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Medium energy probes and the interacting boson model of nuclei

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The Glauber approximation for medium energy scattering of hadronic projectiles from nuclei is combined with the interacting boson model of nuclei to produce a transition matrix for elastic and inelastic scattering in algebraic form which includes coupling to all the intermediate states. We present closed form analytic expressions for the transition matrix elements for the three dynamical symmetries of the interacting boson model. We show the effect of channel coupling on the elastic and inelastic differential cross sections for proton scattering on nuclei which behave like a spherical quadrupole vibrator, a γ-unstable rotor, and both prolate and oblate axially symmetric rotors.

I. INTRODUCTION

Electron scattering from nuclei can be analyzed using distorted wave Born approximation (DWBA) because the electromagnetic interaction with the nucleons inside the nucleus is small compared to the interaction between the nucleons. For hadronic scattering from nuclei the strength of the underlying interaction requires use of a distorted wave impulse approximation (DWIA), based on the two-body t matrix rather than the potential. However for 800 MeV proton scattering from collective nuclei, DWIA is not accurate at high momentum transfer.[1] The coupled channel approach gives improved agreement. These data and calculations[1,2] demonstrate that multiple scattering can be important for medium energy proton scattering from nuclei. How that importance grows with momentum transfer has also been shown theoretically.[2,3]

An alternative to the coupled channels approach calculates the scattering to all orders in the Glauber approximation which is a good approximation for 800 MeV protons.[4,5] For scattering to the yrast band of a strongly deformed nucleus, the scattering matrix can be calculated in the intrinsic frame to all orders, and then the scattering to a particular final state in the yrast band obtained by projecting out the angular momentum of this final state. This generalized Glauber calculation agrees well with data.[2] However, this simple method is valid only for the yrast band in a well-deformed nucleus. In this paper we shall show that the interacting boson model (IBM) (Ref. 6) can be married with the Glauber approximation so that the scattering can be calculated to all orders in closed form for nuclei described by the IBM. The method follows from the fact that the Glauber transition operator is a linear rotation in the IBM space.[7]

In Sec. II we shall review the Glauber or eikonal approximation. We refer the reader to Refs. 4 and 5 for the derivation and the details of the Glauber scattering matrix. In Sec. III, we review the IBM and in Sec. IV, we combine the IBM and Glauber in a natural way. In Secs. V–IX we develop the representation matrix for a single boson, we determine the scattering matrix to all orders in closed form for a spherical quadrupole vibrator, a γ-unstable quadrupole rotor, and an axially symmetric quadrupole rotor, and conclude with a discussion of the large N or "classical" limit. In Sec. X we show the results of sample calculations which illustrate the effects of channel couplings in each of these idealized cases. In a subsequent paper we will report on applications to specific nuclei.

II. THE EIKONAL OR GLAUBER APPROXIMATION

For medium energy proton scattering, the Glauber or eikonal approximation is a good approximation for elastic and inelastic scattering out to moderate angles. Here we apply this approximation to proton scattering on a nucleus described by the IBM which provides a closed form expression for the full coupled channels scattering amplitude. The essence of the IBM is a description of the nuclear degrees of freedom in terms of bosons rather than nucleons. These bosons are introduced into the dynamics via creation and annihilation operators and the algebra of the operators is exploited to solve the problem.

We now wish to consider a proton (mass m) interacting with an IBM nucleus. The most general Hamiltonian we can write down for this is

$$H=\frac{p^2}{2m}+V(r)+W(\mathbf{r},a)+H_T \ , \tag{2.1}$$

where $\mathbf{r}$ is the projectile coordinate. $V(r)$ is the general distorting or optical potential that the target presents to the projectile. It is the part of the projectile-target interaction that is independent of the boson operators. $W(\mathbf{r},a)$ represents the coupling of the projectile to the boson degrees of freedom represented here by the generic boson operator a. W is defined to have vanishing diagonal matrix element in the ground state since that diagonal piece is already in V. Finally, H_T is the target Hamiltonian in the absence of the projectile. For notational simplicity we temporarily suppress the spin degrees of freedom.

The standard eikonal approximation exploits the high energy of the projectile compared with interaction energies (V and W). In addition we neglect the nuclear excitation energy (carried in H_T) which would seem to be an excellent approximation for a few hundred MeV projectile incident on a target with a set of closely coupled collective states with an energy scale of the order of hundreds of kilovolts to a few MeV. Neglect of H_T is equivalent to assuming that the interaction time of the projectile with the nucleus is very short compared with the time for nuclear collective motion. This approximation is analyzed in more detail in a related approach.[8]

If we neglect H_T, the standard eikonal treatment may be applied to the Hamiltonian to yield for the scattering amplitude from initial target state J_iM_i to final J_fM_f with the projectile going from momentum $\mathbf{k}$ to $\mathbf{k}'$ ($\mathbf{k}'=\mathbf{k}+\mathbf{q}$),

$$\langle \mathbf{k}'J_fM_f \mid F \mid \mathbf{k}J_iM_i \rangle = \frac{k}{2\pi i}\int d^2b\, e^{i\mathbf{q}\cdot\mathbf{b}}\langle J_fM_f \mid (e^{i\psi(\mathbf{b})}-1) \mid J_iM_i \rangle \ , \quad (2.2)$$

where the eikonal phase ψ is given by

$$\psi(\mathbf{b}) = -\frac{m}{h^2k}\int_{-\infty}^{\infty} dz[V(\mathbf{b}+\hat{\mathbf{K}}z)+W(\mathbf{b}+\hat{\mathbf{K}}z,a)] \ , \quad (2.3)$$

and where $\hat{\mathbf{K}}$ is a unit vector in the direction of the average momentum

$$\hat{\mathbf{K}} = \frac{\mathbf{k}+\mathbf{k}'}{|\mathbf{k}+\mathbf{k}'|} \ , \quad (2.4)$$

and $\mathbf{b}$ is orthogonal to $\hat{\mathbf{K}}$. We split ψ into two parts,

$$\psi = \overline{\psi} + \hat{\psi} \ . \quad (2.5)$$

The first part is associated with V. It is the average optical model phase. Since V contains no boson operators, $\overline{\psi}$ cannot cause transition between nuclear states. The second part, $\hat{\psi}$, is associated with W and is the transition phase. We can write

$$\langle J_fM_f \mid e^{i\psi(\mathbf{b})}-1 \mid J_iM_i \rangle = T_{fi}(\psi) - \delta_{J_f,J_i;M_f,M_i} \ , \quad (2.6)$$

where T_{fi} is the transition matrix

$$T_{fi} = e^{i\overline{\psi}}U_{fi}(\hat{\psi}) \ , \quad (2.7)$$

and U_{fi} is the nuclear matrix element

$$U_{fi} = \langle J_fM_f \mid \hat{U}(\hat{\psi}) \mid J_iM_i \rangle \quad (2.8)$$

of the transition operator

$$\hat{U}(\hat{\psi}) = e^{i\hat{\psi}(\mathbf{b})} . \quad (2.9)$$

If one expands $\hat{U}$, the first nonvanishing term ($i\hat{\psi}$) gives the distorted wave impulse approximation for the scattering amplitude. The phase $\overline{\psi}$ provides the distorted wave.

However, for a target nucleus described by the IBM, we can do better. We can evaluate the nuclear matrix element (2.8) completely without making a first-order (or impulse) approximation. We note from the definition of $\hat{\psi}(\mathbf{b})$, that the expectation of $\hat{\psi}(\mathbf{b})$ in the ground state vanishes,

$$\langle \text{g.s.} \mid \hat{\psi}(\mathbf{b}) \mid \text{g.s.} \rangle = 0 \quad (2.10)$$

and hence, there is no first-order contribution to the elastic scattering. However, for projectiles for which multiple scattering is important, there can be corrections to the DWIA even in the elastic channel.

The main focus of this paper is determining the conditions for which these multiple scattering corrections are important and how they can be calculated.

III. THE INTERACTING BOSON MODEL OF NUCLEI

The interacting boson model of nuclei[6] (IBM) assumes that the low-lying collective levels of nuclei are composed primarily of $J^\pi=0^+$ and 2^+ coherent pairs of valence nucleons which are approximated as bosons. There are two versions of this model. The original version did not distinguish between neutrons and protons. This version is valid for the lowest collective states of deformed nuclei since these states are primarily symmetric in the neutron and proton degrees of freedom. The second version called IBM-2, does distinguish between neutrons and protons. This version gives a better description of the nuclear states and for some transitions is absolutely necessary. However in this paper we shall consider IBM-1 to simplify the discussion of the concepts. The generalization to IBM-2 is straightforward but increases the number of indices and hence the complexity of exposition.

