Distortion in the β-decay spectrum for low electron kinetic energies

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The observed distortion in the β-decay spectrum of tritium might be reduced by somewhat weaker screening corrections than those used by Simpson. Such weaker corrections seem to be strongly indicated by atomic structure calculations. Other corrections to the Kurie plot are also discussed.

I. INTRODUCTION

The recent observation[1] of a distortion in the β decay of tritium for electron kinetic energies $T<1.5$ keV depends on the choice of the Fermi function $F(Z,W)$. This function enters into the Kurie plot in which the expression

$$K=\left[\frac{N_\beta(Z,W)}{pWF(Z,W)}\right]^{1/2} \tag{1.1}$$

is plotted vs T. Here $N_\beta(Z,W)$ is the measured number of β particles at an energy W and a momentum p, and Z is the charge of the daughter nuclei. In principle, the Fermi function $F(Z,W)$ includes all known effects, such as finite size, screening, radiation, exchange, and higher multipoles. Screening corrections[3-6] will be discussed in the next section. These corrections lower the value of the Fermi function $F(Z,W)$ for β particles of low kinetic energy T. Hence the value of K increases at low T, in comparison with the value of K_0 corresponding to the Fermi function $F_0(Z,W)$ calculated for the Coulomb potential.

The main aim of this paper is to study screening corrections. Should these turn out to be smaller than those used by Simpson,[1] the value of K would increase at low T, so that the hump in the Kurie plot might disappear. In fact, our analysis indicates that this might very probably be the case, so that the observed distortion[1] might have a more conventional origin. However, the uncertainties in the calculation of the Fermi function do not allow one to rule out heavy-neutrino emission completely. Additional experiments in which the screening effect is not so important[2,12,16,17] lead to more complete information about heavy neutrinos.

II. ELECTRON SCREENING CORRECTIONS

Electron screening corrections to β-decay spectra are most conveniently studied by using the exact s-wave solutions to the Klein-Gordon equations.[1,4-6] These may be obtained for a Hulthén model of a completely screened Coulomb field,

$$V_H(r)=-Z_\alpha\lambda e^{-\lambda r}(1-e^{-\lambda r})^{-1} \underset{r\to 0}{\longrightarrow}(Z\alpha/r)(1-\lambda r/2)\ . \tag{2.1}$$

The Klein-Gordon equation is used because the Dirac equation cannot be solved in the closed form for a Hulthén potential. However, there is no reason to expect any peculiar differences associated with the Klein-Gordon equation.[4-6] We use the system of units $\hbar=1$, $c=1$, and $m=1$. As shown in Table I, for the $Z=2$ case, the Fermi functions for a pure Coulomb potential

$$V_C(r)=Z\alpha/r\ , \tag{2.2}$$

calculated by either using the Dirac $[F_0(Z,W)]$ or the Klein-Gordon $[F_C(Z,W)]$ equation, are very close in absolute value and have identical T dependence.

The explicit forms of the Fermi functions for the Hulthén potential (2.1) and the Coulomb potential, which were found[6] by solving the Klein-Gordon equation, are

$$F_D(Z,W)=(\lambda R)^{2s-2}\left[\frac{\Gamma(s+i\nu)\Gamma(s+i\xi)}{\Gamma(s+2ip/\lambda)\Gamma(2s)}\right]^2 \times\left[1-\frac{s^2-\nu\xi}{s}\lambda R\right]\ , \tag{2.3a}$$

$$F_C(Z,W)=(2pR)^{2s-2}e^{\pi y}\left|\frac{\Gamma(s+iy)}{\Gamma(2s)}\right|^2\left[1-2\frac{y}{s}pR\right]\ , \tag{2.3b}$$

where we have used the following abbreviations:

$$y=Z\alpha W/p\ ,$$
$$s=\tfrac{1}{2}+\tfrac{1}{2}\,|\,(1-4\alpha^2Z^2)^{1/2}\,|\ ,$$
$$\nu=\frac{1}{\lambda}[p+(p^2+a-b)^{1/2}]\ ,$$
$$\xi=\frac{1}{\lambda}[p-(p^2+a-b)^{1/2}]\ ,$$
$$a=\alpha^2\lambda^2Z^2\ ,$$
$$b=2\alpha\lambda ZW\ ,$$
$$R=3.107\,527A^{1/3}\,10^{-3}\ .$$

The expressions for the Fermi functions $F_0(Z,W)$ and $F_C(Z,W)$ are similar. One can explicitly show that

TABLE I. Fermi functions.

