding equation cited by Bertsch.[1] The equality of Eq. (3.4) can also be verified by integrating explicitly all integrals involved using the solution (2.8).

For deformed nuclei, the upper limit of radial integrals is no longer a constant radius but a deformed surface which is a function of angles as well as of deformation parameters. Therefore, the angular integrals have to be performed after the radial integrals and thus the result of integrations can be expressed in terms of deformation parameters and the initial frequency of oscillation corresponding to that of a spherical nucleus. For usual quadrupole deformations, the nuclear surface is given by

$$R=R_0\left[1+\sum_\nu\alpha_{2\nu}Y_{2\nu}\right] , \tag{3.5}$$

where $\alpha_{2\nu}$ are collective variables. In an equivalent way we can also define the deformed surface by $R_i=R_0(1+\epsilon_i)$, where $\epsilon_i$ are the increments of three axes which can be expressed in terms of $\alpha_{2\nu}$ as

$$\epsilon_{1(2)}=\epsilon_\pm=-\sqrt{(5/16\pi)}(\alpha_{20}\mp\sqrt{6}\alpha_{22})\ ,\qquad \epsilon_3=\epsilon_0=\sqrt{(5/4\pi)}\alpha_{20}\ . \tag{3.6}$$

When use is made of the deformation parameters $\beta$ and $\gamma$, the increments become simply $\epsilon_i=\frac{2}{3}\delta\cos(\gamma-\frac{2}{3}i\pi)$, where the Nilsson deformation parameter $\delta$ is equal to $(45/16\pi)^{1/2}\beta$.

In the following we use the real version of spherical harmonics $\Phi_{lm}$ instead of $Y_{lm}$, which is defined by[17]

$$\Phi_{l,\pm m}(\theta,\phi)=\sqrt{(1/2)}(_i^{-1})[Y_{lm}(\theta,\phi)\pm(-1)^mY_{l-m}(\theta,\phi)]\ , \tag{3.7}$$

$$\Phi_{l,0}(\theta,\phi)=Y_{l0}(\theta,\phi)\ . \tag{3.8}$$

Generally, all integrals in Eq. (3.4) can be evaluated numerically but the task is laborious leading to results which may screen general features of the giant resonances of deformed nuclei. We therefore derive a formal expression which shows explicitly the effect of static nuclear deformation on the giant resonances of spherical nuclei and which allows us to easily carry out numerical calculations. To this end we expand the radial part of integrands in powers of the collective variable $\alpha_{2\nu}$. Although the calculations of the angular integrals are not straightforward due to the presence of complicated angular functions, the final expressions are always elementary functions of $l$. For example, the lowest order term of the expansion contains the following integral:

$$\int \left\{ \frac{1}{\sin^2\theta} \left[ \frac{\partial^2 \Phi_{l,\pm m}}{\partial\theta\partial\phi} - \cot\theta \frac{\partial \Phi_{l,\pm m}}{\partial\phi} \right]^2 + \frac{\partial^2 \Phi_{l,\pm m}}{\partial\theta^2} \left[ \frac{\partial^2 \Phi_{l,\pm m}}{\partial\theta^2} + l(l+1) \right] \right\} d\Omega = -\tfrac{1}{2} l(l+1) \; . \tag{3.9}$$

The first-order expansion in $\alpha_{2\nu}$ contains an extra spherical harmonics $Y_{2\nu}$ and therefore the angular integrals are further complicated but still integrable analytically. Thus, up to first order of collective variable we get

$$\int \rho \, |\mathbf{u}|^2 d\tau = a_l + b_l \zeta_{l,\pm m} \; , \tag{3.10}$$

$$\lambda \int |\nabla\cdot\mathbf{u}|^2 d\tau + 2\mu \int \sum_{i,j} |e_{ij}|^2 d\tau = c_l + d_l \zeta_{l,\pm m} \; , \tag{3.11}$$

where

$$a_l = \int_0^{R_0} \rho [U_l^2(r) + l(l+1) V_l^2(r)] r^2 dr \; , \tag{3.12}$$

$$b_l = \rho R_0^3 \{ U_l^2(R_0) + [l(l+1) - 3] V_l^2(R_0) \} \; , \tag{3.13}$$

$$\zeta_{l,\pm m} = \frac{1}{(2l-1)(2l+3)} \times \{ [l(l+1) - 3m^2] \varepsilon_0 + \tfrac{1}{2} l(l+1)(\varepsilon_0 + 2\varepsilon_\pm) \delta_{m1} \} \; . \tag{3.14}$$

The explicit forms of $c_l$ and $d_l$ are shown in Appendix B. The expression (3.14) is the geometrical factor arising from the nuclear deformation. The radial integrals $a_l$ and $c_l$ can be evaluated either numerically or analytically using the explicit forms of $U_l(r)$ and $V_l(r)$.

Now the giant resonance frequency of deformed nuclei is given by

$$\omega^2 \simeq \omega_0^2 \left[ 1 - \left[ \frac{b_l}{a_l} - \frac{d_l}{c_l + z_l} \right] \zeta_{l,\pm m} \right] \; , \tag{3.15}$$

where $\omega_0$ is the giant resonance frequency of spherical nuclei having the same mass number and $z_l = U_l^2(R_0) R_0^2 F$. In deriving the expression (3.15), we have neglected a second-order correction to $z_l$, arising from the deformed surface. The formula (3.15) is very simple and transparent. We see thus the effect of deformation through the geometrical factor $\zeta_{l,\pm m}$ multiplied by the constant represented by the terms in the parentheses. When the deformation parameter $\delta$ goes to zero, the geometrical factor vanishes so that the frequency $\omega^2$ becomes $\omega_0^2$.

