Relativistic one-pion-exchange potentials

Leopold Mathelitsch
Institut für Theoretische Physik, Universität Graz, A-8010 Graz, Austria

Humberto Garcilazo*
Kernforschungszentrum Karlsruhe, Institut für Kernphysik und Institut für Experimentelle Kernphysik der Universität Karlsruhe, D-7500 Karlsruhe, Federal Republic of Germany

(Received 10 February 1986)

We compare three different types of relativistic two-body equations, namely the Blankenbecler-Sugar equation, the Gross equation, and an equation derived from relativistic three-body equations. Regarding the one-boson-exchange, the corresponding potentials coincide on mass shell but differ off shell. The potential connected with the three-body equation even shows some cuts in the momentum plane. We analyze the differences for the example of the one-pion-exchange calculating half-off-shell functions and phase shifts. We also include strong πNN form factors and discuss the influence of some of their parametrizations on the phase shifts.

I. INTRODUCTION

Because of the relative large mass of the nucleon a nonrelativistic treatment of the two- and more-nucleon system is justified and works with great success. But for increasing energies one should pay attention to relativistic aspects. The same demand applies when one investigates systems consisting of nucleons and lighter particles, for example pions. In the NNπ system the pion-nucleon subsystem should be treated relativistically and, in order to be consistent, one has to handle the nucleon-nucleon system on the same basis. A correct way to include relativistic effects is to solve the Bethe-Salpeter equation.[1] Because of the complexity of this ansatz just a few attempts have been made to work with this equation—either in the ladder approximation[2] or in full.[3] To introduce some kind of relativity several people derived approximations to the Bethe-Salpeter equation;[4–9] these nonunique approximations were widely used and also extended to more than two-particle systems.[10,11] The differences among these equations and/or the distinction to the nonrelativistic equation were investigated for separable[12,13] or local potentials.[14] In the latter case especially the one-pion exchange (OPE), as the longest-ranged and best-founded component of the nucleon-nucleon interaction, was the object of interest.

Also in the context of the OPE Garcilazo showed how a two-body potential is changed, if one embeds it into a three-body model satisfying both two- and three-body unitarity.[15] This three-body one-pion-exchange potential (three-body OPEP) has an energy-dependent range and it becomes complex above the pion-production threshold. In this paper we want to investigate how this three-body potential with its related equation compares to other relativistic equations. As examples we choose the Blankenbecler-Sugar equation[4] and the Gross equation.[5] The reason for this choice is that the OPEP's, which are derived in these approaches, differ from each other and also from the three-body OPEP. Other relativistic approximations yield one of the above potentials and differ just in the equations.

In Sec. II we state the different models and compare them with each other. We will see that the corresponding OPEP's are the same on-the-mass shell but differ off-mass shell. Moreover, the energy-dependent three-body OPEP shows cuts above the on-shell point and beyond a specific energy also below the on-shell point. The influence of these different potentials and equations on half-shell functions and phase shifts is shown in Sec. III. In the following section we include strong πNN form factors in our calculations investigating two different models. On one hand the usual πNN monopole form factor is taken into account and especially the influence of the—experimentally and theoretically not very well defined—cutoff mass is compared to the differences arising from the relativistic approaches as discussed in Sec. III. But in the relativistic treatments of the OPE not only is the pion off its mass shell but—depending on the equation—also at least one of the nucleons. Therefore we include also a form factor which allows each one of the three particles in the πNN vertex to be off-the-mass shell. Final conclusions are given in Sec. V.

II. RELATIVISTIC ONE-PION-EXCHANGE POTENTIALS

A correct way to calculate the two-hadron interaction in a relativistic formalism is to solve the Bethe-Salpeter equation.[1] The kernel or "potential" therein is the sum of all connected two-particle irreducible diagrams. Because of the complexity of this four-dimensional equation just a few numerical calculations exist: a purely phenomenological treatment used a separable ansatz for the potential. The free parameters in the ansatz were adjusted for each partial wave to reproduce the experimental phase shifts for the nucleon-nucleon and pion-nucleon interaction.[3] Within a meson-theoretical framework the Bethe-Salpeter equation was solved just in the ladder approximation, i.e.,