T (keV)	$F_0(Z,W)$	$F_C(Z,W)$	F_S	F_D ($D=1.45$)	F_D ($D=1$)	F_D ($D=0.8$)
0.5	2.377 496	2.381 442	2.305 235	2.306 430	2.331 925	2.342 893
0.8	2.039 484	2.042 785	1.987 956	1.991 199	2.008 621	2.016 248
1.0	1.911 307	1.914 355	1.868 264	1.871 909	1.886 287	1.892 581
3.0	1.487 362	1.489 572	1.471 175	1.475 031	1.480 312	1.482 682
5.0	1.368 803	1.370 767	1.358 769	1.362 499	1.365 769	1.367 255
10.0	1.255 780	1.257 488	1.250 543	1.254 073	1.255 807	1.256 660
18.0	1.189 803	1.191 345	1.186 754	1.190 176	1.191 276	1.191 317

$$F_D(Z,W) \underset{\lambda\to 0}{\longrightarrow} F_C(Z,W) . \quad (2.4)$$

The dependence of the Fermi function $F_D(Z,W)$ on the Hulthén parameter λ has to be studied in detail.

The Hulthén parameter λ depends on the charge Z, the fine-structure constant α, and on a parameter D (which is also a function of r and Z) as follows:

$$\lambda = 2D\alpha Z^{1/3} . \quad (2.5)$$

It turns out that for a given Z, the parameter D depends only weakly on r. This can be established by a numerical and/or an analytical comparison of the expression (2.1) with some atomic structure calculations.[8–11]

In Ref. 8, a self-consistent field method was used to calculate excited states of helium. In this reference, screening corrections to the Coulomb potential are defined by

$$V(r) = -\frac{\alpha Z}{r} V_Z(r) \; ; \quad V_Z(r) = 1 \; , \quad r = 0 . \quad (2.6)$$

The correction function $V_Z(r)$ is usually tabulated in the literature.[9–11] From the tables of Ref. 8 one can read off $V_Z(r)$ for the configuration $(1s)^2$ for neutral helium atoms as follows:

$$(1s)^2: \; 1.07 \geq D \geq 1.06 \; , \quad 6.85 \leq r \leq 13.7$$

and

$$(1s,2s): \; 0.7 \geq D \geq 0.70 \; , \quad 6.85 \leq r \leq 13.7 .$$

In Ref. 9, numerical values are given for the effective nuclear charges $Z(r)$ in atoms; the values are calculated using the self-consistent field method. The values of D for small r increase with Z, as can be seen from the following examples for $0.01 \leq r \leq 0.1$:

$$Z=5: \; 1.28 \geq D \geq 1.22 \; ,$$

$$Z=6: \; 1.32 \geq D \geq 1.22 \; ,$$

$$Z=19: \; 1.42 \geq D \geq 1.08 \; ,$$

$$Z=32: \; 1.52 \geq D \geq 1.00 .$$

In Ref. 10, analytical fits to Hartree potentials are given for 19 neutral atoms. From these it is possible to deduce the value for $Z=2$ by numerical comparison:

$$Z=2: \; D \cong 1.4 .$$

The results of Ref. 11 are based on nonrelativistic Hartree-Fock-Slater equations. From these results [where $U(X) = V_Z(r)$] one can deduce for $0.01 < x \leq 0,1$ ($x = 1.13 Z^{1/3} r$):

$$Z=2: \; 0.44 \leq D \leq 0.50 \; ,$$

$$Z=20: \; 1.04 \leq D \leq 1.07 .$$

Thus, the Hartree-Fock-Slater potentials lead to smaller D values than the Hartree-Fock approach[18] which gave $D=1.34$ for $Z=2$.

In all cases mentioned above, D always increases with Z. For $Z=2$, this parameter could be in the range

$$1.4 \geq D \geq 0.44 .$$

Smaller values, i.e., $D \approx 1$, seem to be favored.

As discussed below, the analysis of Ref. 1 corresponds to $D=1.45$. This seems to be a correct value for larger Z ($Z>20$) at which one could naturally expect larger screening corrections.

In Table I we give Fermi functions F_D calculated by using the Klein-Gordon equation with $D=0.8$, 1.0, and 1.45. The last value of D corresponds numerically to the Fermi function used by Simpson.[1] The spectrum analysis made in Ref. 1 was based on a very good approximation,

$$F_S(Z,W) \cong F_0(Z,W')\frac{p'W'}{pW} \; ;$$

$$W' = W - V_0 \; , \quad V_0 = 1.45\alpha^2 Z^{4/3} .$$

Inspection of Table I shows that the largest differences between various Fermi functions (i.e., F_0, F_C, and F_D) occur at small T. For example,

$$F_D(D=1) - F_D(D=1.45) = 0.025 \; , \quad T = 0.5 \text{ eV} \; ,$$

$$F_D(D=1) - F_D(D=1.45) = 0.001 \; , \quad T = 18.0 \text{ eV} .$$

This strongly indicates that a change in the parameter D might lead to the disappearance of the observed distortion at low T in the Kurie plot. Exchange corrections[7] work in the same direction. These are due to β-ray emission into bound levels in He^+, while the spectator $(1s)$ electron is knocked into the continuum.