Figure 1 shows the isoscalar giant quadrupole resonance in deformed nuclei of the $A=150$ region. In this figure, the ratio $\omega/\omega_0$ is displayed as a function of the deformation parameter $\delta$ for both prolate and oblate nuclei. We see clearly how the initial giant resonance energy corresponding to that of spherical nuclei is split into different fragments following the values of $m$ and $\delta$. It is worthwhile to remark that the nuclear deformation partially removes the degeneracy of the giant resonance energy and the complete removal of the degeneracy can be achieved when we introduce the concept of the nuclear rotation in addition to the nuclear deformation, as shown[23] in the study of the rotational splitting of the giant resonances. Figure 2 displays the fragmentation of the iso-

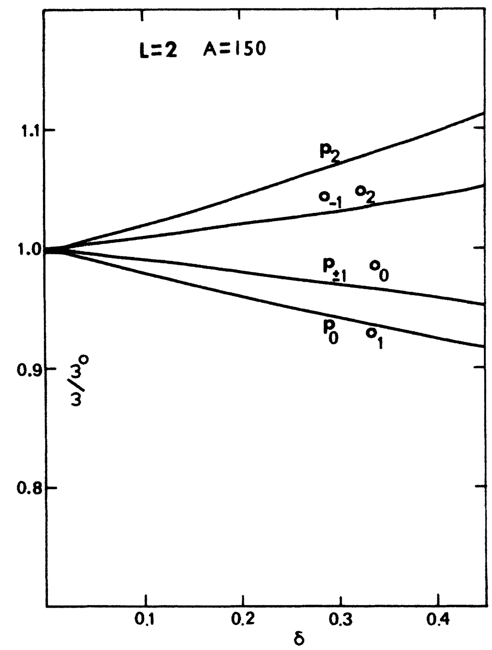


FIG. 1. Isoscalar giant quadrupole resonance of deformed nuclei in the region of $A=150$. The giant resonance frequency of deformed nuclei $\omega$, in units of $\omega_0$, the frequency corresponding to spherical nuclei having the same mass number, is plotted as a function of deformation parameter $\delta$ for both prolate ($\gamma=0°$) and oblate ($\gamma=60°$) figures. The indices after $p$ (prolate) and $o$ (oblate) indicate the projection components $m=0$, $\pm1$, and 2.

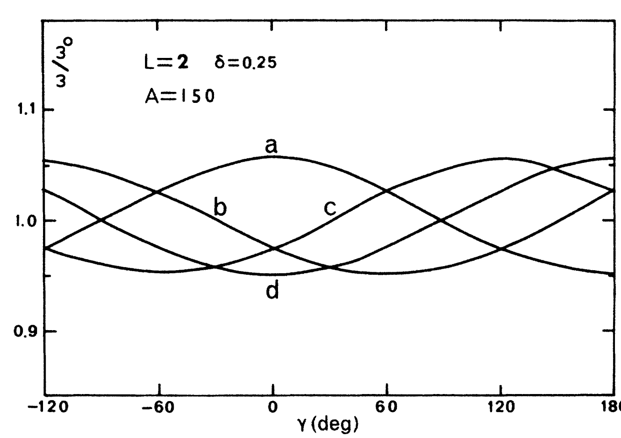


FIG. 2. Fragmentation of the giant quadrupole resonance $\omega$ vs nonaxial deformation parameter $\gamma$ at $\delta=0.25$. The quantity $\omega$ is shown in units of $\omega_0$, the frequency corresponding to spherical nuclei having the same mass number. Curves $a$, $b$, $c$, and $d$ stand for $m=2$, 1, $-1$, and 0, respectively.

scalar giant quadrupole resonance as a function of $\gamma$ for a fixed value of $\delta=0.25$. We see that the resonance energy has generally three fragments, except for the values of $\gamma$ which are not equal to a multiple of 30°. Figure 3 shows the fragmentations of the isoscalar giant dipole and octupole resonances in an axially deformed nucleus. It is noted that the splitting of the giant dipole resonance is very prominent. Figure 4 displays the fragmentation of the isoscalar giant dipole resonance as a function of $\gamma$ for a fixed value of $\delta=0.25$. There are two fragments for axially deformed nuclei and three for triaxial deformation. It is interesting to compare this figure with the corresponding one[17] for the isovector giant dipole resonance. An essential difference that we can bring into relief is that the positions of the resonance energy peaks at $\gamma=0°$ and 60° are interchanged between these two figures. For example, the two peaks of the isoscalar dipole resonance energies at $\gamma=0°$ correspond to those two peaks of the isovector giant dipole resonance energies at $\gamma=60°$.

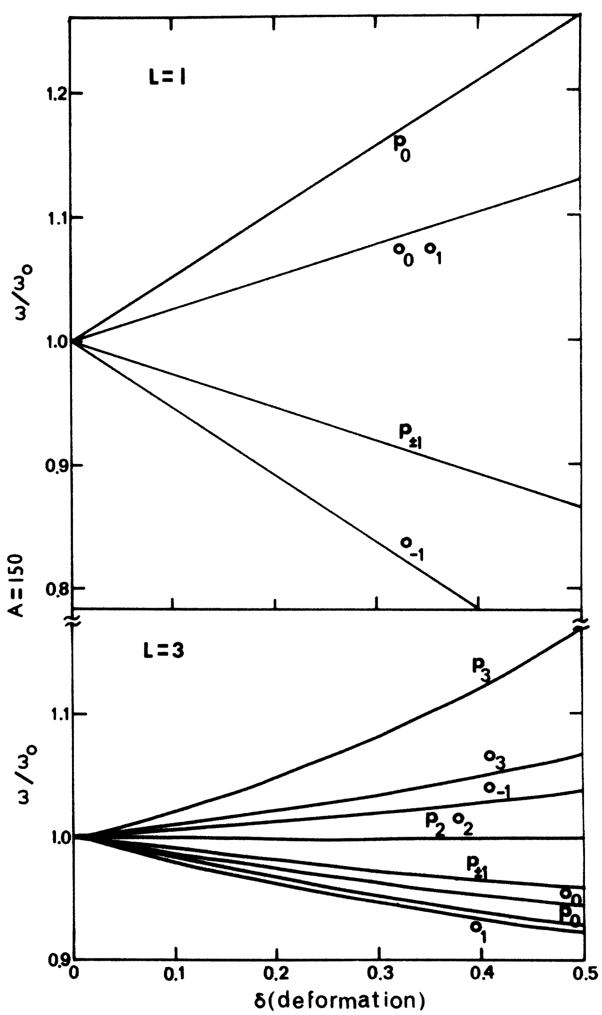


FIG. 3. Same as Fig. 1 for the isoscalar giant dipole and octupole resonances of deformed nuclei.

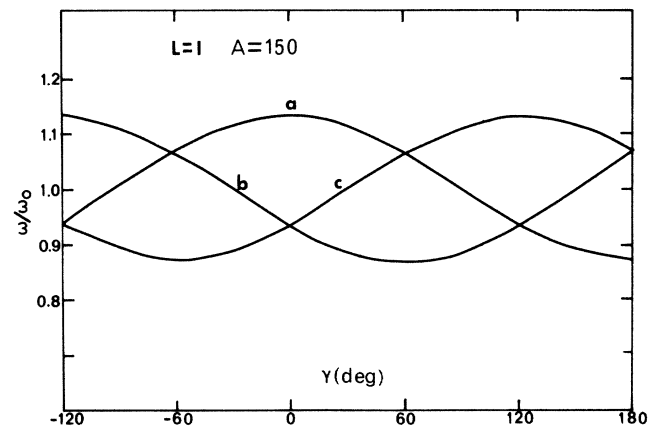


FIG. 4. Same as Fig. 2 for the isoscalar giant dipole resonance. Curves $a$, $b$, and $c$ stand for $m=0$, $-1$, and 1, respectively.

