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Interacting boson model with surface delta interaction between nucleons: Structure and interaction of bosons

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The surface delta interaction is used as an effective nucleon-nucleon interaction to investigate the structure and interaction of the bosons in the interacting boson model. We have obtained analytical expressions for the coefficients of a multipole expansion of the neutron-boson-proton-boson interaction for the case of degenerate orbits. A connection is made between these coefficients and the parameters of the interaction boson model Hamiltonian. A link between the latter parameters and the single boson energies is suggested.

I. INTRODUCTION

The interacting boson model[1,2] has been an important development in nuclear structure physics. In both its forms, IBM-1 and IBM-2, the model has had remarkable phenomenological success. Furthermore, tremendous progress has been made in formulating the theoretical basis of the model. This has allowed a connection to be made between the phenomenological model and the underlying microscopic structure, which is the nuclear shell model.

Much of the theoretical work on the IBM has focused on the calculation of the parameters of the IBM-2 Hamiltonian from a microscopic basis.[3] A reasonable procedure which is often used for doing such a calculation, is the following:

(1) A representation is developed of the bosons in the underlying fermionic, shell model space; i.e., s and d bosons are represented as S and D pairs in the fermion space.

(2) An effective interaction between the S and D pairs in the fermion space is chosen.

(3) A mapping is developed between the fermion picture and the corresponding picture in the boson space.

(4) The parameters of the IBM Hamiltonian are calculated and/or the spectrum of the particular system is determined.

The IBM is a drastic truncation of the full shell-model calculation. However, it is possible to ameliorate the severity of the truncation by refining the above procedure at each step. For example, in addition to s and d bosons, other bosons, such as g, i, k, etc., can be included. It is also possible to include noncollective degrees of freedom as well. It is hoped that much of the effect of configurations that lie outside of the IBM can be taken into account by renormalization of the parameters of the IBM Hamiltonian.

The effective interaction is an important choice to be made because, not only does it determine the interactions between the bosons, but it governs the coupling between the s-d space and degrees of freedom outside the s-d space. This coupling plays an important part in the renormalization of the parameters of the IBM Hamiltonian. It is usual to use a quadrupole-quadrupole interaction but higher multipoles in the interaction are needed and have to be put in by hand. This leads to more parameters in the theory. An attractive feature of the surface delta interaction (SDI) (Ref. 4) is that *all* multipoles are included and are determined by a *single* parameter, the strength.

II. BOSON-BOSON INTERACTION

We will describe a calculation in which we followed a procedure such as outlined in steps 1 through 4 above. We will restrict ourselves to rather schematic situations in which we consider two boson, i.e., four particle configurations, where the nucleons occupy degenerate orbits.

We shall construct the bosons by determining the spectrum arising from placing two identical particles in one or several different degenerate j orbits. If these particles interact via an SDI, then for each value of angular momentum Λ, only a *single* state of each Λ is shifted in energy. All the other states have zero interaction energy. We consider this lowest energy state of each Λ to be a boson of angular momentum Λ. Thus for $\Lambda=0,2,4,\ldots$, we construct $s, d, g, \ldots$, bosons, respectively.

The energy of a Λ boson is given by:[5]

$$e_\Lambda=\sum_j\sum_{j'}\tfrac{1}{2}(2j+1)(2j'+1)\begin{bmatrix} j & j' & \Lambda \\ \frac{1}{2} & -\frac{1}{2} & 0\end{bmatrix}^2 G\ , \quad (1)$$

where the sum j,j' extends over the set of degenerate orbits and G is the strength of the SDI.

For the set of degenerate orbits, we consider several choices: (1) a single j shell; (2) two degenerate orbits, j_1 and j_1+1; (3) two degenerate orbits, j_1 and j_1+2; (4) a degenerate major oscillator shell, i.e., $j_1=\frac{1}{2}, \frac{3}{2}, \ldots, j$.

We now consider a situation in which we have one neutron boson and one proton boson, each constructed in the manner described above, interacting via an SDI in a particular space of degenerate orbits.

The SDI between particles 1 and 2 is written as:[4]

$$V_{\rm SDI}=4\pi G\delta(\Omega_{N_1N_2}) , \quad (2)$$

where G is the strength.