The monopole ($J^\pi=0^+$) boson creation and destruction operators are

$$s^\dagger, s \ , \quad (3.1a)$$

and the quadrupole ($J^\pi=2^+$) boson operators are

$$d_m^\dagger, \tilde{d}_m = (-1)^m d_{-m}; \quad m=-2,-1,0,1,2 \ . \quad (3.1b)$$

The boson Hamiltonian which determines the nuclear wave functions and spectra is primarily composed of a single boson energy and a quadrupole interaction between bosons. This boson Hamiltonian is generally taken to be

$$H = \varepsilon N_d - \kappa Q\cdot Q \ , \quad (3.2)$$

where ε is the single-boson energy and κ the interaction strength. The quadrupole number operator,

$$N_d = \sum_m d_m^\dagger d_m \quad (3.3)$$

counts the number of quadrupole bosons. Thus the first term makes the $J^\pi=2^+$ pair higher in energy than the $J^\pi=0^+$ and hence corresponds to the pairing energy. The quadrupole operator is

$$Q_m = d_m^\dagger s + s^\dagger \tilde{d}_m + \chi[d^\dagger\tilde{d}]_m^{(2)} \ . \quad (3.4)$$

The first term changes monopole bosons into quadrupole bosons and vice versa, while the second term reorients the quadrupole bosons. Hence the quadrupole interaction mixes quadrupole bosons into the ground state thereby producing deformations. Thus, this boson Hamiltonian incorporates the predominant features of the effective nuclear Hamiltonian as determined by phenomenology.

In the IBM, multipole moment operators of the nucleus are expressed in terms of boson operators:

$$T_m^{(l)}=[d^\dagger \tilde{d}]_m^{(l)} ,$$
$$l=0,1,2,3,4, \quad m=-l,\ldots,l , \tag{3.5a}$$

and

$$P_m=s^\dagger \tilde{d}_m+d_m^\dagger s, \quad m=2,1,\ldots,-2 , \tag{3.5b}$$

$$C_m=i(s^\dagger \tilde{d}_m-d_m^\dagger s), \quad m=2,1,\ldots,-2 . \tag{3.5c}$$

These multipole operators are the 35 generators for a special unitary group in six dimensions, SU_6. The general transformation operator in SU_6 is given by

$$\hat{U}(\theta)=e^{i\theta\cdot P+i\bar{\theta}\cdot C+i\sum_l \theta^{(l)}\cdot T^{(l)}} , \tag{3.6}$$

where θ_m, $\bar{\theta}_m$, and $\theta_m^{(l)}$ are the 35 angles of the SU_6 transformation analogous to the three Euler angles for an SU_2 transformation.

The boson Hamiltonian in (3.2) is diagonalized in the space of N boson, where N is one-half the number of valence nucleons. This boson space forms a basis for the symmetric irreducible representation of SU_6 of rank N. Hence the matrix elements of the SU_6 transformation operator (3.6) between boson eigenstates,

$$U_{fi}(\theta)=\langle J_f M_f \mid \hat{U}(\theta) \mid J_i M_i \rangle , \tag{3.7}$$

will be the representation matrix for this irreducible representation of SU_6. That is, this matrix is a generalization of the Wigner D matrix[9] for SU_2.

The SU_6 group has three possible subgroup chains. The first group chain is

$$SU_6 \supset U_5 \supset SO_5 \supset SO_3 . \tag{3.8}$$

The generators of the U_5 subgroup are the multipole operators in (3.5a). The generators of the SO_5 subgroups are those of U_5 with odd multipole operators and have $l=1,3$. The generators of the SO_3 subgroup are the angular momentum operators and have $l=1$. The IBM Hamiltonian (3.2) has a U_5 dynamical symmetry for $\kappa=0$. This symmetry corresponds to the quadrupole spherical vibrator model.

The next group chain is

$$SU_6 \supset SO_6 \supset SO_5 \supset SO_3 . \tag{3.9}$$

The generators of the SO_6 subgroup are the odd multipole generators $l=1,3$ (the SO_5 generators) plus the quadrupole operator in (3.5b). The IBM Hamiltonian has an SO_6 dynamical symmetry for $\varepsilon=0$ and $\chi=0$. This symmetry corresponds to the γ-unstable quadrupole rotor model.

The third group chain is

$$SU_6 \supset SU_3 \supset SO_3 . \tag{3.10}$$

The generators of the SU_3 subgroup are the angular momentum generators ($l=1$) plus the quadrupole operator in (3.4) with $\chi=\pm\sqrt{7}/2$. The Hamiltonian has an SU_3 dynamical symmetry for $\varepsilon=0$ and $\chi=\pm\sqrt{7}/2$. Hence there are two possible SU_3 symmetries, one with negative χ and one with positive χ. Negative χ corresponds to a prolate deformation whereas positive χ corresponds to a oblate deformation.[10] These two choices give the same energy spectrum and transition rates but we shall show that they give different differential cross sections for cases in which multiple scattering is important.

The mapping of fermion operators, such as the density operator onto boson operators is not a completely solved problem. However, there is no doubt that the leading terms in the mapping of a one-body fermion operator, such as the density operator, will be linear in the boson operators defined in (3.5).[11] The tensor properties of these multipole operators, plus the condition that all matrix elements be real, lead to the forms of the total spin zero density,

$$\hat{\rho}^{(0)}(r)=\rho(r)+\alpha(r)P\cdot Y^{(2)}(\hat{\mathbf{r}})+\sum_{l=2,4}\beta_l T^{(l)}\cdot Y^{(l)}(\hat{\mathbf{r}}) , \tag{3.11a}$$

and for the spin vector density,

$$\hat{\rho}^{(1)}(r)=\bar{\alpha}(r)[PY^{(2)}(\hat{\mathbf{r}})]^{(1)}+\sum_{l=2,4}\bar{\beta}_l(r)[T^{(l)}Y^{(l)}(\hat{\mathbf{r}})]^{(1)}+\sum_{l=1,3}\{\gamma_l(r)[T^{(l)}Y^{(l+1)}(\hat{\mathbf{r}})]^{(1)}+\Delta_l(r)[T^{(l)}Y^{(l-1)}(\hat{\mathbf{r}})]^{(1)}\} , \tag{3.11b}$$

where $Y_m^{(l)}(\hat{\mathbf{r}})$ is the spherical tensor of rank l and projection m.

There are additional constraints on the functions if the $B(E\lambda)$ matrix elements are to be reproduced. The electric multipole moment operators are

$$Q_m^{(l)}=\int d^3r\, r^l Y_m^{(l)}(\hat{\mathbf{r}})\hat{\rho}^{(0)}(\mathbf{r}) . \tag{3.12}$$

Hence the $B(E\lambda)$ are given by

$$B(E2)=\left| \int_0^\infty dr\, r^4 \langle J^\pi=2^+ || \alpha(r)P+\beta_2(r)T^{(2)} || 0 \rangle \right|^2 \tag{3.13a}$$

and

$$B(E4)=\left| \int_0^\infty dr\, r^6 \langle J^\pi=4^+ || \beta_4(r)T^{(4)} || 0 \rangle \right|^2 . \tag{3.13b}$$

The microscopic theory of the IBM has made some progress in determining $\alpha(r)$, $\beta_l(r)$, $\gamma_l(r)$, and $\Delta_l(r)$ from the nuclear shell model.[12–14] These functions can also be determined phenomenologically from electron scattering.[15]

IV. THE MARRIAGE OF THE GLAUBER APPROXIMATION AND THE IBM

To combine the IBM and the Glauber or eikonal approximation, we work in impulse approximation and assume that the range of the projectile-nucleon interaction is short compared with the size of the nucleus. The interaction potentials V and W of (2.1) or equivalently the eikonal phase ψ of (2.3) can then be written in terms of the projectile-nucleon forward scattering amplitude $f^{(S)}$ (S is the channel spin) and the density operator $\hat{\rho}^{(S)}$ of (3.11). We then obtain for the phase

$$\psi(\mathbf{b})=\frac{2\pi}{k}\sum_{\tau}\left[f_{\tau}^{(0)}\int\rho_{\tau}^{(0)}(\mathbf{b}+\hat{\mathbf{K}}z)dz+f_{\tau}^{(1)}\int\rho_{\tau}^{(1)}(\mathbf{b}+\hat{\mathbf{K}}z)dz\right], \quad (4.1)$$

where for completeness we have introduced an index τ that differentiates between target protons and neutrons. From (3.11) we see that $\psi(\mathbf{b})$ is then linear in the SU_6 generators. Hence for ψ real, the Glauber transition operator $\hat{U}$ of (2.9) will be in the form of the SU_6 transformation operator given in (3.6). Thus the Glauber transition operator will be a unitary transformation on the bosons in the nucleus, and the matrix $U_{fi}(\hat{\psi})$ given in (2.8) will be the representation matrix for the symmetric representation of SU_6. This means that, like the Wigner D matrix of SU_2, this more general matrix can be calculated in closed form, implying that the scattering in the Glauber approximation can be calculated to *all* orders in the projectile-nucleus coupling within the IBM space. Thus all multiple scattering in which the nucleus is virtually excited to all intermediate states of the IBM is evaluated exactly. This is equivalent to doing a distorted wave coupled channels calculation in that space of states. In general since the scattering amplitudes $f^{(S)}$ are complex, ψ will be complex and the linear transformation will then be just the analytic continuation of a unitary transformation.