Explicit forms of Eq. (3.15) for some multipole values of $l$ are certainly helpful to understand how the deformation affects the giant resonance energies with respect to those of spherical nuclei. For $l=1$, we get

$$\frac{\omega}{\omega_0}\simeq 1\pm\tfrac{2}{3}q_1\delta,\quad m=0,\quad m=-1\ ,$$

$$\simeq 1\mp\tfrac{1}{3}q_1\delta,\quad m=\pm1,\quad m=0,1\ , \tag{3.16}$$

where the upper signs are for prolate deformation and the lower signs are for oblate deformation. Here

$$q_1=\frac{1}{5}\left[\frac{d_1}{c_1}-\frac{b_1}{a_1}\right]$$

which is equal to 0.8 for all nuclei. The formula (3.16) is to be compared with a similar expression[17] for the isovector giant dipole resonance of deformed nuclei, obtained within the framework of the hydrodynamical model. As we have discussed in connection with Fig. 4, the signs in front of the terms with $\delta$ are opposite in the two corresponding formulas. Besides, the constant $q_1$ in the hydrodynamical expression is about 0.9.

Similarly, for the giant quadrupole resonance, we get

$$\frac{\omega}{\omega_0}\simeq 1-\tfrac{2}{3}q_2\delta,\quad m=0$$

$$\simeq 1-\tfrac{1}{3}q_2\delta,\quad m=\pm1$$

$$\simeq 1+\tfrac{2}{3}q_2\delta,\quad m=2\ , \tag{3.17}$$

for prolate deformation and

$$\frac{\omega}{\omega_0}\simeq 1-\tfrac{1}{3}q_2\delta,\quad m=0$$

$$\simeq 1-\tfrac{1}{6}(3m+1)q_2\delta,\quad m=\pm1$$

$$\simeq 1+\tfrac{1}{3}q_2\delta,\quad m=2\ , \tag{3.18}$$

for oblate deformation, where

$$q_2=\frac{1}{7}\left[\frac{b_2}{a_2}-\frac{d_2}{c_2+z_2}\right]$$

which is approximately equal to 0.32 for all nuclei. This value of $q_2$ is to be compared with the corresponding value 0.45 obtained in the hydrodynamical approach[17] for the isovector giant quadrupole resonance.

For the octupole mode of vibration, we have

$$\begin{aligned}\frac{\omega}{\omega_0}&\simeq 1-\tfrac{8}{15}q_3\delta\\ &\simeq 1-\tfrac{2}{15}q_3\delta\\ &\simeq 1\\ &\simeq 1+\tfrac{2}{3}q_3\delta\ ,\end{aligned} \tag{3.19}$$

for prolate deformation and for $m=0$, $\pm 1$, 2, and 3, respectively, and

$$\begin{aligned}\frac{\omega}{\omega_0}&\simeq 1-\tfrac{4}{15}q_3\delta\\ &\simeq 1-\tfrac{1}{5}(2m+1)q_3\delta\\ &\simeq 1\\ &\simeq 1+\tfrac{1}{3}q_3\delta\ ,\end{aligned} \tag{3.20}$$

for oblate deformation and for $m=0$, $\pm 1$, 2, and 3, respectively, where

$$q_3=\frac{1}{6}\left[\frac{b_3}{a_3}-\frac{d_3}{c_3+z_3}\right]$$

which is approximately equal to 0.34 for all nuclei. This value of $q_3$ is to be compared with the corresponding value 0.45 obtained in the hydrodynamical approach.[17] It is to be remarked that the values of $q_l$ are constant for each multipole except for the $A=50$ region where the nuclear elasticity model is to be applied with care.

## IV. GIANT MONOPOLE RESONANCE IN DEFORMED NUCLEI

The fragmentation of the giant monopole resonance which has been observed and which has now been confirmed[4,5] can be explained by the contribution of the surface deformation through the coupling between the monopole and quadrupole modes of vibration. The trial displacement vector $\mathbf{u}$ in Eq. (3.4) now takes the form

$$\mathbf{u}=\Gamma_{00}\mathbf{u}_{00}+\Gamma_{20}\mathbf{u}_{20}+\Gamma_{22}\mathbf{u}_{22}\ , \tag{4.1}$$

where $\mathbf{u}_{lm}$ stands for the displacement vector $\mathbf{u}$ having specific values of $l$ and $m$, and $\Gamma_{lm}$ are the variational parameters which are to be determined from the variational principle. As before, the angular functions in $\mathbf{u}_{lm}$ are the real version of spherical harmonics $\Phi_{l,\pm m}(\theta,\phi)$, defined by Eq. (3.7). It is noted that, owing to the angular momentum coupling rule, the displacement vector $\mathbf{u}_{21}$ cannot be coupled with that of $\mathbf{u}_{00}$, at least in the first order of collective variables $\alpha_{2\nu}$.

Let us first define two quantities which arise from the coupling between the monopole and quadrupole modes of oscillation and which appear in the subsequent formulation in addition to the quantities $a_l$, $b_l$, $c_l$, and $d_l$. These are

$$\begin{aligned}b_{02}&=\sqrt{(2/5)}\rho R_0^3[U_0(R_0)U_2(R_0)]\ ,\\ d_{02}&=-\sqrt{(2/5)}R_0\left\{\lambda j_0(\xi_0)j_2(\xi_2)R_0^2+2\mu\left[r^2\frac{\partial U_0}{\partial r}\frac{\partial U_2}{\partial r}+U_0U_2+U_0(U_2-6V_2)\right]_{r=R_0}\right\}\ ,\end{aligned} \tag{4.2}$$

where $\xi_0$ and $\xi_2$ are the eigenvalues $hR_0$ corresponding to $l=0$ and $l=2$, respectively. The numerator of the variational Eq. (3.4) becomes now

$$\Gamma_{00}^2c_0+(\Gamma_{20}^2+\Gamma_{22}^2)c_2+\tfrac{1}{3}\Gamma_{00}(2\Gamma_{20}\cos\gamma+\sqrt{2}\Gamma_{22}\sin\gamma)\delta d_{02}+\tfrac{4}{21}[(\Gamma_{20}^2-\Gamma_{22}^2)\cos\gamma+\Gamma_{20}\Gamma_{22}\sin\gamma]\delta d_2\ , \tag{4.3}$$