We can also expand the SDI in multipoles, i.e.,

$$V_{\rm SDI}=G[1+5P_2(\cos\Theta_{N_1N_2})+9P_4(\cos\Theta_{N_1N_2})+\cdots] . \quad (3)$$

In a similar fashion, we can make a multipole expansion for the interaction between the neutron boson and proton boson, namely

$$V_{\rm n\text{-}p}=V_{\rm n\text{-}boson-p\text{-}boson} =F_0+5F_2P_2(\cos\Theta_{\rm np})+9F_4P_4(\cos\Theta_{\rm np})+\cdots . \quad (4)$$

We do not specify here a relation between $\Theta_{N_1N_2}$ and $\Theta_{\rm np}$, but require the equality of matrix elements of the two-neutron and two-proton system with those of one-neutron and one-proton boson. Thus we have mapped the fermion picture onto the boson representation. This is known as the Otsuka, Arima, and Iachello (OAI) mapping.[6]

If the angular momentum of each boson is labeled by Λ_ρ, where $\rho=$n or p, then the matrix elements of $V_{\rm n\text{-}boson-p\text{-}boson}$ can be evaluated in terms of the F_0, F_2, F_4, etc. The matrix elements are given by:[5]

$$\langle \Lambda_{\rm n}\Lambda_{\rm p};J \mid V_{\rm n\text{-}p} \mid \Lambda'_{\rm n}\Lambda'_{\rm p};J\rangle=(-)^J\hat\Lambda_{\rm n}\hat\Lambda_{\rm p}\hat\Lambda'_{\rm n}\hat\Lambda'_{\rm p}\sum_K(2K+1)\begin{bmatrix}\Lambda_{\rm p} & \Lambda'_{\rm p} & K\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\Lambda_{\rm n} & \Lambda'_{\rm n} & K\\ 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}\Lambda_{\rm p} & \Lambda_{\rm n} & J\\ \Lambda'_{\rm n} & \Lambda'_{\rm p} & K\end{Bmatrix}F_K , \quad (5)$$

where $\hat\Lambda=(2\Lambda+1)^{1/2}$.

By using an SDI as the origin of the effective n-boson—p-boson interaction, we can evaluate numerically all the matrix elements on the left-hand side of Eq. (5). Diagonalization of the Hamiltonian matrix then yields the two-boson spectrum.

One can also invert Eq. (5) and evaluate the parameters, F_K, of the n-boson—p-boson interaction in terms of the two-boson matrix elements. Notice that all the radial information is contained in the F_K and thus the F_K depend on the structure of the bosons and on the single particle configurations. We will use F_K, F'_K, and F''_K to denote coefficients of the Kth multipole that have different radial components. However, there is considerable simplification if the SDI is used. For example, we find that always:

$$F_0=4G , \quad (6)$$

where G is the strength of the SDI.

The other free parameters are

$$F_2=\sqrt{(1/5)}\langle \Lambda_{\rm n}=0,\Lambda_{\rm p}=0;0 \mid V_{\rm n\text{-}p} \mid \Lambda'_{\rm n}=2,\Lambda'_{\rm p}=2;0\rangle , \quad (7a)$$

$$F'_2=(\tfrac{2}{5})\langle \Lambda_{\rm n}=2,\Lambda_{\rm p}=2;0 \mid V_{\rm n\text{-}p} \mid \Lambda'_{\rm n}=2,\Lambda'_{\rm p}=2;0\rangle -(\tfrac{7}{5})\langle \Lambda_{\rm n}=2,\Lambda_{\rm p}=2;2 \mid V_{\rm n\text{-}p} \mid \Lambda'_{\rm n}=2,\Lambda'_{\rm p}=2;2\rangle+F_0 , \quad (7b)$$

$$F''_2=-\sqrt{(7/10)}\langle \Lambda_{\rm n}=0,\Lambda_{\rm p}=2;2 \mid V_{\rm n\text{-}p} \mid \Lambda'_{\rm n}=2,\Lambda'_{\rm p}=2;2\rangle =-\sqrt{(7/10)}\langle \Lambda_{\rm n}=2,\Lambda_{\rm p}=0;2 \mid V_{\rm n\text{-}p} \mid \Lambda'_{\rm n}=2,\Lambda'_{\rm p}=2;2\rangle, \quad (7c)$$

$$F'_4=(\tfrac{1}{6})\langle \Lambda_{\rm n}=2,\Lambda_{\rm p}=2;0 \mid V_{\rm n\text{-}p} \mid \Lambda'_{\rm n}=2,\Lambda'_{\rm p}=2;0\rangle +(\tfrac{7}{9})\langle \Lambda_{\rm n}=2,\Lambda_{\rm p}=2;2 \mid V_{\rm n\text{-}p} \mid \Lambda'_{\rm n}=2,\Lambda'_{\rm p}=2;2\rangle -(\tfrac{17}{18})F_0 . \quad (7d)$$