If the spin degrees of freedom of the projectile are averaged over in the experiment, only the spin scalar densities and forward scattering amplitudes in (4.1) will survive. In the rest of this discussion we shall consider this case, but realize that for measurements in which projectile spin measurements are made we should use the more general formalism. For convenience then we drop the spin superscript. We note then that only the even multipole generators of SU_6 are involved in the transition operator. We also drop the distinction between neutrons and protons.

For strongly absorbed probes, the representation matrix needed can be greatly simplified. The spherical harmonic of the projectile coordinates can be expanded in powers of (z/b):

$$Y_m^{(l)}(\hat{\mathbf{r}})=Y_m^{(l)}(\hat{\mathbf{b}})+O\left[\frac{z}{b}\right]. \quad (4.2)$$

For a strongly absorbing probe, the integration in (2.2) over impact parameter b will be concentrated primarily at the surface of the nucleus. However, the integration over z in the phase (2.3) covers the entire nucleus. Hence, the first term in (4.2) will dominate. Under this assumption the transition phase becomes,

$$\hat{\psi}(\mathbf{b})=[g(b)P+g_2(b)T^{(2)}]\cdot Y^{(2)}(\hat{\mathbf{b}})+g_4(b)T^{(4)}\cdot Y^{(4)}(\hat{\mathbf{b}}), \quad (4.3a)$$

where

$$g=\frac{2\pi}{k}f\int_{-\infty}^{\infty}dz\,\alpha((b^2+z^2)^{1/2}), \quad (4.3b)$$

$$g_2=\frac{2\pi}{k}f\int_{-\infty}^{\infty}dz\,\beta_2((b^2+z^2)^{1/2}), \quad (4.3c)$$

$$g_4=\frac{2\pi}{k}f\int_{-\infty}^{\infty}dz\,\beta_4((b^2+z^2)^{1/2}). \quad (4.3d)$$

Since in this approximation the spherical tensor is completely outside the integral over z, the transition phase is just a rotation $R(\hat{\mathbf{b}})$ through the Euler angles defined by the projectile of a reduced transition phase,

$$\hat{\psi}(\mathbf{b})=R(\hat{\mathbf{b}})\hat{\psi}^{(0)}R^{\dagger}(\hat{\mathbf{b}}), \quad (4.4a)$$

where the reduced transition phase is,

$$\hat{\psi}^{(0)}=\tilde{g}(b)P_0+\sum_{m=-2}^{2}\phi_m(b)d_m^{\dagger}d_m+\phi_0(b)d_0^{\dagger}d_0, \quad (4.4b)$$

with,

$$\tilde{g}(b)=\sqrt{5/4\pi}\,g(b), \quad (4.4c)$$

$$\phi_m(b)=\frac{(-1)^m}{\sqrt{4\pi}(1+\delta_{m,0})}\sum_{l=2,4}(2l+1)\begin{bmatrix}2 & 2 & l\\ m & -m & 0\end{bmatrix}g_l(b), \quad (4.4d)$$

and

$$\begin{bmatrix}2 & 2 & l\\ m & -m & 0\end{bmatrix}$$

is the Wigner 3-j symbol.[9] These phases are given by

$$\phi_0(b)=\frac{1}{\sqrt{4\pi}}[-(\tfrac{5}{14})^{1/2}g_2(b)+9(\tfrac{1}{70})^{1/2}g_4(b)], \quad (4.5a)$$

$$\phi_2(b)=\frac{1}{\sqrt{4\pi}}[(\tfrac{10}{7})^{1/2}g_2(b)+3(\tfrac{1}{70})^{1/2}g_4(b)], \quad (4.5b)$$

$$\phi_1(b)=-[\phi_0(b)+\phi_2(b)], \quad (4.5c)$$

$$\phi_{-m}(b)=\phi_m(b). \quad (4.5d)$$

Hence, in this approximation the transition operator (2.9) can be written as a product of three-dimensional rotation, a simpler SU_6 transformation, and finally the inverse three-dimensional rotation:

$$\hat{U}(\hat{\psi})=R(\hat{\mathbf{b}})\hat{U}^{(0)}R^{\dagger}(\hat{\mathbf{b}}), \quad (4.6)$$

where $\hat{U}^{(0)}$ is the reduced transition operator and is just

$$\hat{U}^{(0)}=e^{i\hat{\psi}^{(0)}}. \quad (4.7)$$

The matrix $U_{fi}(\psi_l)$ then becomes

$$U_{fi}(\psi)=\sum_{M}D_{MM_f}^{(J_f)}(\hat{\mathbf{b}})D_{MM_i}^{(J_i)*}(\hat{\mathbf{b}})U_{fi}^{(0)}(\psi), \quad (4.8)$$

where the reduced transition matrix is

$$U_{fi}^{(0)}(\psi)=\langle J_f,M\,|\,\hat{U}^{(0)}\,|\,J_i,M\rangle, \quad (4.9)$$

and $D_{MM'}^{(J)}(\hat{\mathbf{b}})$ is the usual Wigner D matrix.[9]

For the target with angular momentum zero, $J_i=0$, we can use (4.5) and the integral representation of the Bessel function, J_M, to do the azimuthal angle integration in (2.2). This leads to a one-dimensional integral:

$$\langle \mathbf{k}',J,M \mid F \mid \mathbf{k},J_i=M_i=0\rangle = \frac{k(-1)^{(J-1)/2}}{2^J}\frac{[(J+M)!(J-M)!]^{1/2}}{\left[\frac{J+M}{2}\right]!\left[\frac{J-M}{2}\right]!}\int_0^\infty b\,db\,J_m(qb)(e^{i\bar{\Psi}}U_{fi}^{(0)}-\delta_{fi})\,. \quad (4.10)$$

Hence our task simplifies to calculating the representation matrix for a much simpler unitary transformation. The transformation of a single boson is derived in the next section.

V. REPRESENTATION MATRIX FOR A SINGLE BOSON

The basic ingredient in deriving the representation matrix for a many-body system is the representation matrix for a single boson. From (4.4b) we see that under the transformation by the reduced transition operator, the quadrupole bosons with $m\neq 0$ are diagonal. However, the $s^\dagger$ and $d_0^\dagger$ are transformed into each other. The best way to derive the transformation matrix is to write these bosons in terms of the eigenbosons of the transition phase:

$$[\hat{\psi}^{(0)},B_\pm^\dagger]=e_\pm B_\pm^\dagger\ . \quad (5.1)$$

It is straightforward to determine that the eigenbosons are

$$B_+^\dagger=\cos u\, s^\dagger+\sin u\, d_0^\dagger\ , \quad (5.2a)$$

$$B_-^\dagger=-\sin u\, s^\dagger+\cos u\, d_0^\dagger\ , \quad (5.2b)$$

where

$$\cos u=\left[\frac{\phi-\phi_0}{2\phi}\right]^{1/2}, \quad (5.2c)$$

$$\sin u=\left[\frac{\phi+\phi_0}{2\phi}\right]^{1/2}, \quad (5.2d)$$

and

$$\phi=(\tilde{g}^2+\phi_0^2)^{1/2}\ . \quad (5.3)$$

The eigenvalues are

$$e_\pm=\phi_0\pm\phi\ . \quad (5.4)$$

Some algebra leads to the results:

$$\hat{U}^{(0)}s^\dagger\hat{U}^{(0)-1}=e^{i\phi_0}(Ws^\dagger+Xd_0^\dagger)\ , \quad (5.5a)$$

$$\hat{U}^{(0)}d_0^\dagger\hat{U}^{(0)-1}=e^{i\phi_0}(Xs^\dagger+W'd_0^\dagger)\ , \quad (5.5b)$$

$$\hat{U}^{(0)}d_m^\dagger\hat{U}^{(0)-1}=e^{i\phi_m}d_m^\dagger,\quad m\neq 0\ , \quad (5.5c)$$

where

$$W(b)=\cos\phi-i(\phi_0/\phi)\sin\phi\ , \quad (5.5d)$$

$$W'(b)=\cos\phi+i(\phi_0/\phi)\sin\phi\ ,$$

$$X(b)=(i\tilde{g}\sin\phi)/\phi\ . \quad (5.5e)$$

Thus we see that the $s^\dagger$ and $d_0^\dagger$ are transformed into each other, while the other bosons only acquire a phase. It is easy to check that this transformation is unitary for ϕ, ϕ_0, and g real.