whereas the denominator of Eq. (3.4) yields the same expression but with $a_0$, $a_2$, $b_{02}$, and $b_2$ in place of $c_0$, $c_2$, $d_{02}$, and $d_2$. Upon introducing the expression (4.3) into Eq. (3.4), we get the equation

$$\left[\sum_{lm,l'm'}^{0,2}\Gamma_{lm}\Gamma_{l'm'}B_{lm,l'm'}\right]\omega^2-\sum_{lm,l'm'}^{0,2}\Gamma_{lm}\Gamma_{l'm'}D_{lm,l'm'}=0\ , \tag{4.4}$$

where $B_{lm,l'm'}$ and $D_{lm,l'm'}$ are functions of $a_{l(l')}$, $b_{l(l')}$, $c_{l(l')}$, $d_{l(l')}$, $b_{ll'}$, $d_{ll'}$, as well as deformation parameters $\gamma$ and $\delta$. This equation can be solved either for the ratio $\omega^2/\omega_{00}^2$ or for the ratio $\omega^2/\omega_{20}^2$ provided that all variational parameters $\Gamma_{lm}$ are known, where $\omega_{00}$ and $\omega_{20}$ are, respectively, the monopole and quadrupole frequencies of oscillation corresponding to spherical nuclei. To solve Eq. (4.4) we note that, in accordance with the requirements of the variational principle, the differentiation of Eq. (4.4) with respect to $\Gamma_{lm}$ must vanish. As a consequence, we obtain three linear homogeneous equations of $\Gamma_{lm}$:

$$
\begin{aligned}
&(a_0\omega^2-c_0)\Gamma_{00}+\tfrac{1}{3}f_{02}\delta\cos\gamma\Gamma_{20}+\frac{\sqrt{2}}{6}f_{02}\delta\sin\gamma\Gamma_{22}=0\ ,\\
&\tfrac{1}{3}f_{02}\delta\cos\gamma\Gamma_{00}+(a_2\omega-c_2+\tfrac{4}{21}g_2\delta\cos\gamma)\Gamma_{20}+\tfrac{2}{21}g_2\delta\sin\gamma\Gamma_{22}=0\ ,\\
&\frac{\sqrt{2}}{6}f_{02}\delta\sin\gamma\Gamma_{00}+\tfrac{2}{21}g_2\delta\sin\gamma\Gamma_{20}+(a_2\omega^2-c_2-\tfrac{4}{21}g_2\delta\cos\gamma)\Gamma_{22}=0\ ,
\end{aligned}
\tag{4.5}
$$

where $f_{02}=b_{02}\omega^2-d_{02}$ and $g_2=b_2\omega^2-d_2$. The system of Eq. (4.5) has a nonzero solution if and only if the determinant constructed with the factors which multiply each variational parameter $\Gamma_{lm}$ in Eq. (4.5) vanishes. The determinantal equation thus obtained can be reformulated so as to yield a third-order equation with respect to the square of the coupled frequency, and this cubic equation gives generally three real solutions for physically meaningful values of the deformation parameters $\gamma$ and $\delta$. The cubic equation takes the form

$$(1-G_1)\omega^6-(\omega_{00}^2+2\omega_{20}^2-G_2)\omega^4+[\omega_{20}^2(2\omega_{00}^2+\omega_{20}^2)-G_3]\omega^2-(\omega_{00}^2\omega_{20}^4-G_4)=0\ , \tag{4.6}$$

where

$$
\begin{aligned}
G_i=&\tfrac{1}{18}A_i\delta^2(1+\cos^2\gamma)+(\tfrac{2}{21})^2B_i\delta^2(1+3\cos^2\gamma)\\
&-\tfrac{2}{189}C_i\delta^3\cos\gamma[2-(3-\sqrt{2})\sin^2\gamma]\ .
\end{aligned}
\tag{4.7}
$$

Here $A_i$, $B_i$, and $C_i$ are one-column—four-row matrices whose elements are functions of $b_2$, $d_2$, $b_{02}$, and $d_{02}$ (see Appendix C).

When $\delta$ is equal to zero, that is for spherical nuclei, $G_i$ is identically zero, and Eq. (4.6) reduces to

$$(\omega^2-\omega_{00}^2)(\omega^2-\omega_{20}^2)^2=0\ , \tag{4.8}$$

which shows that $\omega^2$ can be independently either $\omega_{00}^2$ or $\omega_{20}^2$ and therefore there is no coupling between the monopole and quadrupole modes of oscillation in spherical nuclei.

When $\gamma$ is equal to 0 or $\pi$, all factors which multiply $\Gamma_{22}$ in Eq. (4.5) vanish, except for the last one in the third equation. The determinantal equation then reduces to an equation in which the left-hand side is a product of two factors, of which one is just the factor in front of $\Gamma_{22}$ in the last equation of (4.5). By putting this factor equal to zero we get

$$\omega^2=\frac{c_2\mp\frac{4}{21}d_2\delta}{a_2\mp\frac{4}{21}b_2\delta}\ , \tag{4.9}$$

which is nothing but the uncoupled giant quadrupole resonance frequency for the component of $l=2$, $m=2$ in axially deformed nuclei. It turns out that in the case of $\gamma=0$ or $\pi$, only the component of $l=2$, $m=0$ of the quadrupole oscillations contributes to the coupling with the monopole mode of oscillation. It is noted that in Eq. (4.5) we have neglected the term $z_l$ arising from surface tension and Coulomb interaction. The determinantal equation for $\gamma=0$ or $\pi$ becomes now

$$(1-g_1)\omega^4-(\omega_{00}^2+\omega_{20}^2-g_2)\omega^2+(\omega_{00}^2\omega_{20}^2-g_3)=0\ , \tag{4.10}$$

where

$$
\begin{bmatrix} g_1\\ g_2\\ g_3\end{bmatrix}=\frac{1}{9a_0a_2}\begin{bmatrix} b_{02}^2\\ 2b_{02}d_{02}\\ d_{02}^2\end{bmatrix}\delta^2\mp\frac{4}{21a_2}\begin{bmatrix} b_2\\ \omega_{00}^2b_2+d_2\\ \omega_{00}^2d_2\end{bmatrix}\ .
\tag{4.11}
$$

In the function $g_i$ the upper sign is for $\gamma=0$ and the lower sign is for $\gamma=\pi$. Equation (4.10) is an elementary quadratic equation for $\omega^2$ and its solutions are straightforward. In practice, it is more convenient to solve Eq. (4.10) for the ratio $\omega^2/\omega_{00}^2$.