III. DISTORTIONS IN THE KURIE PLOT

The influence of screening corrections is best seen by comparing the Kurie plots calculated for various values of D, i.e., for the Fermi function F_S used by Simpson[1] and the Fermi function $F_D(Z,W)$ from Eq. (2.3a). The Fermi functions F_S and F_D can be related by

$$F_S = F_D(1-2\tilde{\delta}) ,$$
$$\tilde{\delta} = \frac{F_D - F_S}{2F_D} , \qquad (3.1)$$

with

$$\frac{\Delta K}{K} = \frac{\left[\frac{N}{pWF_S}\right]^{1/2} - \left[\frac{N}{pWF_D}\right]^{1/2}}{\left[\frac{N}{pWF_S}\right]^{1/2}} = 1-(1-2\tilde{\delta})^{1/2}$$
$$\cong \tilde{\delta} \; ; \; \tilde{\delta} \ll 1 . \qquad (3.2)$$

If the Kurie-plot distortion is due to screening corrections, then (3.2) represents an approximate estimate of the quantity $\Delta K/K$ from Ref. 1. In numerical calculations we have used the screening correction factors

$$f_S = \frac{F_S}{F_0(Z,W)} , \; f_D = \frac{F_D(Z,W)}{F_C(Z,W)} . \qquad (3.3)$$

In order to be able to compare the relative energy dependence, we have normalized all f_N's to f_D ($D \equiv 1$) at 18 keV. The quantity given in Table II is

$$\tilde{\delta} = \frac{f_D - f_S}{2f_D} . \qquad (3.4)$$

The kinetic energy dependence of this quantity can be compared with $\Delta K/K$ from Ref. 1.

Additional distortions to the Kurie plot come from the so-called "forbidden" corrections. This name is sometimes used to denote higher multipoles appearing in the expansion of leptonic (β-ray) wave functions. A general term in a multipole expansion is proportional to the matrix element[13–15] of a tensor operator $T_{J\Lambda}$:

$$\langle f \,|\, r^n T_{J\Lambda} \,|\, \rangle , \qquad (3.5)$$

where J is the overall spin and Λ is the orbital angular momentum change. The usual contributions, which have been considered in the present discussion, appear with the matrix elements of the operators r^0T_{00} and r^0T_{10}. In a higher-multipole expansion, the operators r^2T_{00}, r^2T_{10}, rT_{11}, and r^2T_{12} also appear. The new value for (1.1) is then

$$\tilde{K} = \left[\frac{N(Z,W)}{pWF(Z,W)C(Z,W)}\right]^{1/2} . \qquad (3.6)$$

The general form of the correction factor $C(Z,W)$ is

$$C(Z,W) = 1 + aW + b/W + cW^2 . \qquad (3.7a)$$

Here a, b, c, and d depend on the matrix elements (3.5), on weak magnetism, on induced pseudoscalar, etc. Somewhat lengthy expressions are well known in the literature; see, for example, Refs. 13–15.

A simple shell-model estimate of the matrix-element ratios[13] leads to

$$C(Z,W) = g_V^2 \,|\, \langle T_{00}\rangle \,|^2 \left[C_F + (g_A/g_V)^2 \left| \frac{\langle T_{10}\rangle}{\langle T_{00}\rangle} \right|^2 C_{\rm GT} \right] , \qquad (3.7b)$$

$$C_F = 1 + \frac{2}{M}(W_0 + \xi) + a_0 \left[\frac{W_0}{2} + \xi \right] - \frac{2}{M} W - \left[\frac{1}{M} + \frac{a_0}{3} \right] \Big/ W ,$$

where

$$a_0 = \frac{2}{M} \frac{\langle rT_{11}\rangle}{\langle T_{00}\rangle} + 4 \frac{\langle r^2T_{12}\rangle}{\langle T_{00}\rangle}$$

and

$$C_{\rm GT} = c_1 + c_2/W + c_3 W ,$$

with the abbreviations:

$$c_1 = 1 - \left[a_1 + \frac{c_3}{2} \right] (W_0 - 3\xi) - \frac{1}{3M} \left[1 - \frac{\langle T_{12}\rangle\sqrt{2}}{\langle T_{10}\rangle} \right] \xi + \left| \frac{g_V}{g_A} \right| \frac{4.7}{3M} \left[2 + \frac{\langle T_{12}\rangle\sqrt{2}}{\langle T_{10}\rangle} \right] \xi ,$$