The representation matrix for many bosons can be derived from the transformation of the single bosons given in (5.5). The general result has been derived in terms of a five-dimensional integration.[16] We present in the next section some special results which can be given in closed form.

VI. TRANSITION MATRIX FOR A QUADRUPOLE VIBRATOR

The dynamical symmetry group of the quadrupole spherical vibrator is the U_5 group as noted in (3.8). The ground state of the quadrupole vibrator is a boson condensate of N monopole bosons, where N is one-half the number of valence nucleons. The excited states are labeled by the number n of quadrupole bosons, or "phonons," with the energy of the states increasing approximately linearly with n. For a given n there is a multiplet of states with the quantum number, τ, designating the number of quadrupole bosons not coupled in pairs to angular momentum zero, the quantum number, n_Δ, designating the number of quadrupole bosons coupled in triplets to angular momentum zero, and of course the angular momentum, J, and its projection, M. The allowed values of these quantum numbers are

$$\tau=n,n-2,\ldots,0 \text{ or } 1\ , \quad (6.1)$$

and for each τ the allowed angular momenta and n_Δ are determined by all possibilities of partitioning τ,

$$\tau=3n_\Delta+\lambda\ , \quad (6.2)$$

where n_Δ and λ are integers, and then the allowed J are

$$J=2\lambda,2\lambda-2,2\lambda-3,\ldots,\lambda\ . \quad (6.3)$$

The eigenstates of the quadrupole vibrator in terms of these quantum numbers are monomials of rank N-n in the monopole bosons and of rank n in quadrupole bosons:

$$|N,n,\tau,n_\Delta,J,M\rangle=\eta_{Nn\tau}(s^\dagger)^{N-n}(d^\dagger\cdot d^\dagger)^{n-\tau/2}\times|\tau,\tau,\tau,n_\Delta,J,M\rangle\ . \quad (6.4a)$$

The normalization $\eta_{Nn\tau}$ is

$$\eta_{Nn\tau}=\left[\frac{(2\tau+3)!!}{(N-n)!(n+\tau+3)!!(n-\tau)!!}\right]^{1/2}, \quad (6.4b)$$

and the state with $N=\tau$ is a complicated function of the quadrupole bosons but has the property:

$$\tilde{d}\cdot\tilde{d}\,|\tau,\tau,\tau,n_\Delta,J,M\rangle=0\ , \quad (6.4c)$$

which is consistent with the definition of the τ quantum number which is that it is the number of quadrupole bosons not coupled to angular momentum zero.

The ground state of the quadrupole vibrator is then

$$|\tilde{0}\rangle \equiv |N,n=\tau=J=M=0\rangle = \frac{(s^\dagger)^N}{\sqrt{N!}}|0\rangle , \quad (6.5)$$

where $|0\rangle$ denotes the doubly-magic core. If we operate on this ground state with the reduced transition operator (4.7), then

$$\hat{U}^{(0)}|\tilde{0}\rangle = e^{iN\phi_0}\frac{(Ws^\dagger + Xd_0^\dagger)^N}{\sqrt{N!}}|0\rangle , \quad (6.6)$$

where we have used (5.5a). If we take the matrix element with respect to a general final state with n quadrupole bosons, given in (6.4), we can use the binomial theorem to expand (6.6) in powers of quadrupole bosons. The reduced transition matrix for scattering from the ground state to an excited state is then

TABLE I. Coefficients $A_{\tau n_\Delta J}$ for lowest values of τ.

τ	n_Δ	J	A
0	0	0	1
1	0	2	1
2	0	2	$-[\frac{1}{7}]^{1/2}$
2	0	4	$[\frac{9}{35}]^{1/2}$
3	0	6	$[\frac{3}{77}]^{1/2}$
3	0	4	$-3[\frac{2}{385}]^{1/2}$
3	0	3	0
3	1	0	$-[\frac{1}{105}]^{1/2}$

$$\langle N,n,\tau,n_\Delta,J,M=0|\hat{U}^{(0)}|N,n=\tau=n_\Delta=J=M=0\rangle = \left[\frac{N!(2\tau+3)!!}{(N-n)!(n+\tau+3)!!(n-\tau)!!}\right]^{1/2} \times e^{iN\phi_0}W^{N-n}X^n A_{\tau n_\Delta J} , \quad (6.7a)$$

where $A_{\tau n_\Delta J}$ is a constant which is independent of the Glauber phases ψ. The constant is

$$A_{\tau n_\Delta J} = \frac{\langle \tau,\tau,\tau,n_\Delta,J,0|(d_0^\dagger)^\tau|0\rangle}{\tau!} . \quad (6.7b)$$

The coefficients $A_{\tau n_\Delta J}$ are given in Table I for the lowest values of τ.

We note that the $J^\pi = 3^+$ state cannot be excited. This result, which holds for any odd angular momentum state within the IBM space, follows from the approximation made in (4.2). Hence, the degree to which any unnatural parity state is excited is a measure of the validity of this approximation. The results in (6.7) give the amplitude for exciting the individual states in a spherical quadrupole vibrator. The average probability for exciting the states in an n-phonon multiplet has been given previously (Ref. 17).

VII. GAMMA UNSTABLE ROTOR

The eigenfunctions for the SO_6 dynamical group chain do not conserve the number of monopole or quadrupole bosons. Instead of the quantum number n used in the quadrupole vibrator, the SO_6 quantum number is σ and the allowed values are

$$\sigma = N, N-2, \ldots, 0 \text{ or } 1 . \quad (7.1)$$

Since SO_5 is also a subgroup of SO_6 as well as U_5, the remaining quantum numbers are the same as for a quadrupole vibrator with the allowed values of τ being

$$\tau = \sigma, \sigma-1, \ldots, 0 . \quad (7.2)$$

The eigenfunctions are given by

$$|N,\sigma,\tau,n_\Delta,J,M\rangle = \bar{\eta}_{N\sigma}(I^\dagger)^{N-\sigma/2}|\sigma,\sigma,\tau,n_\Delta,J,M\rangle , \quad (7.3a)$$

where

$$\bar{\eta}_{N\sigma} = \left[\frac{(2\sigma+4)!!}{(N+\sigma+4)!!(N-\sigma)!!}\right]^{1/2} \quad (7.3b)$$

and

$$I^\dagger = (s^\dagger)^2 - d^\dagger \cdot d^\dagger \quad (7.3c)$$

is an SO_6 invariant. That is, the SO_6 generators commute with this four-boson operator. In particular, for the quadrupole operator (3.5b),

$$[P_m, I^\dagger] = 0 . \quad (7.4)$$

The state for $N=\sigma$ is

$$|\sigma,\sigma,\tau,n_\Delta,J,M\rangle = \Sigma_p D_p(\sigma\tau)(s^\dagger)^{\sigma-\tau-2p} \times (I^\dagger)^p|\tau,\tau,\tau,n_\Delta,J,M\rangle , \quad (7.5a)$$

where

$$D_p(\sigma\tau) = \left[\frac{2^{\sigma+1}(\sigma-\tau)!(2\tau+3)!!}{(\sigma+1)!(\sigma+\tau+3)!}\right]^{1/2}\left[-\frac{1}{4}\right]^p \times \frac{(\sigma+1-p)!}{(\sigma-\tau-2p)!p!} . \quad (7.5b)$$

Using the above eigenfunctions and the basic transformations (5.5) we can derive the reduced transition matrix.