For the arbitrary value of $\gamma$ we solve directly Eq. (4.6). By putting

$$\omega^2=\Omega^2+\tfrac{1}{3}(\omega_{00}^2+2\omega_{20}^2-G_2)/(1-G_1)\ ,$$

Eq. (4.6) can be transformed to

$$\Omega^6+p\Omega^2+q=0\ , \tag{4.12}$$

where $p=-\frac{1}{3}a^2+b$ and $q=2(\frac{1}{3}a)^3-\frac{1}{3}ab+c$. The functions $a$, $b$, and $c$ are given by

$$a=-\frac{\omega_{00}^2+2\omega_{20}^2-G_2}{1-G_1},\quad b=\frac{\omega_{20}^2(2\omega_{00}^2+\omega_{20}^2)-G_3}{1-G_1}\ ,$$

and

$$c=-\frac{\omega_{00}^2\omega_{20}^4-G_4}{1-G_1}\ .$$

Since the inequality

$$(\tfrac{1}{3}p)^3+(\tfrac{1}{2}q)^2<0$$

is satisfied for physically meaningful values of $\gamma$ and $\delta$, Eq. (4.12) has generally three real solutions. The coupled frequency $\omega^2$ is finally given by

$$
\begin{aligned}
\omega^2=&2(-1)^{n-1}\sqrt{-(1/3)p}\,\cos\tfrac{1}{3}[\alpha+(n-1)\pi]\\
&+\frac{\omega_{00}^2+2\omega_{20}^2-G_2}{3(1-G_1)},\quad n=1,2,3\ ,
\end{aligned}
\tag{4.13}
$$

where

$$\alpha=\cos^{-1}\left[\frac{\frac{1}{2}q}{\sqrt{-(1/3)p}}\right]\ .$$

Figure 5 displays two ratios $\omega/\omega_{00}$ and $\omega/\omega_{20}$ as a function of deformation parameter $\delta$ for the values of $\gamma=0°$ and 60°. In this figure, where we have used the notations $\omega_0$ and $\omega_2$ instead of $\omega_{00}$ and $\omega_{20}$, Pr and Ob stand for the prolate and oblate deformations, respectively, and the explicit assignment $L=2$ indicates that the states remain uncoupled. The most significant feature in this figure is the appearance of the lower component of the ratio $\omega/\omega_0$

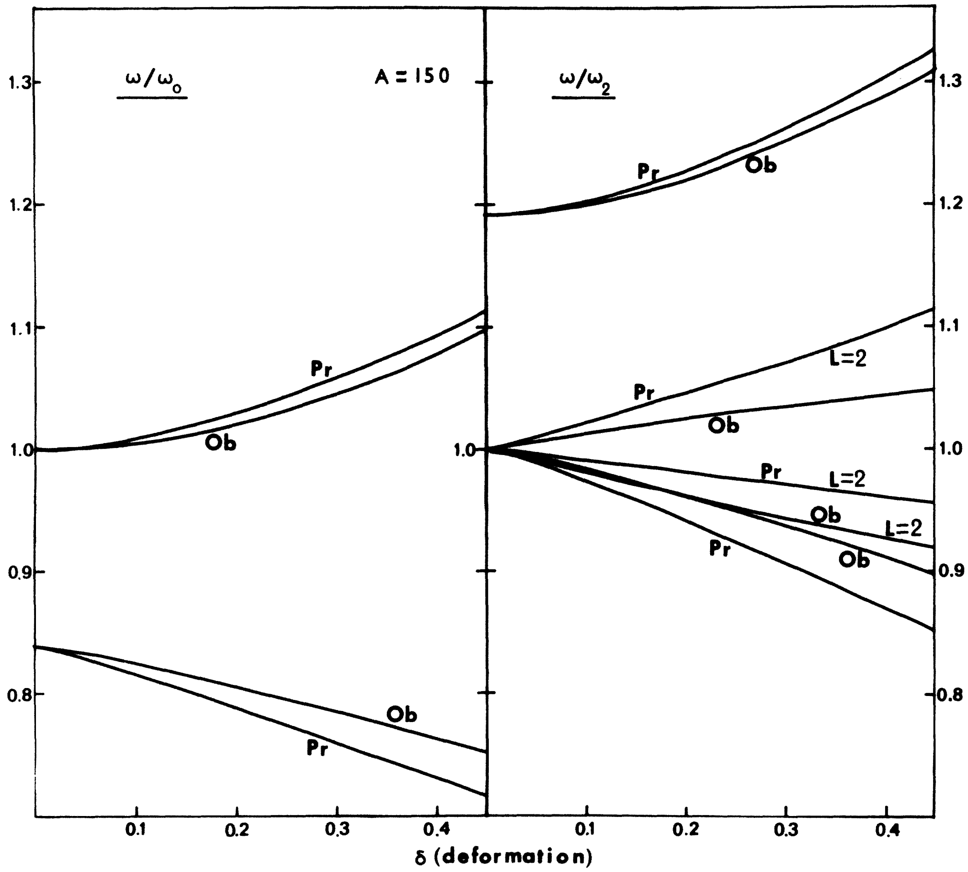


FIG. 5. Fragmentations of the giant monopole and quadrupole resonances versus deformation parameter δ. In order to see the effect of the coupling between the monopole and quadrupole modes of oscillation on the initial frequencies, the coupled frequency is plotted separately for the giant monopole resonance and for the giant quadrupole resonance. For actual coupled resonance energies see Fig. 6. See the text for further details.

and the higher component of the ratio $\omega/\omega_2$. Because of the coupling, assignments of a definite multipole, either monopole or quadrupole, to these states would not make much sense. By comparing with Fig. 1, we see how the coupling affects the resonance energies of the quadrupole modes of oscillation in deformed nuclei. When we multiply the phenomenological giant monopole and quadrupole resonance energies, namely $80A^{-1/3}$ and $64A^{-1/3}$ MeV corresponding to those of spherical nuclei, we can estimate practically the variation of the resonance energies as a function of deformation parameter δ. In the nuclear elasticity approach the giant monopole and quadrupole resonance energies amount to $79A^{-1/3}$ and $66A^{-1/3}$ MeV for the region of $A=150$. Figure 6 shows the calculated coupled giant resonance energies for $\gamma=0°$, 30°, and 60°, that is, prolate, triaxial, and oblate deformations, by assuming a fixed value of the Nilsson deformation parameter δ. The dashed lines indicate the positions of the components of $l=2$, $m=0$ before coupling. The explicit assignments of $L=2$ for 64.4, 66.1, and 66.7 MeV mean that these states having $m=1$ or $m=2$ ($\gamma=0°$) are pure and remain unchanged. As we see, the energy difference between the states in each deformation is more than 3 MeV, except for oblate nuclei where the first two resonance energies are very near each other so that one may

