Application of a number-conserving boson expansion theory to Ginocchio's SO(8) model

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A boson expansion theory based on a number-conserving quasiparticle approach is applied to Ginocchio's SO(8) fermion model. Energy spectra and $E2$ transition rates calculated by using this new boson mapping are presented and compared against the exact fermion values. A comparison with other boson approaches is also given.

I. INTRODUCTION

The Bardeen-Copper-Schriefer (BCS) approximation has long been known as a powerful and efficient approach in describing various aspects of nuclear physics.[1] The main reason for its broad and successful use lies in the introduction of the quasiparticle (QP) description and ensuing simplification of the calculations. As is well known, however, the BCS approximation entails two related problems: (i) the nonconservation of particle number, and (ii) the presence of spurious components in the states. Although it is difficult to estimate the amount of error in BCS, it is reasonable to expect that it remains relatively small so long as $N \gg n_q$ and $2\Omega_{\rm eff} \gg n_q$, where N is the particle number, $\Omega_{\rm eff}$ is the effective degeneracy, and n_q is the QP number.

These two conditions, however, pose serious restrictions on the region of applicability of the BCS theory. For this reason, over the years various number-projection methods have been developed to remove the BCS problems[2] mentioned above. However, the use of these methods normally results in loss of the simplicity of the BCS theory, and, more generally, of the advantages of the QP description.

Recently, Li[3] proposed a new method, called a number conserving quasiparticle (NCQP) approach. Like earlier approaches,[2] the NCQP method again starts with the BCS theory and restores number conservation and removes the spurious components through the introduction of appropriate projection operators. This method, however, differs significantly from earlier ones[2] in that the projection operator is incorporated in the new forms of the various operators, while the states maintain the BCS form. Thus, with the NCQP method, number conserving calculations can be performed essentially without giving up any of the advantages of the QP description. In simple models, such as the $1j$ shell model which was the subject of our recent paper,[4] and Ginocchio's SO(8) (Ref. 5) model, which we treat in the present paper, the NCQP theory is exact, while it becomes approximate in realistic many-j cases. However, even in the latter case this new method succeeds in removing most of the BCS error.

Because of its simplicity, the BCS formalism has been considered a very convenient basis for more sophisticated calculations. In the past 20 years or so, a number of boson formalisms,[6,7] which may in general be called boson expansion theories (BET's), have been developed with the BCS approximation as a starting point. We shall refer to these theories as BCS + BET. In particular, the BCS + BET of Kishimoto and Tamura (KT) has been successfully used for realistic calculations to fit collective low-lying states for a number of nuclei.[7] (We shall refer to the first and second papers of Ref. 7, respectively, as KT1 and KT2.) In these cases, the conditions that $2\Omega_{\rm eff} \gg n_q$ and $N \gg n_q$ were fairly well satisfied and thus the BCS approximation was expected to be reasonably accurate. As remarked above, however, these two conditions set some restrictions on the region of applicability of the BCS theory, and thus of BCS + BET.

In recent years a major effort was made to improve the KT boson expansion. The results were presented in Ref. 8, which we shall refer to as KT3. In this paper a rather general formulation of the BET was presented that included the previous KT formalism as a special case. More recently, Tamura,[9] using the results of KT3, set forth a quite convenient bosonization procedure, called a term-by-term bosonization (TTB) method.

The BET formalism of Refs. 8 and 9 contains significant improvements, but it still relies on the use of the BCS approximation, as was the case with the previous KT work. Thus it has the same limited region of applicability. However, the NCQP method is now available, and it can be incorporated into our BET with relative ease. We thus have a new BET, which we may call the NCQP + BET, which is free from the restriction due to the BCS approximation.

The general formulation of NCQP + BET will be given in a forthcoming paper.[10] We note, however, that NCQP + BET for a $1j$ shell model has already been worked out in Ref. 4. There we also confirmed the good accuracy and fast convergence of the new BET. In the present paper, we shall present NCQP + BET for the Ginocchio SO(8) model.[5] This model has three limiting cases in which exact fermion calculations can be carried out analytically, and thus provides us with a good testing ground for the various theories, including BET.

Ginocchio's SO(8) model has been used by Arima *et al.*[11,12] to assess the accuracy of the boson mapping of Otsuka *et al.*,[13,14] BCS + BET (Ref. 7), and the boson expansion of Belyaev *et al.*[15] The main purpose of the present paper is to test the NCQP + BET in the same model and compare its performance with the performance of the boson theories listed above.

A note is in order here, regarding our work of Ref. 4. There we presented two different techniques to derive a boson formalism for the $1j$ shell model problem. The first formalism, which was called SR + BET, was based on the use of the seniority reduction (SR) scheme,[16] while the other was the NCQP + BET. These two formalisms were shown to be exact and completely equivalent, but in our discussion major emphasis was placed on the SR technique, rather than on the NCQP method. We did this because the major goal of Ref. 4 was to compare our use of the SR with that made previously by Otsuka *et al.*[13,14] In the Ginocchio model, again both of the above techniques can be adopted and their equivalence proven. However, in the present paper we shall concentrate on the NCQP approach. We want to show in some detail how NCQP + BET replaces BCS + BET and compare its performance with that of other boson theories.