In the SO_6 limit, $\chi=0$ and $g_4=0$. Hence the angles ϕ will take the simple dependence,

$$\phi_0=0,\ \phi=\tilde{g}\ , \tag{7.6a}$$

$$W=W'=\cos\phi,\ X=i\sin\phi\ . \tag{7.6b}$$

Since the quadrupole operator (3.5b) is a generator of the SO_6 group, it will not connect different representations of SO_6, and consequently the transition matrix will not either. Furthermore because of (7.4) and (7.3a), the transition matrix will not depend on N. Hence we shall have

$$\langle N,\sigma',\tau',n'_\Delta,J',M \mid \hat{U}^{(0)} \mid N,\sigma,\tau,n_\Delta,J,M\rangle=\delta_{\sigma',\sigma}\langle \sigma,\sigma,\tau',n'_\Delta,J',M \mid \hat{U}^{(0)} \mid \sigma,\sigma,\tau,n_\Delta,J,M\rangle\ . \tag{7.7}$$

Thus the scattering shall be entirely within the SO_6 multiplet. The ground state has $\tau=n_\Delta=J=M=0$ and $\sigma=N$. With many steps of algebra the reduced transition matrix can be shown to be

$$\langle N,N,\tau,n_\Delta,J,M=0 \mid \hat{U}^{(0)} \mid N,N,\tau=n_\Delta=J=M=0\rangle=\left[\frac{3\cdot 2^{N+1}N!}{(N+3)(N+2)}\right]^{1/2} A_{\tau n_\Delta J}X^\tau \times\sum_p D_p(N\tau)(\cos\phi)^{N-\tau-2p}\ . \tag{7.8}$$

This expression can be written in two additional, but equivalent ways, one in terms of Gegenbauer polynomials,[18] $C_n^{(\alpha)}(x)$,

$$\langle N,N,\tau,n_\Delta,J,M=0 \mid \hat{U}^{(0)} \mid N,N,\tau=n_\Delta=J=M=0\rangle=\left[\frac{3(N-\tau)!(2\tau+3)!!}{(N+3)(N+2)(N+1)(N+\tau+3)!}\right]^{1/2}(2\tau+2)!! \times X^\tau A_{\tau n_\Delta J}C_{N-\tau}^{(\tau+2)}(\cos\phi)\ , \tag{7.9}$$

and the hypergeometric function,[18] $F(a,b;c;z)$,

$$\langle N,N,\tau,n_\Delta,J,M=0 \mid \hat{U}^{(0)} \mid N,N,\tau=n_\Delta=J=M=0\rangle=\left[\frac{3(N+\tau+3)!}{(N+3)(N+2)(N+1)(N-\tau)!(2\tau+3)!!}\right]^{1/2} X^\tau A_{\tau n_\Delta J} \times F\left[-\frac{(N-\tau)}{2},\frac{N+\tau}{2}+2;\tau+\tfrac{5}{2};\sin^2\phi\right]\ . \tag{7.10}$$

From the last expression, since $F(a,b;c;0)=1$, it is easy to confirm that the reduced transition matrix reduces to $\delta_{\tau,0}$ for $\phi=0$, as it should since this is the case of no scattering.

VIII. AXIALLY SYMMETRIC ROTOR

In the limit of an axially symmetric rotor, the number of quadrupole bosons is not conserved just as for the γ-unstable rotor. In addition, since the SO_5 group is not in the subgroup sequence (3.10), the quantum numbers τ and n_Δ are also not conserved. The SU_3 quantum numbers (λ,μ) for the ground state band are $(\lambda,\mu)=(2N,0)$ and the allowed angular momentum are

$$J=0,2,4,\ldots,2N\ . \tag{8.1}$$

The SU_3 ground state band is generated from a boson condensate of intrinsic bosons:[10]

$$\mid N;i\rangle=(B_i^\dagger)^N\mid 0\rangle\ , \tag{8.2a}$$

where for $i=o$ (oblate)

$$B_o^\dagger=s^\dagger+\frac{1}{\sqrt{2}}[d_0^\dagger+\sqrt{3}/2(d_2^\dagger+d_{-2}^\dagger)]\ , \tag{8.2b}$$

and for $i=p$ (prolate)

$$B_p^\dagger=s^\dagger+\sqrt{2}d_0^\dagger\ . \tag{8.2c}$$

We use the fact that an oblate intrinsic boson can be written as a rotation about the x axis of a boson with zero z projection,

$$B_o^\dagger=R_x(s^\dagger-\sqrt{2}d_0^\dagger)R_x^\dagger\ . \tag{8.2d}$$

Then the SU_3 states in the ground state multiplet will just be projected from this condensate of intrinsic bosons:

$$\mid N,(2N,0),J,M\rangle=(\mp 1)^{J/2}\frac{2J+1}{8\pi^2\Lambda_j}\int d\Omega\, R(\Omega)D_{M0}^{(J)}(\Omega)(s^\dagger\mp\sqrt{2}d_0^\dagger)^N\mid 0\rangle\ , \tag{8.3a}$$

where $R(\Omega)$ is a rotation about the three Euler angles $(\theta_1,\theta_2,\theta_3)$, and $D^{(J)}_{M0}(\Omega)$ is the Wigner D matrix. The normalization Λ_J is

$$\Lambda_J=\left[\frac{3^N(2J+1)N!(2n)!}{(2N-J)!!(2N+J+1)!!}\right]^{1/2} . \quad (8.3b)$$

The upper sign is for positive χ (oblate) and the lower sign for negative χ (prolate).

In the SU_3 limit,

$$g_2=\pm\frac{\sqrt{7}}{2}g,\ g_4=0 , \quad (8.4a)$$

$$\phi=\frac{3}{2\sqrt{2}}\bar{g},\ \phi_0=\mp\tfrac{1}{3}\phi,\ \phi_1=\phi_0,\ \phi_2=-2\phi_0 , \quad (8.4b)$$

and

$$W=\cos\phi\pm\frac{i}{3}\sin\phi , \quad (8.5a)$$

$$W'=\cos\phi\mp\frac{i}{3}\sin\phi , \quad (8.5b)$$

$$X=\frac{2\sqrt{2}}{3}i\sin\phi . \quad (8.5c)$$

Here we also see the difference between the different SU_3 subgroups. While ϕ is independent of the sign of χ, the ϕ_m have the same magnitude but different signs for the two choices.

Because the SU_3 eigenstates are given in terms of a boson condensate, the most convenient way to derive the representation matrix is to write the rotated intrinsic boson in terms of the eigenbosons (5.2); in fact the intrinsic boson is proportional to one of the eigenbosons. In each case, the eigenbosons transform as

$$\hat{U}^{(0)}(s^\dagger\mp\sqrt{2}d_0^\dagger)\hat{U}^{(0)\dagger}=e^{4i\phi_0}(s^\dagger\mp\sqrt{2}d_0^\dagger) , \quad (8.6a)$$

$$\hat{U}^{(0)}(\sqrt{2}s^\dagger\pm d_0^\dagger)\hat{U}^{(0)\dagger}=e^{-2i\phi_0}(\sqrt{2}s^\dagger\pm d_0^\dagger) . \quad (8.6b)$$

The matrix element of $\hat{U}^{(0)}$ between the states will then reduce to the four-dimensional integral:

$$\langle N,(2N,0),J,M \mid \hat{U}^{(0)} \mid N,(2N,0),J=M=0\rangle=\frac{(\mp1)^{J/2}}{(4\pi)^2}\left[\frac{(2J+1)(2N-J)!!(2N+J+1)!!}{(2N)!}\right]^{1/2} e^{iN\phi_0}$$
$$\times\int\int\int\int d\theta_1 d\theta_1' \sin\theta_2 d\theta_2 \sin\theta_2' d\theta_2' P_J(\cos\theta_2)$$
$$\times[\cos\theta_2\cos\theta_2' e^{(3i\phi_0/2)}+\sin\theta_2\sin\theta_2'\cos(\theta_1-\theta_1')e^{-(3i\phi_0/2)}]^{2N} . \quad (8.7)$$

Some algebra produces a reduced transition matrix in terms of a hypergeometric function:

$$\langle N,(2N,0),J,M=0 \mid \hat{U}^{(0)} \mid N,(2N,0),J=M=0\rangle=(\mp1)^{J/2}\left[\frac{2^{J/2}(2J+1)N!(2N+J+1)!!}{(2N+1)!!\left[N-\frac{J}{2}\right]!}\right]^{1/2}\frac{(J-1)!!}{(2J+1)!!}$$
$$\times e^{-2iN\phi_0}(e^{6i\phi_0}-1)^{J/2}F\left[-\left[N-\frac{J}{2}\right],\frac{J+1}{2};J+\tfrac{3}{2};1-e^{6i\phi_0}\right] . \quad (8.8)$$

This can also be written in terms of Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$,[18]

$$\langle N,(2N,0),J,M=0 \mid \hat{U}^{(0)} \mid N,(2n,0),J=M=0\rangle=(\mp1)^{J/2}\left[\frac{(2J+1)N!\left[N-\frac{J}{2}\right]!}{2^{J/2}(2N+1)!!(2N+J+1)!!}\right]^{1/2}(J-1)!!\,2^N$$
$$\times e^{-2iN\phi_0}(e^{6i\phi_0}-1)^{J/2}P_{N-(J/2)}^{[J+1/2,-(N+1)]}(x) , \quad (8.9a)$$

where

$$x=2e^{6i\phi_0}-1 . \quad (8.9b)$$

The difference in oblate and prolate comes from the fact that ϕ_0 differs in sign for each case [Eq. (8.4b)].