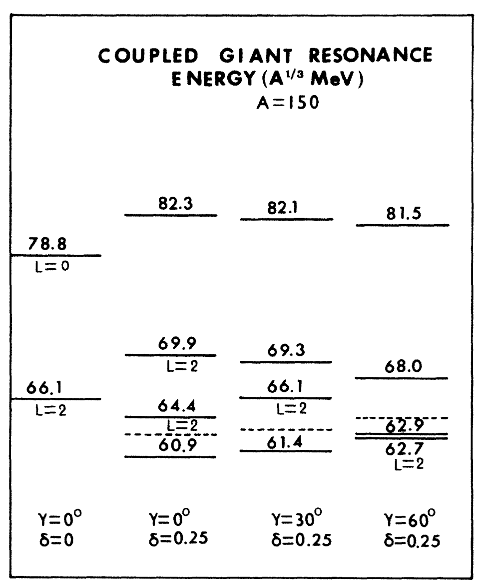


FIG. 6. Coupled giant resonance energies for $l=0$ and $l=2$. Two energy values in the extreme left are those for spherical nuclei and other energy values are calculated using the deformation parameters $\gamma=0°$, 30°, and 60° at $\delta=0.25$.

observe three peaks at about $63A^{-1/3}$, $68A^{-1/3}$, and $81.5A^{-1/3}$ MeV, respectively, instead of four peaks.

It is important at this stage to evaluate the coupling strength between the monopole and quadrupole modes of oscillation. One method of studying the coupling strength in the present approach is to calculate the overlaps between the trial coupled state $\mathbf{u}$ and the individual participants $\mathbf{u}_{lm}$. When we define a reduced coupled displacement vector $\mathbf{u}'$ by $\mathbf{u}'=\mathbf{u}/\Gamma_{00}$, the coupling strength can be expressed as

$$\frac{\langle \mathbf{u} \mid \Gamma_{2m}\mathbf{u}_{2m}\rangle}{\langle \mathbf{u}\mid \mathbf{u}\rangle}=\frac{\left\langle \mathbf{u}' \left| \frac{\Gamma_{2m}}{\Gamma_{00}}\mathbf{u}_{2m}\right.\right\rangle}{\langle \mathbf{u}'\mid \mathbf{u}'\rangle} . \tag{4.14}$$

Once the values of $\omega$ are obtained from either Eq. (4.6) or (4.10), the ratio of the variational parameters $\Gamma_{2m}/\Gamma_{00}$ can be calculated from Eq. (4.5). For example, for $\gamma=0°$, we get a simple expression

$$\frac{\Gamma_{20}}{\Gamma_{00}}=-\frac{(b_{02}\omega^2-d_{02})\delta}{(a_2+\frac{4}{21}b_2)\omega^2-(c_2+\frac{4}{21}d_2\delta)} . \tag{4.15}$$

For oblate nuclei we get

$$\frac{\Gamma_{20}}{\Gamma_{00}}=6\frac{\sqrt{2}\frac{a_0\omega^2-c_0}{f_{02}}g_2-\frac{7}{12}f_{02}\delta}{21(a_2\omega^2-c_2)+(2-\sqrt{2})g_2\delta} , \tag{4.16}$$

$$\frac{\Gamma_{22}}{\Gamma_{00}}=-2\sqrt{6}\left[\frac{a_0\omega^2-c_0}{f_{02}\delta}+\frac{1}{6}\frac{\Gamma_{20}}{\Gamma_{00}}\right] . \tag{4.17}$$

Table II shows the ratios $\Gamma_{20}/\Gamma_{00}$ and $\Gamma_{22}/\Gamma_{00}$ for prolate and oblate deformations as a function of $\delta$. The values of the coupling strength defined by Eq. (4.14) as well as the resonance energies are also given in this table. The resonance energies without coupling strength are those for uncoupled states, namely the components with $m=1$ for both the prolate and oblate deformations and the com-

TABLE II. Coupled resonance energies, ratios $\Gamma_{20(22)}/\Gamma_{00}$, and overlaps between the coupled state and the individual participants to the coupling. The energies without coupling strengths are for the states which remain uncoupled. $a$, $b$, and $c$ refer to $\langle \mathbf{u} \mid \Gamma_{00}\mathbf{u}_{00}\rangle/\langle \mathbf{u}\mid\mathbf{u}\rangle$, $\langle \mathbf{u} \mid \Gamma_{20}\mathbf{u}_{20}\rangle/\langle \mathbf{u}\mid\mathbf{u}\rangle$, and $\langle \mathbf{u} \mid \Gamma_{22}\mathbf{u}_{22}\rangle/\langle \mathbf{u}\mid\mathbf{u}\rangle$, respectively.