To make contact with the IBM, we rewrite $V_{\rm n\text{-}p}$ in terms of the usual IBM parameters,[1,2] i.e.,

$$V_{\rm n\text{-}p}=E_0+\kappa Q_\pi^{(2)}\cdot Q_\nu^{(2)}+M_{\nu\pi}+V_x . \quad (8a)$$

Here E_0 is a constant, $Q_\rho^{(2)}$ is the quadrupole operator given by:

$$Q_\rho^{(2)}=(d_\rho^\dagger s_\rho+s_\rho^\dagger\tilde d_\rho)^{(2)}+\chi_\rho(d_\rho^\dagger\tilde d_\rho)^{(2)} , \quad \rho={\rm n,p} \quad (8b)$$

and $M_{\nu\pi}$ is the Majorana operator given by:

$$M_{\nu\pi}=\xi_2(d_\nu^\dagger s_\pi^\dagger-d_\pi^\dagger s_\nu^\dagger)^{(2)}(\tilde d_\nu s_\pi-\tilde d_\pi s_\nu)^{(2)} +\sum_{K=1,3}\xi_K(d_\nu^\dagger d_\pi^\dagger)^{(K)}(\tilde d_\nu\tilde d_\pi)^{(K)} . \quad (8c)$$

V_x represents extra terms that are usually neglected in the IBM Hamiltonian. We can write V_x as:[7]

$$V_x=\sum_{M=0,2,4}Y_M(d_\nu^\dagger d_\pi^\dagger)^{(M)}(\tilde d_\nu\tilde d_\pi)^{(M)} . \quad (8d)$$

We can link the parameters of $V_{\rm n\text{-}p}$ given in Eq. (7) with the usual IBM parameters given in Eq. (8). These expressions are given in Appendix A.

In an earlier publication,[5] one of us (Moszkowski) showed that for certain matrix elements, the coefficients of the n-boson—p-boson interaction are related to the single boson energies. It can be shown that for matrix elements were $\Lambda_{\rm n}=\Lambda_{\rm p}=0$ and $\Lambda'_{\rm n}=\Lambda'_{\rm p}=K\neq0$ [see Eq. (5)], the parameters of $V_{\rm n\text{-}p}$ in Eq. (4) are given by:

$$F_K=4G\frac{e_K}{e_0} , \quad (9)$$

where e_K and e_0 are the energies of $J=K$ and $J=0$ bosons, respectively. The energies e_K are given explicitly by Eq. (3). A derivation of the result (9) based on a physical interpretation of the bosons is given in Appendix B.

Equation (9) also holds for matrix elements that have $\Lambda_{\rm n}=0$, $\Lambda_{\rm p}=K\neq0$, $\Lambda'_{\rm n}=K$, $\Lambda'_{\rm p}=0$ as well as for matrix elements having $\Lambda_{\rm n}=K\neq0$, $\Lambda_{\rm p}=0$, $\Lambda'_{\rm n}=0$, $\Lambda'_{\rm p}=K$.

The parameters F_K for matrix elements involving $v=2$

bosons only, i.e., no s bosons, are somewhat more difficult to obtain analytically but can still be obtained in closed form, at least for degenerate configurations with an SDI. These expressions are given in Ref. 5.

To summarize, we have established a connection between the parameters F_K of $V_{\text{n-boson-p-boson}}$ in Eq. (4) and the parameters of the IBM Hamiltonian in Eq. (8). Furthermore, we have a numerical procedure for calculating the F_K [see Eq. (7)]. However, the interesting feature is that, for certain matrix elements, the F_K have a simple, analytical form, namely that given by Eq. (9). The goal now is to see if the exact result given by Eq. (9) for the case where s bosons are involved can be exploited to obtain simple results for the case where only d bosons are involved. If this is possible, we will have established a simple, analytical procedure for determining the parameters of the IBM Hamiltonian in terms of the strength of the SDI.