IX. THE LARGE N LIMIT

For N large the interacting boson model approaches the geometric collective model.[10] In this case the reduced transition matrix becomes special functions. In taking this limit, we must pay attention to the fact that the function $\alpha(r)$ is normalized so that the $B(E2)$ from the ground state to the first excited state is reproduced [see Eq. (3.13a)]. Therefore α and hence g [Eq. (4.3b)] scale as the matrix element of the quadrupole operator. Also, for proton scattering, the forward scattering amplitude f is predominately imaginary. Hence we introduce the reduced $\bar{g}$,

$$\tilde{g}=\frac{i\bar{g}}{\langle 2_1^+||Q||0_1^+\rangle}\ . \tag{9.1}$$

The reduced matrix element $\langle 2_1^+||Q||0_1^+\rangle$ can be derived from the reduced transition matrix by taking the $g\to 0$ limit. For U_5

$$\langle 2_1^+||Q||0_1^+\rangle_5=\sqrt{5N}\ , \tag{9.2a}$$

for SU_6,

$$\langle 2_1^+||Q||0_1^+\rangle_6=\sqrt{N(N+4)}\ , \tag{9.2b}$$

and for SU_3,

$$\langle 2_1^+||Q||0_1^+\rangle_3=\sqrt{2N(N+3/2)}\ . \tag{9.2c}$$

Therefore, since $\bar{g}$ is fixed by the $B(E2)$, for N large $\tilde{g}$ becomes small. Hence in taking the N large limit, we assume $\tilde{g}$ small, but $\bar{g}$ not necessarily small. For example in the U_5 limit,

$$\cos\tilde{g}\simeq 1-\frac{\tilde{g}^2}{2}\ , \tag{9.3}$$

and then we have, assuming $\mathcal{X}=0$ and hence the relations (7.6),

$$\langle N,n,\tau,n_\Delta,J,M=0\,|\,\hat{U}^{(0)}\,|\,N,n=\tau=n_\Delta=J=M=0\rangle \underset{N\to\infty}{\longrightarrow} \left[\frac{(2\tau+3)!!}{(n+\tau+3)!!(n-\tau)!!}\right]^{1/2} A_{\tau n_\Delta J}\frac{\bar{g}^n}{\sqrt{5}}e^{\bar{g}^2/10}\ , \tag{9.4a}$$

$$\bar{g}\ll\sqrt{5N}\ ,\quad n\ll N\ , \tag{9.4b}$$

which agrees with the geometrical model.[4]

For SO_6 we take the limit in the hypergeometrical function using the relation that $(N)_p=\Gamma(N+p)/\Gamma(N)\to N^p$. The result is

$$\langle N,N,\tau,n_\Delta,J,M=0\,|\,\hat{U}^{(0)}\,|\,N,N,\tau=n_\Delta=J=M=0\rangle \underset{N\to\infty}{\longrightarrow} [3(2\tau+3)!!]^{1/2}A_{\tau n_\Delta J}(-1)^\tau\frac{i_{\tau+1}(\bar{g})}{\bar{g}}\ , \tag{9.5a}$$

$$\bar{g}\ll N,\quad \tau\ll N\ , \tag{9.5b}$$

where i_l is the modified spherical Bessel function of order l.[18]

In the SU_3 limit, the hypergeometric function becomes a confluent hypergeometric function[18]

$$\langle N,(2N,0),J,M=0\,|\,\hat{U}^{(0)}\,|\,N,(2N,0),J=M=0\rangle \underset{N\to\infty}{\longrightarrow} (2J+1)^{1/2}\frac{(J-1)!!}{(2J+1)!!}(3\bar{g})^{J/2}e^{\mp(\bar{g}/2)}M\left[\frac{J+1}{2},J+\frac{3}{2},\pm\frac{3\bar{g}}{2}\right]\ , \tag{9.6a}$$

for

$$\bar{g}\ll\sqrt{2}N,\quad J\ll 2N\ . \tag{9.6b}$$

X. RESULTS OF CALCULATIONS

We are interested in the size of the coupled channel effects in elastic and inelastic scattering. This will depend trivially on the size of the coupling and more substantially on the nuclear model, the number of bosons, and the momentum transfer q. First, we should comment that the effects of channel coupling can be viewed as a correction to the reduced transition matrix $U_{fi}^{(0)}$, (4.9), expressed as a power series in the coupling, i.e., as a power series in $\bar{g}$ of Eq. (9.1). The importance of these terms are known[2,3] to grow with q, so our question may be expressed as what is the nature of the power series, and how fast do the corrections appear?

We analyze the effects of channel coupling by considering the ratio R_f of the reduced transition matrix $U_f^{(0)}(\bar{g})$, (4.9), which includes all the coupled channel effects, to the leading order (in $\bar{g}$) nonvanishing term in $U_f^{(0)}$ which we call $u_f^{(0)}$:

$$R_f=\frac{U_f^{(0)}(\bar{g})}{u_f^{(0)}}\ , \tag{10.1}$$

where we have assumed that the initial state is the ground state of the nucleus. If no coupled channels contribute, then $R_f=1$. However, we should keep in mind, that in general $u_f^{(0)}\sim(\bar{g})^n$ where n can be greater than one; that is, the leading order may not be zeroth or first order in $\bar{g}$, but may be some higher order.

In the U_5 limit we see from (9.4a), that as $N\to\infty$, R_f is state independent:

$$R_f^{U_5}=R^{U_5}=e^{\bar{g}^2/10}\ . \tag{10.2}$$

For elastic scattering and $N\to\infty$ in each of the IBM limits R can be expanded to give

$$R^{U_5}=e^{+(\bar{g}^2/10)}=1+\tfrac{1}{10}\bar{g}^2+\tfrac{1}{200}\bar{g}^4+\cdots\ , \tag{10.3a}$$

$$R_0^{SO_6}=\frac{3i_1(\bar{g})}{g}=1+\frac{\bar{g}^2}{10}+\frac{\bar{g}^4}{200}+\cdots\ , \tag{10.3b}$$

and

$$R_0^{SU_3}=e^{\mp(\bar{g}/2)}\sum_{n=0}^{\infty}\frac{(\pm\frac{3}{2}\bar{g})^n}{(2n+1)n!}$$

$$=1+\frac{\bar{g}^2}{10}\pm\frac{\bar{g}^3}{105}-\tfrac{11}{448}\bar{g}^4+\cdots . \quad (10.3c)$$

Note that there are no terms linear in $\bar{g}$. The terms quadratic in $\bar{g}$ are the same in all three cases by virtue of having fitted $\bar{g}$ to the $B(E2)$. Furthermore, odd powers in $\bar{g}$ appear only in the SU_3 limit. Physically, these odd terms arise because of large quadrupole self-couplings in the 2^+ and higher states. In the SO_6 limit such terms are zero since this limit corresponds to a γ-unstable rotor which has zero quadrupole moment. For finite N there are odd terms in the U_5 limit only if $\chi\neq 0$. In practice these terms are always small, and they vanish as $N\rightarrow\infty$ because they are of single particle rather than collective strength.

These features are illustrated in Figs. 1 and 2 where we have plotted the potential model ($R=1$) and the complete elastic cross sections for 800 MeV proton scattering from a fictional nucleus. This nucleus has the radius (5.85 fm) with diffusivity (0.575 fm) of ^{148}Sm (a vibrator), but a quadrupole coupling consistent with that of ^{154}Sm (a rotator). For the U_5 calculations a value of $\chi=-0.60$, appropriate to ^{144}Sm, is used. We chose such a composite example because there is no single nucleus for which all three IBM symmetries apply. We also chose a rather large $B(E2)$ to make the coupling effects visible for the lowest states. Had we chosen the $B(E2)$ of ^{148}Sm they would not have been. The proton-nucleon interaction is characterized by a zero range interaction with total cross section $\sigma_T=46$ mb and ratio of real to imaginary part $r=-0.38$. (These are not free values but have been modified to correct the effective central potential depth to the relativistic impulse approximation value.) We take a Woods-Saxon nuclear density and its derivative for the transition density. These are the inputs used in all calculations presented here.