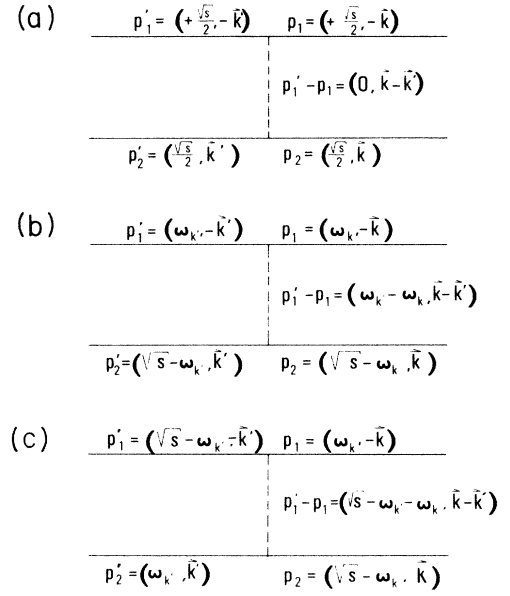

FIG. 1. The kinematics of the interaction of two particles of mass m via the exchange of a single particle in the c.m. frame [$\sqrt{s}$ is the total energy and $\omega_k = (m^2 + k^2)^{1/2}$]. (a) Both particles are equally off their mass shell; (b) one particle is always on its mass shell; (c) the particles are alternatively on their mass shell.

the potential was restricted to the exchange of a single particle.[2]

In order to circumvent the difficulties of the Bethe-Salpeter equation while keeping "some" relativity, several people proposed approximations to this equation. They all have in common reducing the four-dimensional equation to a three-dimensional one. Such a reduction is not unique and different choices yield different relativistic equations.[4–9] The differences between the various reductions compared to either the full relativistic Bethe-Salpeter equation or to the nonrelativistic Schrödinger or Lippman-Schwinger equations were discussed in several papers.[9,12–14]

One can divide these reductions in two families according to the prescription for the exchanged particle: If it bears just a (three-dimensional) momentum, both particles are equally off their mass shells in intermediate states. This is illustrated in Fig. 1(a), where two particles with equal mass m are interacting via the exchange of a particle with mass μ. The total energy is given by $\sqrt{s}$. The corresponding potential is given in general by

$$V(p_1,p_2;p_1',p_2') = \frac{\lambda^2}{(p_1' - p_1)^2 - \mu^2} , \tag{1}$$

where λ is the coupling constant (this corresponds, strictly speaking, to the exchange of a scalar particle but also to the one-pion exchange in the case of spin-singlet states after removing the delta function at $\mathbf{r}=0$).

For the case of Fig. 1(a), Eq. (1) gives

$$V^A(\mathbf{k},\mathbf{k}';s) = -\frac{\lambda^2}{(\mathbf{k}-\mathbf{k}')^2 + \mu^2} . \tag{2}$$

This prescription is chosen in the reductions by Thompson,[8] by Woloshyn and Jackson,[9] and by Blankenbecler and Sugar.[4]

A second family is characterized by the constraint that one particle is restricted to its mass shell. That means that there is a relative-energy dependence, a retardation, in the interaction. The kinematics is shown in Fig. 1(b) and the potential is given in the form

$$V^B(\mathbf{k},\mathbf{k}';s) = -\frac{\lambda^2}{(\mathbf{k}-\mathbf{k}')^2 - (\omega_k - \omega_{k'})^2 + \mu^2} , \tag{3}$$

with

$$\omega_k = (m^2 + k^2)^{1/2} .$$

This unsymmetric reduction was chosen by Gross,[5] by Erkelenz and Holinde,[6] and by Kadyshevski.[7]

A different approach to a relativistic description of a two-body reaction was followed by Garcilazo.[15] He restricted himself—and we will do the same in the following—to the one-pion exchange between two nucleons [that means that in Eqs. (1)–(3) m is the nucleon mass and μ the pion mass]. Garcilazo derived an OPEP in the framework of a three-body model of nucleon-nucleon scattering, i.e., the nucleon is treated as a bound state of a pion and a nucleon. The model chosen was introduced by Aaron, Amado, and Young[10] and satisfies both two- and three-body unitarity. These features lead to an OPEP which is different from cases A and B [Eqs. (2) and (3)], namely,

$$V^C(\mathbf{k},\mathbf{k}';s) = -\frac{\lambda^2}{(\mathbf{k}-\mathbf{k}')^2 - (\sqrt{s} - \omega_k - \omega_{k'})^2 + \mu^2} . \tag{4}$$

The kinematics of this potential can be seen in the diagram Fig. 1(c). The basis of this diagram is the fact that the Aaron, Amado, and Young formalism requires one particle on its mass shell. The iterations within the Faddeev equations generate the on-mass-shell constraint to alternate from one nucleon to the other. The main difference of the potential C compared to A and B is the explicit appearance of the total energy $\sqrt{s}$. This causes specific features of V^C that we are going to discuss next.