III. RESULTS

Table I shows a comparison of the ratios:

$$\frac{F'_K}{F_0} \text{ and } \frac{e_K}{e_0}\ ;\ K=2,4 \tag{10}$$

which are equal when Eq. (9) holds. Numerical values are given for the four different configurations of single particle orbitals that we have considered. Table II gives the analytical expressions for these ratios.

These results indicate that Eq. (9) holds approximately for the case of a single j shell and for the case of a degenerate major oscillator shell, i.e., $j_i=\frac{1}{2}, \frac{3}{2}, \ldots, j$. The agreement is slightly worse for the case of two degenerate j shells j_1 and j_1+2. The case of two degenerate j shells having j_1 and j_1+1 apparently causes the greatest violation of Eq. (9). In the limit that $j \rightarrow \infty$, we see that these ratios are equal. This limit, namely that the Λ's of the bosons are small compared to the maximum single particle j, can also be interpreted as a semiclassical limit.

The approximate agreement of the ratios in Eq. (10) means that, at least for certain cases, we have a way of estimating the values of the parameters of the IBM Hamiltonian by knowing the values of the single boson energies.

The task ahead is to generalize these simple estimates to the more realistic case of nondegenerate single particle orbits. Furthermore, it would be interesting to know if the approximate connection between the parameters of the IBM Hamiltonian and the single boson energies has any greater significance.

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TABLE I. Coefficients of a multipole expansion of the boson-boson interaction for different configurations.

(i) Single degenerate j shell				
j	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$
F'_2	0	0.3265	0.5805	0.7199
$4e_2/e_0$	0.8000	0.9143	0.9524	0.9697
F'_4	0	0.1837	0.3265	0.4050
$4e_4/e_0$	0	0.3809	0.4675	0.5035

(ii) Two degenerate j shells j and $j+1$				
j	$\frac{1}{2},\frac{3}{2}$	$\frac{3}{2},\frac{5}{2}$	$\frac{5}{2},\frac{7}{2}$	$\frac{7}{2},\frac{9}{2}$
F'_2	1.9600	0.1836	0.5378	0.7126
$4e_2/e_0$	1.6000	1.1429	1.0667	1.0390
F'_4	0	0.3266	0.4444	0.4918
$4e_4/e_0$	0	1.1429	0.7273	0.6474

(iii) Two degenerate j shells j and $j+2$			
j	$\frac{1}{2},\frac{5}{2}$	$\frac{3}{2},\frac{7}{2}$	$\frac{5}{2},\frac{9}{2}$
F'_2	1.7411	2.0797	2.0111
$4e_2/e_0$	1.8857	2.2730	2.3780
F'_4	0.1354	0.4954	0.6299
$4e_4/e_0$	0.2857	0.7350	1.4831

(iv) Degenerate oscillator shell $j_i=\frac{1}{2}, \frac{3}{2}, \ldots, j$				
j	$\frac{1}{2},\frac{3}{2}$	$\frac{1}{2}$ to $\frac{5}{2}$	$\frac{1}{2}$ to $\frac{7}{2}$	$\frac{1}{2}$ to $\frac{9}{2}$
F'_2	1.9600	2.4693	2.7777	2.9835
$4e_2/e_0$	1.6000	2.2857	2.6667	2.9091
F'_4	0	0.6748	1.3611	1.8390
$4e_4/e_0$	0	0.9524	1.5758	1.9953

TABLE II. Boson-boson multipole coefficients (analytic expressions).