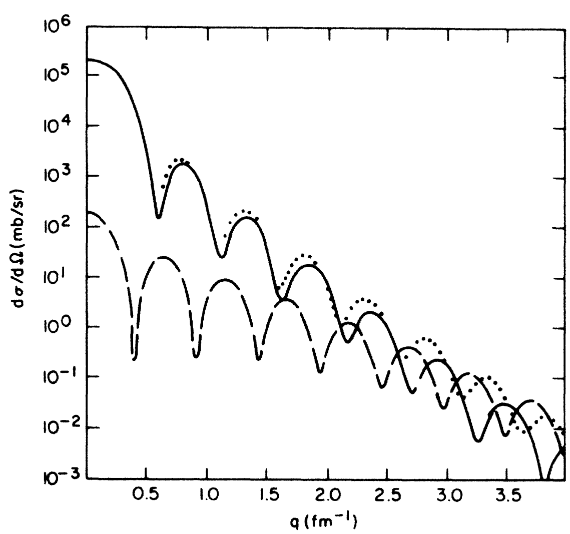

FIG. 1. Calculated elastic scattering cross section of an 800 MeV proton from a fictional Sm nucleus possessing either U_5 or SO_6 symmetry. The dotted line is the cross section without and the solid line with channel coupling. The dashed line is the square of the dispersive contribution to the cross section. The parameters for this calculation are given in the text. Out to momentum transfer of 4 fm^{-1} differences between SO_6 and U_5 are not visible.

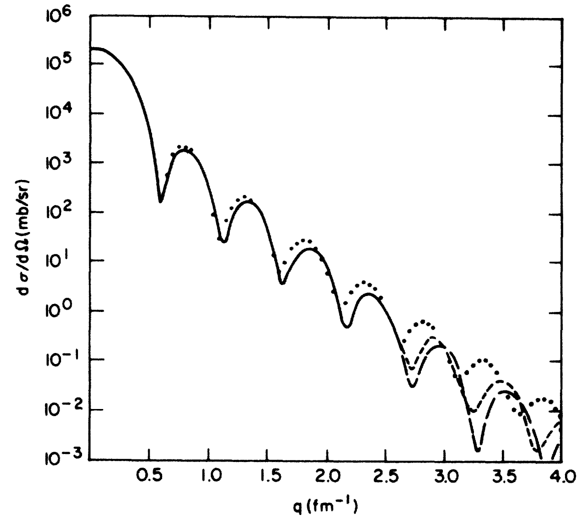

FIG. 2. Calculated elastic scattering cross section of an 800 MeV proton from a fictional Sm nucleus assumed to have SU_3 symmetry. The dotted line is the cross section without and the solid line with channel coupling. Beyond a momentum transfer of 2.5 fm^{-1} the coupled channel calculation is sensitive to the oblate or prolate shape of the nucleus and hence the solid line there splits into a short dashed or long dashed line, respectively. The parameters for this calculation are given in the text.

In Fig. 1 we present our computed elastic cross section with (solid line) and without (dotted line) the effects of coupled channels for either the U_5 or SO_6 limit of the IBM. There is no visible difference between these two IBM symmetries. We also computed the dispersive correction to the amplitude (the difference between the amplitude with and without channel coupling), which we call Δf, square it and present it for comparison as the dashed line in Fig. 1. We see that even the differences between the U_5 and SO_6 corrections are invisible. The channel coupling corrections themselves begin to affect the cross section near $q=1.0$ fm^{-1}, and dominate the cross section beyond $q=3.0$ fm^{-1}. For q less than 4.0 fm^{-1} the differences between the U_5 and SO_6 models are not visible.

In Fig. 2 we present computed elastic cross sections for prolate (long dash) and oblate (short dash) SU_3 nuclei. For comparisons, the elastic cross section with no multiple quadrupole scattering is also presented. The prolate and oblate calculations begin to differ due to the third-order term near $q=2.5$ fm^{-1}.

We can understand the results in Figs. 1 and 2 by using the analytic-stationary phase methods.[19,20] If we separate the scattering amplitude into near and far side contributions, $\bar{g}$ in the expansion of R, (10.3), can be evaluated at the stationary phase point. With a derivative or Tassie form for the transition density, this makes $\bar{g}$ proportional to iqb_0 where $b_0=c+i\pi z$ (c is the nuclear radius and z the diffusivity) is the singular point of the Woods-Saxon form. It is the fact of i that makes the positive quadratic corrections in R lead to a reduction of the elastic cross section.

In summary, we find effects due to $\bar{g}^2$ by $q=1.0$ fm^{-1}, due to $\bar{g}^3$ by $q=2.5$ fm^{-1}, and no hint of effects due to $\bar{g}^4$ out to $q=4.0$ fm^{-1}. It is only in these $\bar{g}^4$ terms that the difference between U_5 and SO_6 would begin to be seen. These calculations have been carried out to large values of q using a large coupling strength. We conclude that for all but the most deformed nuclei, dispersive corrections to elastic scattering can be characterized model independently in terms of the coupling to the nearby collective states. Of course, varying the radial dependence of the coupling potential would change the elastic scattering, but for a given choice it would be the same for U_5 and SO_6 nuclei, and nearly the same for prolate and oblate SU_3 nuclei.

We now turn to the effects of channel coupling on excitation of the first or yrast 2_1^+ state. The large N limit of the dispersion correction factor, R, for U_5 is given by (10.3a) and for SO_6 and SU_3 is

$$R_{2_1}^{SO_6}=15\frac{i_2(\bar{g})}{\bar{g}^2}=1+\frac{\bar{g}^2}{14}+\frac{\bar{g}^4}{504}\ , \tag{10.4a}$$

and

$$R_{2_1}^{SU_3}=e^{\mp(\bar{g}/2)}\sum_{n=0}^{\infty}\frac{15(\pm\frac{3}{2}\bar{g})^n}{(2n+3)(2n+5)n!}$$

$$=1\pm\frac{\bar{g}}{7}+\frac{\bar{g}^2}{14}+\cdots\ . \tag{10.4b}$$

Note that again odd terms appear only in the SU_3 case, but now there is a linear term. Also the quadratic terms in the U_5 and SO_6 cases are slightly different.

In Fig. 3 we show DWIA and complete (with full channel coupling) cross sections for excitation of the 2_1^+ yrast state in the U_5 and SO_6 cases. Also shown is the square of the magnitude of the correction (the difference between the DWIA and coupled channel amplitude) to the amplitude for the U_5 case. This correction was calculated with $\chi=0.6$, which introduces a linear correction for finite N. We have taken $N=8$ in this case, only to illustrate the possible effects of this linear term. If we set $\chi=0$, the total dispersive correction to the amplitude is increased slightly. For SO_6 the total dispersive correction is somewhat smaller than the plotted curve. All three calculations (U_5, $\chi=-0.60$, U_5, $\chi=0$, and SO_6) lead to 2_1^+ cross sections identical on the scale of this figure.

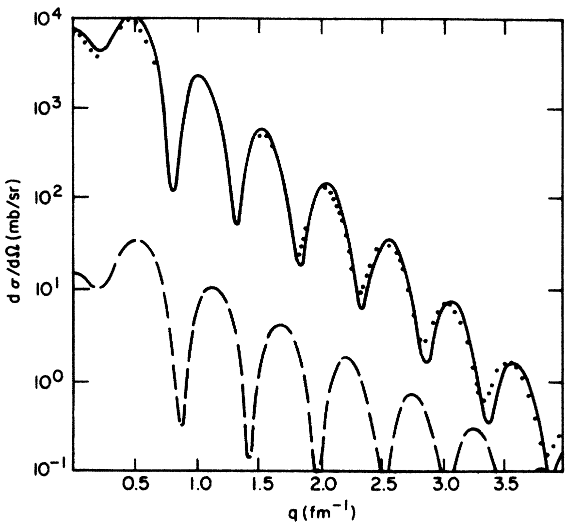

FIG. 3. Calculated cross section for excitation of the first 2^+ state in a fictional Sm nucleus, possessing either SO_6 or U_5 symmetry, by an 800 MeV proton. The dotted line is the cross section (DWIA) without channel coupling. The solid line is the full cross section with dispersive corrections. The square of the corrections themselves is the dashed line. The parameters are discussed in the text. The U_5 and SO_6 cases are identical on the scale of the figure.

As was the case in elastic scattering, there are no visible differences between the U_5 and SO_6 corrected cross sections, partly because the effects of channel coupling are very small for inelastic scattering. This may seem difficult to understand, since for the U_5 case R is state independent for large N. Nevertheless the result persists independent of variations in the details of the couplings, and is also true for odd parity excitations.[3] The relative insensitivity of inelastic scattering to channel coupling arises from the fact that though corrections to both elastic and inelastic scattering fall with momentum transfer at about the same rate, since elastic cross sections fall faster than inelastic, dispersive effects become important more rapidly in the elastic case.