The potential V^C can be written in the form

$$V^C(\mathbf{k},\mathbf{k}';s) = -\frac{\lambda^2}{2[(\mathbf{k}-\mathbf{k}')^2 + \mu^2]^{1/2}} \left\{ \frac{1}{\sqrt{s} - (m^2+k^2)^{1/2} - (m^2+k'^2)^{1/2} + [\mu^2 + (\mathbf{k}-\mathbf{k}')^2]^{1/2}} - \frac{1}{\sqrt{s} - (m^2+k^2)^{1/2} - (m^2+k'^2)^{1/2} - [\mu^2 + (\mathbf{k}-\mathbf{k}')^2]^{1/2}} \right\} . \tag{5}$$

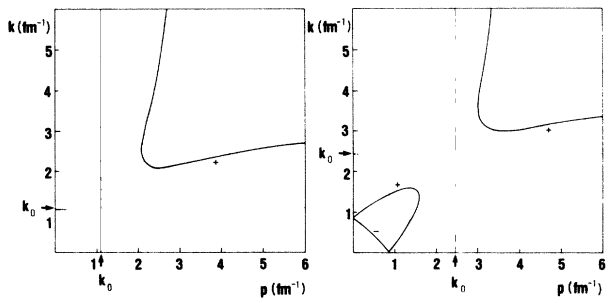

FIG. 2. Positions of the cuts of the potential $V^C(p,k;s)$ at laboratory energies of (a) 100 MeV, (b) 500 MeV.

If the on-shell momentum is k_0 [$\sqrt{s}=2(m^2+k_0^2)^{1/2}$] one can see from Eq. (5) that for k, $k'<k_0$ the first part of the sum remains positive, whereas the second part can give rise (but not necessarily) to some singularities. For k, $k'>k_0$ it can happen that the denominator of the first part is equal to zero. In order to be able to investigate the possible singularity structure, we perform a partial wave decomposition of V^C. This is shown explicitly in the Appendix and the results are illustrated in Fig. 2 for two energies.

One can see that the singularities occur just for full off-shell values ($k_0\neq k$, $k_0\neq k'$). As derived in the Appendix the singularities for k, $k'>k_0$ always exist, whereas the singularities for k, $k'<k_0$ appear only above the pion-production threshold. These singularities are typical for the treatment of the three-particle systems above breakup threshold.[16] In our case they are related to the fact that the potential V^C is derived in the framework of a three-body model of the two-nucleon interaction.

In the next section we will investigate how these singularities affect two-nucleon quantities such as half-shell functions and phase shifts.

III. PHASE SHIFT CALCULATION

For calculating phase shifts from the potentials of Eqs. (2)–(4), we first have to decide into which equations to embed our potentials. As stated already in Sec. II, the reduction of the Bethe-Salpeter equation to a specific three-dimensional one is not unique and depends on the propagator chosen.

Besides the comparison among the relativistic cases we want to contrast our results also to the nonrelativistic case. The nonrelativistic OPEP is the same as the one of case A, where the exchanged particle bears just three-momentum; the adequate equation is the Lippman-Schwinger equation

$$T(p,k;k^2)=V^A(p,k)-\int_0^\infty \frac{q^2dq}{q^2-k^2-i\epsilon}V^A(p,q)T(q,k;k^2) \quad (6)$$

written in the half-off-shell form.

Among the relativistic equations with potential V^A (proposed by Thompson,[8] by Woloshyn and Jackson,[9] and by Blankenbecler and Sugar[4]) we have chosen the Blankenbecler-Sugar equation, since it is used most frequently. Aside from the derivation from the Bethe-Salpeter equation, one can get the Blankenbecler-Sugar equation also via the minimal-relativity transformation

$$\hat{T}(p,k;k^2)=\frac{m}{(p^2+m^2)^{1/4}}T(p,k;k^2)\frac{m}{(k^2+m^2)^{1/4}}\ ,$$
$$\hat{V}(p,k;k^2)=\frac{m}{(p^2+m^2)^{1/4}}V(p,k;k^2)\frac{m}{(k^2+m^2)^{1/4}}\ , \quad (7)$$

applied to Eq. (6). The result is

$$T(p,k;k^2)=V^A(p,k)-\int_0^\infty \frac{m^2q^2dq}{(m^2+q^2)^{1/2}(q^2-k^2-i\epsilon)}\times V^A(p,q)T(q,k;k^2)\ . \quad (8)$$

Also for the potential of case B we have chosen the equation, which is stated most in the literature, namely the Gross equation[5]

$$T(p,k;k^2)=V^B(p,k)-\int_0^\infty \frac{m^2q^2dq}{2(m^2+q^2)^{1/2}(m^2+k^2)^{1/2}[(m^2+q^2)^{1/2}-(m^2+k^2)^{1/2}-i\epsilon]}V^B(p,q)T(q,k;k^2)\ . \quad (9)$$

Other possibilities would have been the reductions given by Kadyshevski[7] or by Erkelenz and Holinde.[6] The derivation of potential V^C by Garcilazo using the three-body model also ends up with Eq. (9).