| $\delta$ | 0.15 | | | | 0.25 | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | Prolate ($\gamma=0°$) | | | | |
| $E$ ($A^{-1/3}$ MeV) | 80.22 | 68.23 | 65.05 | 63.15 | 82.25 | 69.85 | 64.42 | 60.85 |
| $\Gamma_{20}/\Gamma_{00}$ | −0.350 | | | 4.689 | −0.481 | | | 3.345 |
| $a$ | 0.963 | | | 0.099 | 0.937 | | | 0.173 |
| $b$ | 0.037 | | | 0.901 | 0.063 | | | 0.827 |
| | | | | Oblate ($\gamma=60°$) | | | | |
| $E$ ($A^{-1/3}$ MeV) | 79.82 | 67.26 | 64.21 | 64.11 | 81.50 | 67.94 | 62.93 | 62.65 |
| $\Gamma_{20}/\Gamma_{00}$ | −0.180 | −4.256 | 4.064 | | −0.245 | −3.359 | | 2.722 |
| $\Gamma_{22}/\Gamma_{00}$ | −0.272 | 8.778 | 3.708 | | −0.427 | 5.706 | | 2.557 |
| $a$ | 0.951 | 0.012 | 0.066 | | 0.898 | 0.021 | | 0.137 |
| $b$ | 0.008 | 0.208 | 0.537 | | 0.009 | 0.284 | | 0.507 |
| $c$ | 0.041 | 0.781 | 0.397 | | 0.093 | 0.695 | | 0.356 |
| $\delta$ | 0.35 | | | | 0.45 | | | |
| | | | | Prolate ($\gamma=0°$) | | | | |
| $E$ ($A^{-1/3}$ MeV) | 84.69 | 71.61 | 63.81 | 58.57 | 87.47 | 73.54 | 63.22 | 56.40 |
| $\Gamma_{20}/\Gamma_{00}$ | −0.566 | | | 2.791 | −0.623 | | | 2.489 |
| $a$ | 0.922 | | | 0.226 | 0.917 | | | 0.264 |
| $b$ | 0.078 | | | 0.774 | 0.083 | | | 0.736 |
| | | | | Oblate ($\gamma=60°$) | | | | |
| $E$ ($A^{-1/3}$ MeV) | 83.74 | 68.56 | 61.83 | 61.01 | 86.45 | 69.14 | 60.81 | 59.36 |
| $\Gamma_{20}/\Gamma_{00}$ | −0.279 | −3.011 | | 2.159 | −0.294 | −2.836 | | 1.850 |
| $\Gamma_{22}/\Gamma_{00}$ | −0.559 | 4.435 | | 2.096 | −0.671 | 3.753 | | 1.855 |
| $a$ | 0.849 | 0.027 | | 0.201 | 0.800 | 0.027 | | 0.256 |
| $b$ | 0.004 | 0.349 | | 0.480 | 0.005 | 0.404 | | 0.459 |
| $c$ | 0.147 | 0.624 | | 0.319 | 0.195 | 0.569 | | 0.285 |

ponents with $m=2$ for the prolate deformation. As was expected, the contribution to the higher energy state comes mostly from the monopole mode of oscillation, whereas the contribution to the lower energy state comes generally from the quadrupole mode of oscillation.

## V. SUMMARY AND CONCLUSION

In this paper, we have investigated the isoscalar giant resonances of deformed nuclei within the framework of the nuclear elasticity. In particular, we have shown how the coupling between the monopole and quadrupole modes of oscillation modifies the initial fragmentation of the giant quadrupole resonance and thus gives rise to the fragmentation of the giant monopole resonance. We have first briefly reviewed the method of derivation of the eigenvalue equation for elastic vibrations of spherical nuclei and this supplies a basis for the variational procedure we have subsequently developed for the discussion of the giant resonances of deformed nuclei. The variational equation we have solved allowed us to express the frequency of vibration in a simple form which displays explicitly the dependence of the resonance energies on deformation parameters.

Although we have introduced surface tension and Coulomb force in the course of formulation of the variational equation, we have not included the effects of these forces in our numerical calculations. As pointed out by Wong and Azziz,[2] the inclusion of surface tension and Coulomb repulsion does not produce much changes in the energies and characteristics of the elastic vibrational states. Furthermore, we have not considered the damping of the giant resonances. The introduction of the damping due to the two-body viscosity leads[2] to the Lamé-Navier-Stokes equation of motion and the extension of the present method to such a viscoelastic system is beyond the scope of the present investigation. The giant resonances of fast rotating nuclei in the framework of the elastic model will be discussed in a forthcoming paper.

The author wishes to thank Dr. P. Ring for helpful discussions.

## APPENDIX A

In spherical polar coordinates, the antisymmetric strain tensors $e_{ij}$ $(i\neq j)$ of Eq. (2.3) take the forms

$$e_{12}=e_{21}=e_{r\theta}=\frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{1}{r}u_\theta\right],$$

$$e_{23}=e_{32}=e_{\theta\phi}=\frac{1}{2}\left[\frac{1}{r}\frac{\partial u_\phi}{\partial\theta}-\frac{1}{r}\cot\theta u_\phi+\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}\right],$$

$$e_{31}=e_{13}=e_{\phi r}=\frac{1}{2}\left[\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\phi}{\partial r}-\frac{1}{r}u_\phi\right].$$

These complete the expression (2.12) for $e_{ij}$.

In the absence of surface tension and Coulomb repulsion, the two ratios $C_l/A_l$ obtained from Eqs. (2.13) and (2.14) are given by

$$\frac{\left[\frac{\eta}{\xi}\right]^2 j_l(\xi)+\frac{4}{\xi}\frac{dj_l(\xi)}{d\xi}-\frac{2l(l+1)}{\xi^2}j_l(\xi)}{2l(l+1)\frac{d}{d\eta}\left[\frac{j_l(\eta)}{\eta}\right]}$$

and

$$-\frac{2\frac{d}{d\xi}\left[\frac{j_l(\xi)}{\xi}\right]}{\frac{d^2}{d\eta^2}j_l(\eta)+\frac{l(l+1)-2}{\eta^2}j_l(\eta)}.$$

By equating these two expressions we obtain the eigenvalue Eq. (2.15).

## APPENDIX B

In this appendix we first derive Eq. (3.1). The dilatation $\Delta$ of Eq. (2.3) in spherical polar coordinates is simply

$$\Delta=\nabla\cdot\mathbf{u}=e_{rr}+e_{\theta\theta}+e_{\phi\phi}.$$

With this dilatation we get

$$(\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u})=(\lambda+2\mu)\left[\frac{\partial\Delta}{\partial r}\mathbf{n}_r+\frac{1}{r}\frac{\partial\Delta}{\partial\theta}\mathbf{n}_\theta+\frac{1}{r\sin\theta}\frac{\partial\Delta}{\partial\phi}\mathbf{n}_\phi\right].$$

The radial part $\mathbf{n}_r$ of this relation can be reformulated so as to give

$$\frac{\partial}{\partial r}T_{rr}+\frac{1}{r}\frac{\partial}{\partial\theta}T_{r\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}T_{\phi r}+\frac{2\mu}{r}(2e_{rr}-e_{\theta\theta}-e_{\phi\phi}+\cot\theta e_{r\theta})+\mu[\nabla(\nabla\cdot\mathbf{u})-\nabla^2\mathbf{u}]\mathbf{n}_r,$$

where the stress-strain tensors $T$ take the forms $T_{\alpha\alpha}=\lambda\Delta+2\mu e_{\alpha\alpha}$ and $T_{\alpha\beta}=2\mu e_{\alpha\beta}$ $(\alpha\neq\beta)$ with $\alpha,\beta=r$, $\theta$, or $\phi$. Similarly, the $\mathbf{n}_\theta$ part can be written as

$$\frac{\partial}{\partial r}T_{r\theta}+\frac{1}{r}\frac{\partial}{\partial\theta}T_{\theta\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}T_{\theta\phi}+\frac{1}{r}[2\mu(e_{\theta\theta}-e_{\phi\phi})\cot\theta+3T_{r\theta}]+\mu[\nabla(\nabla\cdot\mathbf{u})-\nabla^2\mathbf{u}]\mathbf{n}_\theta.$$