	j	$j-1,j$	$\frac{1}{2}$ to j
$\frac{e_2}{e_0}$	$\frac{1}{4}\cdot\frac{(2j-1)(2j+3)}{(2j)(2j+2)}$	$\frac{1}{4}\cdot\frac{(2j-1)(2j+1)}{(2j-2)(2j+2)}$	$\frac{(2j-1)}{(2j+2)}$
$\frac{F_2'}{F_0}$	$\frac{1}{4}\cdot\frac{(2j-3)^2(2j+5)^2}{(2j)^2(2j+2)^2}$	$\frac{1}{4}\cdot\frac{(2j-4)^2(2j+4)^2}{(2j-2)^2(2j+2)^2}$	$\frac{(4j-1)^2}{(4j+4)^2}$
$\frac{e_4}{e_0}$	$\frac{9}{64}\cdot\frac{(2j-3)(2j-1)(2j+3)(2j+5)}{(2j-2)(2j)(2j+2)(2j+4)}$	$\frac{9}{64}\cdot\frac{(2j-3)(2j-1)(2j+1)(2j+3)}{(2j-4)(2j-2)(2j+2)(2j+4)}$	$\frac{1}{8}\cdot\frac{(2j-3)(2j-1)(16j+35)}{(2j)(2j+2)(2j+4)}$
$\frac{F_4'}{F_0}$	$\frac{9}{64}\cdot\frac{(2j-3)^2(2j+5)^2}{(2j)^2(2j+2)^2}$	$\frac{9}{64}\cdot\frac{(2j-3)^2(2j+5)^2}{(2j-2)^2(2j+2)^2}$	

APPENDIX A: IBM PARAMETERS IN TERMS OF MULTIPOLE COEFFICIENTS

The parameters of the IBM Hamiltonian in Eq. (8) can be expressed in terms of the parameters F_K given in Eq. (7). These are displayed below.

$$H_0=F_0\ , \tag{A1}$$

$$\kappa=F_2\ , \tag{A2}$$

$$\xi_2=0\ , \tag{A3}$$

$$\chi=\chi_\pi=\chi_\nu=-\sqrt{(10/7)}F_2''/F_2\ , \tag{A4}$$

$$\xi_1=-(\tfrac{5}{7})F_2'+(\tfrac{12}{7})F_4'+(\tfrac{1}{2})F_2\chi^2\ , \tag{A5}$$

$$\xi_3=(\tfrac{40}{49})F_2'+(\tfrac{9}{49})F_4'-(\tfrac{4}{7})F_2\chi^2\ , \tag{A6}$$

$$Y_0=(\tfrac{10}{7})F_2'+(\tfrac{18}{7})F_4'-F_2\chi^2\ , \tag{A7}$$

$$Y_2=-(\tfrac{15}{49})F_2'+(\tfrac{36}{49})F_4'+(\tfrac{3}{14})F_2\chi^2\ , \tag{A8}$$

$$Y_4=(\tfrac{20}{49})F_2'+(\tfrac{1}{49})F_4'-(\tfrac{2}{7})F_2\chi^2\ . \tag{A9}$$

Other authors[8] have been coming to the conclusion that $\xi_2=0$, based on fits to data. Of course, all these arguments apply only in spherical nuclei and not in the deformed region.

APPENDIX B: BOSON-BOSON INTERACTION FOR NUCLEONS IN DEGENERATE ORBITS WITH SDI

This is a derivation of the expression

$$F_K=4Ge_K/e_0 \tag{B1}$$

based on an analysis of the nucleon pair wave functions, and in particular of the correlations induced by the surface delta interaction (SDI). For the sake of mathematical simplicity, we neglect spin in our considerations; including spin does not change the results.

We consider now two particles in a collection of degenerate orbits interacting via an SDI with strength G. For each value of angular momentum L, only a *single* state of each L is shifted in energy. We consider this lowest energy state of each L to be a boson of angular momentum L. The energy of this shifted state is

$$e_L=\sum_l\sum_{l'}(2l+1)(2l'+1)\begin{bmatrix} l & l' & L \\ 0 & 0 & 0\end{bmatrix}^2 G\ . \tag{B2}$$

The corresponding wave function is

$$\Psi_{LM}(1,2)=(\mathcal{N}/\sqrt{\Omega})\sum_l\sum_{l'}\hat{l}\hat{l}'\begin{bmatrix} l & l' & L \\ 0 & 0 & 0\end{bmatrix}\sum_m\sum_{m'}\begin{bmatrix} l & l' & L \\ m & m' & M\end{bmatrix}Y_l^m(\Theta_1,\phi_1)Y_{l'}^{m'}(\Theta_2,\phi_2)\ , \tag{B3}$$

where $\hat{l}=(2l+1)^{1/2}$, $\Omega=\sum(2l+1)$ is the total pair degeneracy, and $\mathcal{N}$ is a normalization constant. The labels 1 and 2 refer to the two particles, respectively.