For the numerical solution of Eqs. (6), (8), and (9) we used the prescription of Kowalski and Noyes[17] and for the constant λ we have taken the value

$$\lambda^2=\frac{g^2}{8\pi^2}\frac{\mu^2}{m^2}=0.003\,975\ , \quad (10)$$

which corresponds to a coupling constant of $g^2=14.5$.

The differences among the relativistic potentials [Eqs. (2)–(4)] and the associated equations [Eqs. (8) and (9)] are shown in Fig. 3 for the phase shifts and in Fig. 4 for the half-off-shell functions for angular momenta $L=0$ and 1. The phase shifts are shown for laboratory energies up to 500 MeV and the Kowalski-Noyes function $f(p,k)$ (Ref. 17) is drawn for a laboratory energy of 300 MeV ($k=1.9$ fm^{-1}).

In comparison to the nonrelativistic result from the Lippmann-Schwinger equation [Eq. (6)] Fig. 3 shows that all three relativistic potentials (and equations) give almost the same relativistic correction. It also shows that the effect is not negligible (at the order of ten percent at

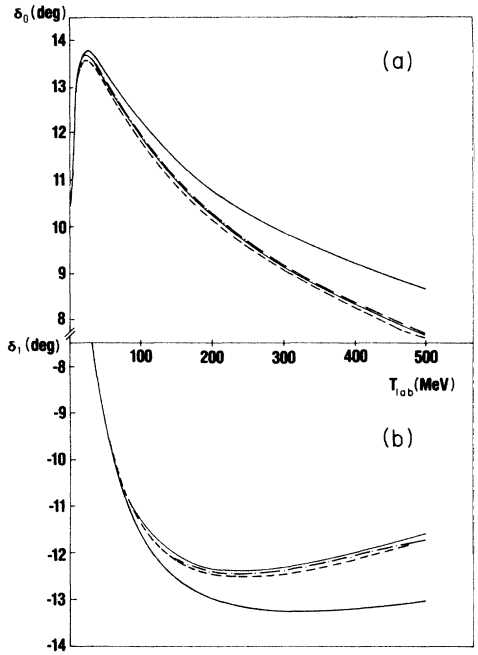

FIG. 3. Phase shifts of the one-pion exchange for the (a) S wave, (b) P wave. The solid line gives the result for the Lippmann-Schwinger equation, the dashed line for the Blankenbecler-Sugar equation, the dotted line for the Gross equation, and the dashed-dotted line for the three-body equation.

$T_{\rm lab}=500$ MeV) and that the relativistic results are similar to a nonrelativistic treatment for a potential with reduced attraction. Both statements also hold for higher partial waves ($L\leqq 5$) for the case of the OPEP.

The half-shell functions (Fig. 4) show a different behavior. There, the results of the Lippman-Schwinger equation and the Blankenbecler-Sugar equation are almost identical and the same is true for the Gross and the three-body equations at an off-shell momentum of $p\gtrsim 2$ fm^{-1}. Since these functions are divided by their on-shell values, the differences connected with the different phase shifts do not show up in such a pronounced way and features of the potential itself appear: Exactly the same potential is put into the nonrelativistic and the Blankenbecler-Sugar equations and the potentials according to the Gross and three-body equations are similar due to the fact that the one nucleon is on its mass shell.

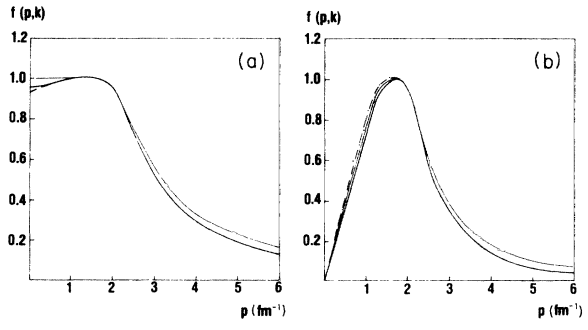

FIG. 4. Kowalski-Noyes half-shell function $f(p,k)$ at a laboratory energy of 300 MeV for the (a) S wave, (b) P wave. Description as in Fig. 3. The results for the Lippmann-Schwinger and Blankenbecler-Sugar equations are indistinguishable on this scale, likewise the results from the Gross and three-body equations for momenta $p\gtrsim 3$ fm^{-1}.