In the same way we can transform the $\mathbf{n}_\phi$ part. We have

$$\frac{\partial}{\partial r}T_{\phi r}+\frac{1}{r}\frac{\partial}{\partial\theta}T_{\theta\phi}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}T_{\phi\phi}+\frac{1}{r}(3T_{\phi r}+2T_{\theta\phi}\cot\theta)+\mu[\nabla(\nabla\cdot\mathbf{u})-\nabla^2\mathbf{u}]\mathbf{n}_\phi.$$

By remarking the vector relation $\nabla(\nabla\cdot\mathbf{u})-\nabla^2\mathbf{u}=\nabla\times\nabla\times\mathbf{u}$ and using the identity

$$2\mu(2e_{rr}-e_{\theta\theta}-e_{\phi\phi})=2T_{rr}-T_{\theta\theta}-T_{\phi\phi}\ ,$$

we see that Eq. (2.4) transforms to Eq. (3.1).

We next show how to derive the variational Eq. (3.4). When we multiply Eq. (3.1) by $\mathbf{u}^*$ and integrate the result over the volume, the right-hand side becomes simply

$$-\omega^2\int\rho\,|\,\mathbf{u}\,|^2 d\tau\ .$$

As for the left-hand side, we first integrate by part those integrals having the derivatives in the integrands and then apply the boundary conditions (2.11) or (2.17). In this way we get, for example,

$$\int u_r^*\left[\frac{\partial}{\partial r}T_{rr}+\frac{1}{r}\frac{\partial}{\partial\theta}T_{r\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}T_{r\phi}\right]d\tau$$
$$=-\lambda\int\frac{1}{r^2}\frac{\partial}{\partial r}(u_r^* r^2)\Delta d\tau-2\mu\int\left\{\frac{1}{r^2}\frac{\partial}{\partial r}(u_r^* r^2)e_{rr}+\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(u_r^*\sin\theta)e_{r\theta}+\frac{\partial u_r^*}{\partial\phi}e_{r\phi}\right]\right\}d\tau\ ,$$

$$\int u_\theta^*\left[\frac{\partial}{\partial r}T_{r\theta}+\frac{1}{r}\frac{\partial}{\partial\theta}T_{\theta\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}T_{\theta\phi}\right]d\tau$$
$$=-\lambda\int\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta^*\sin\theta)\Delta d\tau-2\mu\int\left\{\frac{1}{r^2}\frac{\partial}{\partial r}(u_\theta^* r^2)e_{\theta r}+\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(u_\theta^*\sin\theta)e_{\theta\theta}+\frac{\partial u_\theta^*}{\partial\phi}e_{\theta\phi}\right]\right\}d\tau\ ,$$

$$\int u_\phi^*\left[\frac{\partial}{\partial r}T_{\phi r}+\frac{1}{r}\frac{\partial}{\partial\theta}T_{\theta\phi}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}T_{\phi\phi}\right]d\tau$$
$$=-\lambda\int\frac{1}{r\sin\theta}\frac{\partial u_\phi^*}{\partial\phi}\Delta d\tau-2\mu\int\left\{\frac{1}{r^2}\frac{\partial}{\partial r}(u_\phi^* r^2)e_{\phi r}+\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(u_\phi^*\sin\theta)e_{\phi\theta}+\frac{\partial u_\phi^*}{\partial\phi}e_{\phi\phi}\right]\right\}d\tau\ .$$

Adding together these results and the rest of the integrals, we finally obtain

$$-\lambda\int(\nabla\cdot\mathbf{u}^*)(\nabla\cdot\mathbf{u})d\tau-2\mu\int(e_{rr}^*e_{rr}+e_{\theta\theta}^*e_{\theta\theta}+e_{\phi\phi}^*e_{\phi\phi}+2e_{r\theta}^*e_{r\theta}+2e_{\theta\phi}^*e_{\theta\phi}+2e_{\phi r}^*e_{\phi r})d\tau\ .$$

The explicit forms of the quantities $c_l$ and $d_l$ in Eq. (3.11) are given by

$$c_l=\lambda\int[-A_lj_l(hr)]^2r^2dr+2\mu\int\left\{\left[\frac{\partial U_l}{\partial r}\right]^2r^2+U_l^2+[U_l-l(l+1)V_l]^2+\tfrac{1}{2}l(l+1)\left[U_l+r\frac{\partial V_l}{\partial r}-V_l\right]^2-l(l+1)V_l^2\right\}dr\ ,$$

$$d_l=R_0^3[-A_lj_l(hR_0)]^2+2\mu R_0\left\{R_0^2\left[\frac{\partial U_l}{\partial r}\right]^2_{r=R_0}+U_l^2(R_0)+[U_l(R_0)-l(l+1)V_l(R_0)]^2-[7l(l+1)-12]V_l^2(R_0)\right\}\ .$$

The integration of the first term in $c_l$ is straightforward and the second integral can also be evaluated analytically giving a very lengthy expression.

## APPENDIX C

The quantities $A_i$, $B_i$, and $C_i$ in Eq. (4.7) have the forms

$$\begin{bmatrix}A_1\\A_2\\A_3\\A_4\end{bmatrix}=\frac{1}{a_0a_2}\begin{bmatrix}b_{02}^2\\2b_{02}d_{02}+\omega_{20}^2b_{02}^2\\d_{02}^2+2\omega_{20}^2b_{02}d_{02}\\\omega_{20}^2d_{02}^2\end{bmatrix},\quad\begin{bmatrix}B_1\\B_2\\B_3\\B_4\end{bmatrix}=\frac{1}{a_2^2}\begin{bmatrix}b_2^2\\2b_2d_2+\omega_{00}^2b_2^2\\d_2^2+2\omega_{00}^2b_2d_2\\\omega_{00}^2d_2^2\end{bmatrix},\quad\begin{bmatrix}C_1\\C_2\\C_3\\C_4\end{bmatrix}=\frac{1}{a_0a_2^2}\begin{bmatrix}b_2b_{02}^2\\2b_2b_{02}d_{02}+b_{02}^2d_2\\b_2d_{02}^2+2b_{02}d_{02}d_2\\d_{02}^2d_2\end{bmatrix}.$$


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