In the SO_6 limit R is a purely even function of the coupling for any N. This is also true of the large N limit of U_5. For the SU_3 limit on the other hand R contains all powers of the coupling. In Fig. 4 we present the DWIA and complete cross sections for excitation of the 2_1^+ state for prolate and oblate SU_3 nuclei. Dispersive effects are visible at all momentum transfers, but these effects are much larger in the prolate than the oblate case. This comes about because the linear and quadratic terms work

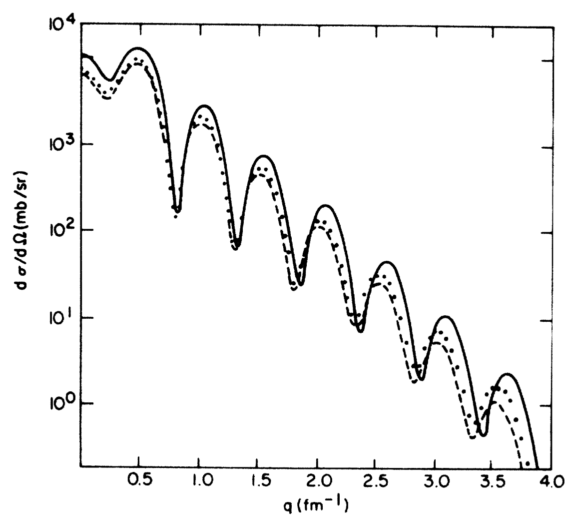

FIG. 4. Calculated cross section for excitation of the first 2^+ state in a fictional Sm nucleus, assumed to possess SU_3 symmetry, by an 800 MeV proton. The dotted line is the cross section (DWIA) without channel coupling. The solid line (dashed line) is the cross section with channel coupling for the prolate (oblate) case. The parameters are discussed in the text.

together in the former case and tend to cancel in the latter.

Again the analytic-stationary phase method can be used in (10.4b) to illuminate the results in Fig. 4. The $\bar{g}$ term becomes $\pm\lambda iqb_0$, where λ is a real constant depending on the $B(E2)$ and kinematic factors. We can then write in the near side amplitude

$$1\pm\frac{\bar{g}}{7}=1\pm\frac{\lambda iqb_0}{7}\cong e^{\pm(\lambda iqb_0/7)} . \quad (10.5)$$

Combining this with the e^{iqb_0} that comes from the amplitude,[19,20] the effect of $\bar{g}$ is to change b_0 to $b_0[1\pm(\lambda/7)]$. Hence the plus sign (oblate) corresponds to a larger radius and diffusivity and therefore decreases the cross section and moves minima inward, while the minus sign (prolate) corresponds to a smaller radius and diffusivity with the opposite affect in the cross section.

It is also interesting to study excitation of higher 2^+ states. This can be done easily in the IBM. For the second 2_2^+ state the dispersive ratio for U_5 is given by (10.3a) and for SO_6 by

$$R_{2_2}^{SO_6}=\frac{105 i_3(\bar{g})}{\bar{g}^3}=1+\frac{\bar{g}^2}{18}+\frac{\bar{g}^4}{792}+\cdots . \quad (10.6)$$

In Fig. 5 we present the cross section for excitation of the 2_2^+ state in the SO_6 limit. There is no direct population of this state; in lowest order it is a two-step process. Similar results apply for the excitation of the 2_2^+ state in the U_5 limit. The dispersive corrections, which are fourth order, are quite small. Comparing Figs. 3 and 5 indicates that for real nuclei

$$B(E2;0_1^+\to 2_2^+)/B(E2;0_1^+\to 2_1^+)\le 0.01$$

would be required for the two-step effect to be important and not overwhelmed by an admixture of a one-step effect. Many Pt isotopes have ratios in that range. Interference between weak one-step and two-step effects would allow determination of the sign of the one-step process and might contain other nuclear structure information.

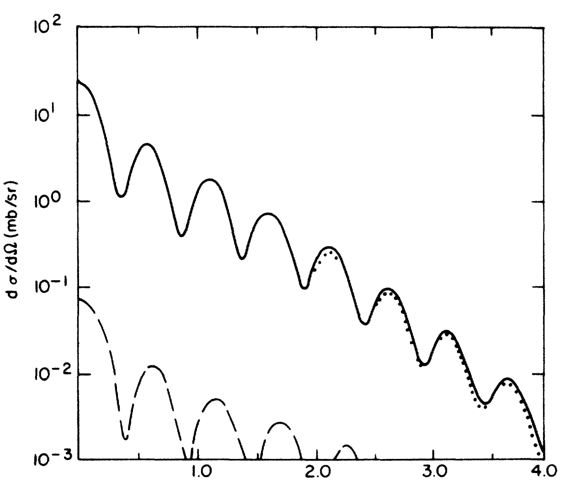

FIG. 5. Calculated cross section for excitation of the 2_2^+ state in a fictional Sm nucleus, assumed to possess SO_6 symmetry, by an 800 MeV proton. The dotted line is the leading contribution to the cross section, in this case second order or two-step, and the solid line is the cross section with channel coupling in all orders. The dashed line shows the square of the magnitude of the dispersive corrections. The parameters are given in the text.

XI. SUMMARY AND CONCLUSIONS

We have seen that by exploiting the algebraic simplicity of the interacting boson model combined with the eikonal approximation and the adiabative assumption we can derive closed form expressions for proton nucleus elastic and inelastic scattering that take channel coupling among the IBM states exactly into account. This is equivalent to a full coupled channel calculation, but is simple to implement and is physically more transparent. In the eikonal-IBM treatment, the Glauber or eikonal exponent is linear in the generators of the IBM SU_6 symmetry group. There are then two major technical steps in calculating the scattering amplitude. The first involves calculating the matrix element of the transition operator between nuclear states. Since that operator is an exponential of the generator this calculation is equivalent to calculating the representation matrix of the appropriate IBM transformation. This can be done in closed form for nuclei that have one of the three IBM subgroups as a dynamical symmetry. For nuclei which do not have these dynamical symmetries, the representation matrix can be calculated numerically. The second step involves evaluating the impact parameter integral over the appropriate combination of Bessel function and representation matrix. This is a simple one-dimensional numerical integral. In this paper we have shown how the two steps are carried out with particular emphasis on the representation matrix for the three special IBM symmetries: U_5, SO_6, and SU_3.

Having developed the formalism we turned to a numerical example to see the effects of the channel coupling. Since our purpose is exploratory and pedagogical, we did not attempt to address particular nuclei, that will come in later work, but rather we took a composite example based on a samarium isotope with a large $B(E2)$ but which can be described alternatively as a vibrator (U_5), γ unstable rotor (SO_6), or axial symmetric rotor (SU_3).

One principal purpose is to examine the effects of channel coupling or of dispersive effects as they are also called. For elastic scattering the lowest-order dispersive effects are second order in the channel coupling and these second-order effects depend only on the $B(E2)$. Third-order terms are only important in the SU_3 case, where they distinguish oblate from prolate deformations. It is well established that channel coupling effects grow in importance with momentum transfer.[2,3] We find that in our numerical example, which has a large $B(E2)$, second-order contributions to the elastic scattering become visible around momentum transfer $q=1.0$ fm^{-1} and third-order around $q=2.5$ fm^{-1}. Fourth-order and higher processes are not significant until at least $q=4.0$ fm^{-1} if at all. Hence for reasonable values of momentum transfer, dispersive effects are important but they depend largely on the $B(E2)$ and in particular do not distinguish the SO_6 and U_5 cases for fixed $B(E2)$.

For inelastic scattering to the first 2^+ state the effects of channel coupling depend on the nature of the nucleus. For both the spherical quadrupole vibrator (U_5) and the γ-unstable rotor (SO_6) dispersive corrections become important at relatively high momentum transfer ($\geqq 2.5$ fm). This follows from the fact that both these limits correspond to γ-unstable nuclei and hence the correction to the DWIA begins at second order. Thus for γ-soft or γ-unstable nuclei we do not expect large dispersive corrections to inelastic scattering. However for nuclei which are γ stable (SU_3) we find first-order dispersive corrections to the leading order, and these are important at *all* angles.

For inelastic scattering to the non-yrast state in the U_5, SO_6, and SU_3 symmetry limits, the leading excitation process is second order (two-step) and dispersive effects are small particularly for U_5 and SO_6. For more realistic situations there could be admixtures of representations and hence first and second order could be comparable.

The combination of the interacting boson model and eikonal scattering theory yields a simple but powerful approach to intermediate energy proton scattering from nuclei exhibiting a wide class of collective motion. In particular the ease with which channel coupling is included permits systematic studies even when strong coupling is important. We plan to apply these methods to a range of specific nuclei spanning the different IBM symmetries. There are a number of generalizations of this method to other hadronic probes, to odd nuclei, and even to the scattering from molecules[21] that are also under investigation.

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