The OPE phase shifts at the lower partial waves are not realistic in the sense that, e.g., the $L=0$ shift is too small by a factor of 5 compared to results of nucleon-nucleon phase-shift analyses. To allow a comparison of our relativistic equations also for more realistic phase shifts for low partial waves, we follow a simple method shown by Woloshyn and Jackson,[9] namely multiplying the mass of the "pion" by a factor of 3 and consequently raising the constant λ^2 by a factor of 9 [see Eq. (6)]. The resulting S- and P-phase shifts can be seen in Fig. 5. There the relation between the phase shifts is different from the one seen in Fig. 3. In the S wave the transition from the Lippmann-Schwinger to the Blankenbecler-Sugar treatment is more or less the same as in the case of the pure OPE. The difference between these two calculations comes just from the kinematical factor of the "minimal relativity" [Eq. (7)]. However, the transition to the Gross and to the three-body equations leads to a different behavior, where the phase shift is increased at lower and decreased at higher energies as compared to the nonrelativistic case. In the P waves the "ordering" among the phase shifts does not change (that means the relative effect is the same as for the OPE), but the difference between the three relativistic equations becomes larger.

In conclusion we can say that the relativistic effect is not negligible and that for the pure OPE the differences

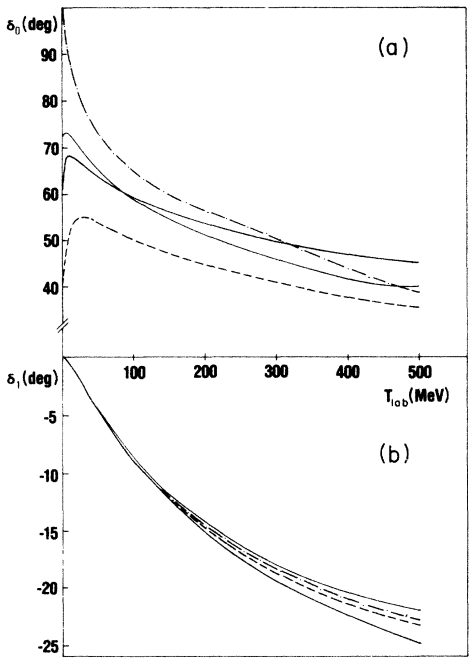

FIG. 5. Same as Fig. 3 with the mass of the exchanged particle multiplied by a factor of 3 and the coupling constant λ^2 raised by a factor of 9.

among the various relativistic equations are considerably smaller than the deviation from the nonrelativistic result. But one cannot generalize this statement, since a variation of the underlying potential can yield different results in the various equations.[18]

IV. INFLUENCE OF πNN FORM FACTORS

Strongly connected to the investigation of OPEP's are questions concerning the treatment of the πNN vertex, namely, whether one should include strong πNN form factors and if so, which parametrization is appropriate.[19] In this section we want to investigate how the uncertainty related to the strong form factors compares to the results caused by the different relativistic treatments of the OPE.

Since in the Blankenbecler-Sugar equation the pion possesses only three-momentum, the commonly used parametrization of the strong form factor for the πNN vertex is the monopole fit

$$F_{\pi NN}(\mathbf{q}^2)=\frac{\Lambda^2}{\Lambda^2+\mu^2+\mathbf{q}^2} \tag{11}$$

with $\mathbf{q}$ being the three-dimensional momentum of the pion.

Λ, the so-called cutoff mass or parameter, is not very well determined either experimentally or theoretically. The discrepancy in the Goldberger-Treiman relation leads to a low value of $\Lambda\sim 600-800$ MeV, but $\Lambda\sim 700$ MeV was also deduced from an analysis of nucleon-nucleon charge exchange scattering[20] and from calculations with quark bag models.[21] On the other hand, studies with realistic nucleon-nucleon potentials show a high value for Λ ($\Lambda\approx 1200-1500$ MeV) (Ref. 22) and also dispersion relations indicate $\Lambda\gtrsim 1000-1400$ MeV.[23]

In Fig. 6 the OPE S-wave phase shift is shown calculated with the Lippmann-Schwinger and the Blankenbecler-Sugar equations with and without inclusion of a form factor. For the form factor we took the expression (11) with a cutoff mass of $\Lambda=700$, 1000, and 1300 MeV, respectively. One can see that the addition of a strong form factor yields an effect larger than the one from the inclusion of relativistic components. Moreover, the variation due to the different cutoff masses has more or less the size of the transition from the nonrelativistic to the relativistic equation.