$$a_1 = -\frac{1}{3M} \left[1 + 2 \frac{\langle irY_1 \sigma \mathbf{p}\rangle}{\langle T_{10}\rangle\sqrt{3}} \right] ,$$

$$c_2 = a_1 - c_3/2 ,$$

$$c_3 = \frac{2}{3M} \left| \frac{g_V}{g_A} \right| \left[9.4 + 2 \frac{\langle irT_{11}(p)\rangle}{\langle T_{10}\rangle} - \frac{8M}{3} \frac{\langle r^2T_{10}\rangle}{\langle T_{10}\rangle} - \frac{4\sqrt{2}M\langle r^2T_{12}\rangle}{3\langle T_{10}\rangle} \right] .$$

TABLE II. Kurie-plot distortions $(\Delta K/K) \times 10^3$.

T (keV)	$\Delta K/K$[a]	$\tilde{\delta}$ ($D=1.45$)[b]	$\tilde{\delta}$ ($D=1$)[b]	$\tilde{\delta}$ ($D=0.8$)[b]	$\tilde{\delta}$ ($D=0.5$)[b]
0.5	4.9	−1.32	3.66	5.67	8.82
0.8	4.2	−0.77	3.09	4.75	7.20
1.0	3.6	−0.57	2.74	4.17	6.31
2.0		−0.39	1.61	2.46	3.74
3.0		−0.22	1.09	1.67	2.57
5.0		−0.14	0.59	0.91	1.43
10.0		−0.065	0.17	0.28	0.46
18.0					

[a]The two-neutrino formula from Ref. 1, corrected by a factor of $\frac{1}{2}$.

[b]Equation (3.4), with additional explanations in the text.

These corrections are very small for β transitions with small end-point energy W_0. For tritium, where $W_0 = 18.6$ keV, the correction to $\Delta K/K$ is 1.77×10^{-3} at $T = 1$ keV and 1.79×10^{-3} at $T = 18$ keV. It is practically constant through the whole T range and it cannot influence the $\Delta K/K$ ratios.

In Fig. 1 we compare our calculated $\tilde{\delta}$'s with the $\Delta K/K$ ratio given in Fig. 3(c) of Ref. 1. From Fig. 1 and Table II one may conclude that it might be possible to reduce the observed distortion by simply using somewhat weaker screening corrections than those employed by Simpson. Such weaker corrections seem to be strongly suggested by atomic structure calculations, in particular those carried out in Ref. 11.

Screening corrections are significant for $T < 3$ keV, decreasing sharply with energy. However, screening corrections exist continuously through the whole energy range. The corrections due to heavy-neutrino emission show a hump at $T < 1.6$ keV if the heavy-neutrino mass is $M_2 = 17.1$ keV. In the experiment these two effects have to be distinguished by their shapes.

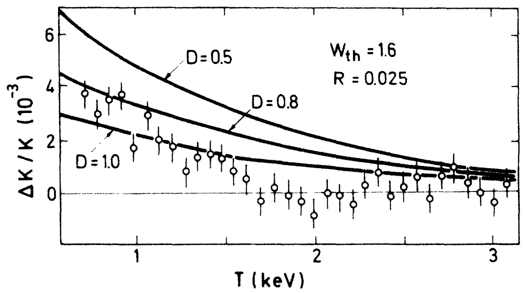

FIG. 1. Full curves show $\tilde{\delta}$'s calculated from (3.2) and normalized at $T = 5$ keV. The curves are not fits to experimental points (Ref. 1); the experimental points are drawn for comparison of general trends.

IV. CONCLUSION

As has been illustrated above, the observed distortion[1] in the tritium spectrum will change if one uses somewhat weaker screening corrections than those used in Ref. 1. However, the uncertainty in the parameter D, given by (2.7), is too large to draw any definite conclusion. Spectrum-shape measurements can, in principle, discriminate heavy-neutrino contributions from the effects due to screening corrections. However, a more practical check might be forthcoming[2,12] or has already been given[16,17,19] by the different experiments. The measurements[16,17] of the β spectrum of ^{35}S are mainly concerned with electrons of relatively high kinetic energies, 110 keV $< T <$ 170 keV. At these energies screening corrections are less important. The analysis of the experimental data for ^{35}S is more influenced by the shape correction factor $C(Z,W)$ [Eq. (3.7)] than the analysis of the tritium spectrum. The limits for the branch to 17-keV neutrinos are either 0.4% (Ref. 16) or 0.2% (Ref. 17). The analysis of the internal bremsstrahlung[19] in the electron capture decay of ^{55}Fe gave 0.7% as the upper limit. These results are in agreement with the theoretical possibility that the distortion reported by Simpson[1] might be mainly due to screening corrections.

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