For $L=0$ or s bosons, in particular, we have the following:

$$\begin{aligned}\Psi_{00}(1,2)&=(\mathcal{N}/\sqrt{\Omega})\sum_l\sum_m(-)^m Y_l^m(\Theta_1,\phi_1)Y_l^{-m}(\Theta_2,\phi_2)\\ &=(\mathcal{N}/4\pi\sqrt{\Omega})\sum_l(2l+1)P_l(\cos\Theta_{12})\end{aligned} \tag{B4}$$

using the addition theorem for spherical harmonics.

Appropriately normalized, this becomes

$$\Psi_{00}(1,2)=(1/\sqrt{\Omega})\sum_l(2l+1)P_l(\cos\Theta_{12})\ . \tag{B5}$$

For an S-pair, the two particle density is

$$\rho(1,2)=|\Psi_{00}(1,2)|^2$$
$$=(1/\Omega)\sum_{l}\sum_{l'}(2l+1)(2l'+1)P_l(\cos\Theta_{12})P_{l'}(\cos\Theta_{12})$$
$$=(1/\Omega)\sum_{L}\sum_{l}\sum_{l'}(2L+1)(2l+1)(2l'+1)\begin{bmatrix} l & l' & L \\ 0 & 0 & 0\end{bmatrix}^2 P_L(\cos\Theta_{12}) \quad \text{(B6)}$$

Notice that setting $L=0$ in (B2) gives

$$e_0=\Omega G$$

and the comparison of (B2) and (B6) yields:

$$\rho(1,2)=\sum_L (e_L/e_0)(2L+1)P_L(\cos\Theta_{12}) \ . \quad \text{(B7)}$$

Thus the two particle wave function for an S pair is simply related to the SDI energies for *all* the pairs.

The two nucleons forming an S-state pair have conjugate wave functions, and thus the same single particle densities. Let us define the "intrinsic" single particle density by:

$$\rho_I(1)=\sum_L (e_L/e_0)^{1/2}(2L+1)P_L[\cos(\Theta_1-\Theta_{\rm I})] \ , \quad \text{(B8)}$$

where $\Theta_{\rm I}$ can be regarded as the angular coordinate of boson I.

The intrinsic two particle density is simply the product:

$$\rho_{\rm I}(1,2)=\rho_{\rm I}(1)\rho_{\rm I}(2) \ . \quad \text{(B9)}$$

The actual two particle density may be obtained by averaging over all possible directions $\Theta_{\rm I}$ of the boson, i.e.,

$$\rho(1,2)=\int \rho_{\rm I}(1,2)d\Omega_{\rm I}/4\pi \ . \quad \text{(B10)}$$

Next, we use the intrinsic densities to calculate the interaction between bosons. Let I and II denote, respectively, the angular coordinates of the two bosons. Nucleons 1 and 2 are associated with boson I, while 3 and 4 are associated with boson II. Now 1 and 2 have the same densities, and so do 3 and 4. Thus the boson-boson interaction is four times the interaction energy between any pair, as long as each boson has one nucleon represented; for example, nucleons 1 and 3. For an SDI, nucleons 1 and 3 only interact if they are at the same angle.

Thus we can write:

$$V_{\text{boson-boson}}=4G\int \rho_{\rm I}(1)\rho_{\rm II}(1)d\Omega_1/4\pi$$
$$=4G\sum_L\sum_{L'}(e_L/e_0)^{1/2}(e_{L'}/e_0)^{1/2}(2L+1)(2L'+1)P_L[\cos(\Theta_1-\Theta_{\rm I})]$$
$$\times P_{L'}[\cos(\Theta_1-\Theta_{\rm II})]d\Omega_1/4\pi$$
$$=4G\sum_L (e_L/e_0)(2L+1)P_L[\cos(\Theta_{\rm I}-\Theta_{\rm II})] \quad \text{(B11)}$$

If we now expand $V_{\text{boson-boson}}$ as a multipole expansion:

$$V_{BB}=\sum_K F_K(2K+1)P_K[\cos(\Theta_{\rm I}-\Theta_{\rm II})] \quad \text{(B12)}$$

we verify the result that was to be proven:

$$F_K=4Ge_K/e_0 \ . \quad \text{(B1)}$$

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