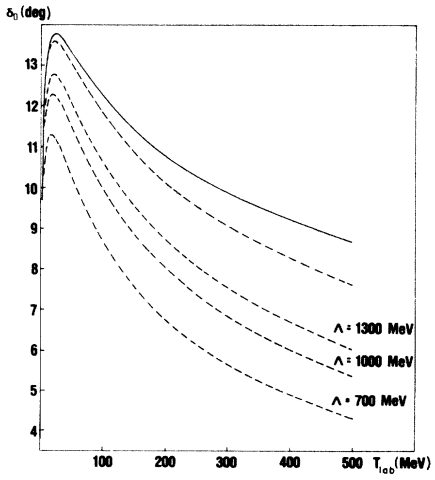

FIG. 6. S-wave phase shifts for the one-pion exchange with the Lippmann-Schwinger equation (solid line), the Blankenbecler-Sugar equation without hadronic form factors (dashed line), and with strong form factors (dotted lines) with cutoff masses of 700, 1000, and 1300 MeV, respectively.

The question arises whether the parametrization of Eq. (11) is the correct one, since just the pion is treated off mass shell, whereas at least one nucleon is also off the mass shell in the relativistic equations considered in this paper (see Fig. 1). Bryan, Dominquez, and VerWest proposed a model for a hadronic form factor, which is based on features of the dual and Regge models and which allows one, two, or all three particles to be off mass shell.[24] Applied to the OPE of the nucleon-nucleon scattering their form factor reads

$$F(p_1^2,p_2^2,q^2)=\frac{\Lambda^2}{\Lambda^2-(q^2-\mu^2)}\frac{\Lambda^2}{\Lambda^2-(p_1^2-m^2)}\times\frac{\Lambda^2}{\Lambda^2-(p_2^2-m^2)}\ , \tag{12}$$

where p_1, p_2, and q are the four-momenta of the two nucleons and the pion, respectively, and $\Lambda^2=1.25$ GeV^2 is a universal range, which is the same for all hadrons.

Inserting the kinematics of our interactions $A-C$ into Eq. (12) yields OPEP's with full off-shell form factors. The potential used in the Blankenbecler-Sugar equation is then given by

$$V_F^A(\mathbf{k},\mathbf{k}';s)=\left[\frac{\Lambda^2}{\Lambda^2-\frac{s}{4}+m^2+\mathbf{k}^2}\right]^2\times\left[\frac{\Lambda^2}{\Lambda^2-\frac{s}{4}+m^2+\mathbf{k}'^2}\right]^2\times\left[\frac{\Lambda^2}{\Lambda^2+\mu^2+(\mathbf{k}-\mathbf{k}')^2}\right]^2 V^A(\mathbf{k},\mathbf{k}';s)\ , \tag{13}$$

the Gross potential reads

$$V_F^B(\mathbf{k},\mathbf{k}';s)=\frac{\Lambda^2}{\Lambda^2-s+2\sqrt{s}\,\omega_k}\frac{\Lambda^2}{\Lambda^2-s+2\sqrt{s}\,\omega_{k'}}\times\left[\frac{\Lambda^2}{\Lambda^2+\mu^2-(\omega_k-\omega_{k'})^2+(\mathbf{k}-\mathbf{k}')^2}\right]^2\times V^B(\mathbf{k},\mathbf{k}';s)\ , \tag{14}$$

and the potential derived in the three-body formalism looks like

$$V_F^C(\mathbf{k},\mathbf{k}';s)=\frac{\Lambda^2}{\Lambda^2-s+2\sqrt{s}\,\omega_k}\frac{\Lambda^2}{\Lambda^2-s+2\sqrt{s}\,\omega_{k'}}\times\left[\frac{\Lambda^2}{\Lambda^2-(\sqrt{s}-\omega_k-\omega_{k'})^2+\mu^2+(\mathbf{k}-\mathbf{k}')^2}\right]^2\times V^C(\mathbf{k},\mathbf{k}';s)\ . \quad (15)$$

We have calculated OPE phases with these modified potentials and—as a first result—the differences in the phases due to the various relativistic treatments are even smaller than for the calculation without form factors. Therefore, we do not give all of the curves, but just show bands for the S- and P-wave phase shifts. Each band consists of the results for the potentials V^A, V^B, and V^C without a form factor [Eqs. (2)—(4)] and with inclusion of the off-shell form factors [Eqs. (13)—(15)]. In addition, we have switched off the part of the form factors in Eqs. (13)—(15) which led to the nucleons off-mass shell (i.e., the first and second term in each expression).

Figure 7 again shows the large effect of the inclusion of strong form factors for the phase shifts. Switching off the nucleon off-shell parts in the form factors always leads to an increase of the phase shifts, as already explained in Ref. 24. The size of this effect is dependent on the partial wave—the difference is at least a factor of 2 larger for the P wave than for the S wave (percentage wise), and may, of course, also depend on the underlying potential.

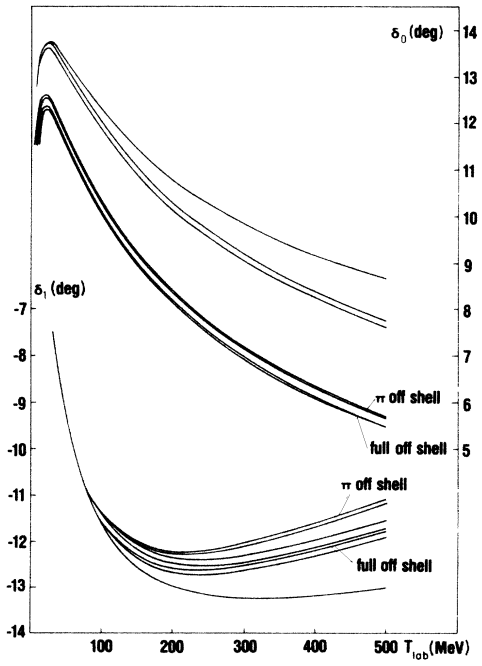

FIG. 7. S- and P-wave phase shifts for the one-pion exchange for the Lippmann-Schwinger equation (solid line), for the relativistic equations discussed in the text without strong form factors (shadowed area), and with full off-shell and "pion-off-shell" form factors (double lines).

V. CONCLUSION

We have shown that the one-pion-exchange potential, which was derived by Garcilazo via a relativistic three-body model of the two-nucleon interaction, deviates from other relativistic expressions for the same exchange mechanism. In particular, the three-body OPEP is energy dependent and shows a singular structure for full off-shell momenta. There can be two closed cuts in the momentum plane, but one cut just comes in at scattering energies above the one-pion-production threshold.

We have compared the resulting half-shell functions and phase shifts of the three-body potential (with its connected equation) to the corresponding results of the Lippmann-Schwinger, Blankenbecler-Sugar, and Gross equations. Whereas features of the individual potentials are reproduced in the half-shell functions, the one-pion-exchange phase shifts are very similar for all three relativistic equations, but show some deviations from the nonrelativistic result. This finding, however, is dependent on the potential and also on the parameters of a given potential.

We have also included strong form factors for the πNN vertex. Their effect by far exceeds the differences caused by the various relativistic equations. Even the uncertainty due to the ill-determined value of the cutoff parameter is larger than these relativistic deviations. The inclusion of full off-shell hadronic form factors yields smaller phase shifts than for the case where just the pion is treated off the mass shell in the form factor. The size of this effect can even be larger (depending on the considered partial wave) than the one caused by the inclusion of the (pion) form factor usually used.

ACKNOWLEDGMENTS

We want to thank Dr. W. Plessas for valuable discussions and for reading the manuscript. This work was supported in part by the Bundesministerium für Forschung und Technologie of the Federal Republic of Germany.

APPENDIX

A partial wave decomposition

$$V_L(k,k')=2\pi\int_{-1}^{+1}dx\ V(\mathbf{k},\mathbf{k}')P_L(x) \quad (A1)$$

with $x=\cos(\sphericalangle\,\mathbf{k},\mathbf{k}')$ and P_L the Legendre function of the first kind, applied to the potential of Eq. (4) gives

$$V_L^C(k,k')=-2\pi\frac{\lambda^2}{kk'}Q_L(x_0)\ . \quad (A2)$$

There Q_L is the Legendre function of the second kind and

$$x_0=\frac{k^2+k'^2+\mu^2-[\sqrt{s}-(m^2+k^2)^{1/2}-(m^2+k'^2)^{1/2}]^2}{2kk'}\ . \quad (A3)$$

The singularities of V_L^C are based on the Legendre function, i.e., they appear at $x_0=\pm1$. Inserting this condition into Eq. (A3) we get

$$(k\mp k')^2+\mu^2=[\sqrt{s}-(m^2+k^2)^{1/2}-(m^2+k'^2)^{1/2}]^2\ . \quad (A4)$$

The solution of this equation for k as a function of k' gives

$$k=\frac{\mp 4k'U\pm[16k'^2U-4(4k'^2-V^2)(U^2-m^2V^2)]^{1/2}}{2(4k'^2-V^2)} \tag{A5}$$

with

$$U=s+2m^2-\mu^2-2\sqrt{s}\,(m^2+k'^2)^{1/2}\,,$$

$$V=2[\sqrt{s}-(m^2+k'^2)^{1/2}]\,.$$

The result is shown in Fig. 2 for different values of $\sqrt{s}$. The plus sign indicates that these parts of the curve originate from the condition $x_0=+1$, the minus sign corresponds to $x_0=-1$.

More insight into the structure of these singularities is obtained calculating the extrema of Eq. (A4) with $x_0=+1$ (the curve given by $x_0=-1$ has no minimum for k, $k'>0$). Derivation of Eq. (A4) with respect to k' yields a relation between k and k'

$$k_{\min}=\left[1\pm\frac{\mu}{m}\right]k'_{\min}\,. \tag{A6}$$

Inserting relation (A6) into Eq. (A3) gives the result

$$k^2_{\min}=\frac{1}{4s}\,|s-\mu^2|\;|s-(2m\pm\mu)^2|\,. \tag{A7}$$

From this equation one can see that not all singularities exist for arbitrary s. The condition $k^2_{\min}\geqq 0$ is fulfilled by the first parenthesis and, if the case with the minus sign is taken, also by the second parenthesis. Therefore the right part of the singularities always exists. For the left part (the case with the plus sign) there exists a threshold energy, namely

$$s_0=(2m+\mu)^2\,. \tag{A8}$$

That means that these singularities occur just for energies above pion production threshold.

*On leave from Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional, México 14 D.F., México.

[1]E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).

[2]C. Schwartz and C. Zemach, Phys. Rev. **141**, 1454 (1966); M. J. Levine, J. Wright, and J. A. Tjon, *ibid.* **157**, 1416 (1967); L. J. Gammel, M. T. Menzel, and W. R. Wortman, Phys. Rev. D **3**, 2175 (1971); J. Fleischer and J. A. Tjon, Nucl. Phys. **B84**, 375 (1975); Phys. Rev. D **15**, 2537 (1977).

[3]K. Schwarz, J. Fröhlich, L. Streit, and H. F. K. Zingl, Acta Phys. Austriaca **53**, 191 (1980); J. Fröhlich, K. Schwarz, L. Streit, and H. F. K. Zingl, Phys. Rev. C **25**, 2591 (1982).

[4]R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966).

[5]F. Gross, Phys. Rev. **186**, 1448 (1969).

[6]K. Erkelenz and K. Holinde, Z. Naturforsch. **280**, 353 (1973).

[7]V. G. Kadyshevski, Nucl. Phys. **B6**, 125 (1968); M. Bawin and J. P. Lavine, *ibid.* **B49**, 610 (1970).

[8]R. H. Thompson, Phys. Rev. D **1**, 110 (1970).

[9]R. M. Woloshyn and A. D. Jackson, Nucl. Phys. **B64**, 269 (1973).

[10]R. Aaron, R. D. Amado, and J. E. Young, Phys. Rev. **174**, 2022 (1968).

[11]F. Gross, Phys. Rev. C **26**, 2226 (1982).

[12]K. Schwarz, H. F. K. Zingl, and L. Mathelitsch, Acta Phys. Austriaca **51**, 269 (1979).

[13]J. Fröhlich, K. Schwarz, and H. F. K. Zingl, Phys. Rev. C **27**, 265 (1983).

[14]G. E. Brown, A. D. Jackson, and T. S. Kuo, Nucl. Phys. **A133**, 481 (1969); A. D. Jackson and J. A. Tjon, Phys. Lett. **32B**, 9 (1970); M. Fortes and A. D. Jackson, Nucl. Phys. **A175**, 449 (1971); G. E. Brown and A. D. Jackson, *The Nucleon-Nucleon Interaction* (North-Holland, Amsterdam, 1976), Chap. VI; F. Gross, Phys. Rev. C **26**, 2203 (1982).

[15]H. Garcilazo, Phys. Lett. **99B**, 185 (1981).

[16]E. W. Schmid and H. Ziegelmann, *The Quantum Mechanical Three-Body Problem*, edited by H. Stumpf (Pergamon, New York, 1974), p. 99; W. M. Kloet and R. R. Silbar, Nucl. Phys. **A338**, 281 (1980).

[17]K. L. Kowalski, Phys. Rev. Lett. **15**, 798 (1965); H. P. Noyes, *ibid.* **15**, 538 (1965).

[18]A dependence of relativistic effects from the potentials has also been shown in Ref. 12 for the case of separable interactions.

[19]J. L. Friar, B. F. Gibson, and G. L. Payne, Phys. Rev. C **30**, 1084 (1984).

[20]K. Bongardt, H. Pilkuhn, and H. G. Schlaile, Phys. Lett. **52B**, 271 (1973).

[21]A. W. Thomas, Adv. Nucl. Phys. **13**, 1 (1983).

[22]K. Holinde, Phys. Rep. **68**, 121 (1981).

[23]W. Nutt and B. Loiseau, Nucl. Phys. **B104**, 98 (1976).

[24]R. A. Bryan, C. A. Dominguez, and B. J. VerWest, Phys. Rev. C **22**, 160 (1980).