II. NUMBER CONSERVING BET FOR GINOCCHIO'S SO(8) MODEL

In this section we first present the basic formulas pertaining to Ginocchio's SO(8) model,[5] and then derive the NCQP + BET. In order to elucidate the relation between the BCS approximation and the NCQP approach, we discuss also the BCS + BET before presenting the NCQP + BET.

A. The SO(8) model

In the SO(8) model the single-nucleon angular momentum j is separated into a pseudo-orbital angular momentum $l\geq 0$ and a pseudo-spin $i=\frac{3}{2}$, $\mathbf{j}=\mathbf{l}+\mathbf{i}$. Next, a set of operators is constructed which form a closed algebra. These operators are defined as follows:

$$B^{\dagger}_{\lambda\mu}=\sum_{jj'}\frac{1}{\sqrt{\Omega}}(-)^{\lambda+l+j'+3/2}\hat{j}\hat{j}'W(jj'\tfrac{3}{2}\tfrac{3}{2};\lambda l)[a^{\dagger}_{j}a^{\dagger}_{j'}]_{\lambda\mu}\ , \tag{2.1a}$$

$$B_{\lambda\mu}=(B^{\dagger}_{\lambda\mu})^{\dagger}\ , \tag{2.1b}$$

$$C^{\dagger}_{kq}=\sum_{jj'}\sqrt{2/\Omega}(-)^{k+l+j'+3/2}\hat{j}\hat{j}'W(jj'\tfrac{3}{2}\tfrac{3}{2};kl)[a^{\dagger}_{j}\tilde{a}_{j'}]_{kq}\ . \tag{2.1c}$$

In (2.1) $\lambda=0$ and 2 and $k=0$, 1, 2, and 3, W is a Racah coefficient, and [ ] denotes the usual angular momentum coupling. Furthermore $\Omega=2(2l+1)$. Equation (2.1) defines in all 28 operators which form a complete set of generators of the special orthogonal group in eight dimensions, SO(8).

Note that we use here slightly different definitions for the pair operators than those adopted in Refs. 5 and 11. The correspondence is easily established and is given by

$$B^{\dagger}_{00}=\frac{1}{\sqrt{\Omega}}S^{\dagger},\quad B^{\dagger}_{2\mu}=\frac{1}{\sqrt{\Omega}}D^{\dagger}_{\mu},\quad C^{\dagger}_{kq}=\frac{1}{\sqrt{2\Omega}}P^{(k)}_{q}\ .$$

We find the use of the normalized operators (2.1) more convenient, because the algebra of the group SO(8) can be given in a compact form as

$$[B_{\lambda\mu},B^{\dagger}_{\lambda'\mu'}]=\delta_{\lambda\lambda'}\delta_{\mu\mu'}-\sum_{k}P^{(kq)}_{\lambda\bar{\mu};\lambda'\mu'}C^{\dagger}_{kq}\ , \tag{2.2a}$$

$$[C^{\dagger}_{kq},B^{\dagger}_{\lambda\mu}]=\sum_{\lambda'\mu'}P^{(kq)}_{\lambda\bar{\mu};\lambda'\mu'}B^{\dagger}_{\lambda'\mu'}\ , \tag{2.2b}$$

$$[C^{\dagger}_{kq},C^{\dagger}_{k'q'}]=-\tfrac{1}{2}\sum_{k''q''}[1-(-)^{k+k'+k''}]P^{(k''q'')}_{kq,k'q'}C^{\dagger}_{k''q''}\ . \tag{2.2c}$$

In (2.2) we have introduced the quantity

$$P^{(kq)}_{\lambda\mu;\lambda'\mu'}=\sqrt{8/\Omega}\hat{\lambda}\hat{\lambda}'(\lambda\mu\lambda'\mu'\,|\,kq)W(\lambda\lambda'\tfrac{3}{2}\tfrac{3}{2};k\tfrac{3}{2})\ . \tag{2.2d}$$

Note that $W(22\frac{3}{2}\frac{3}{2};2\frac{3}{2})=0$. Thus if $\lambda=\lambda'=2$, the summation in (2.2a) ranges over $k=0$, 1, and 3; namely the $k=2$ term vanishes.

Using the operators of (2.1), one can construct the following schematic shell model Hamiltonian:[5,11]

$$H=G_0\Omega B^{\dagger}_{00}B_{00}+G_2\Omega\sum_{\mu}B^{\dagger}_{2\mu}B_{2\mu}+\tfrac{1}{2}b_2\Omega\sum_{q}C^{\dagger}_{2q}C_{2\bar{q}}+\tfrac{1}{2}\Omega\sum_{k=1,3}b_kC^{\dagger}_{kq}C_{k\bar{q}}\ . \tag{2.3}$$

In Ref. 11, the last term in (2.3) was neglected. We shall do the same in the present paper.

The general Hamiltonian (2.3) has three, exactly solvable, limiting cases,[5,11] which can be obtained by properly choosing the relative strengths of the interaction. The condition that $G_2=b_2$ gives the $SO_5\times SU_2$ limit; $G_0=G_2$ gives the SO_6 limit; and finally $G_0=b_2$ gives the SO_7 limit. The numerical calculations presented in Sec. III will refer to these three limiting cases for $l=5$ (which makes $\Omega=22$).

B. BCS + BET

The BCS + BET for the Ginocchio model was presented in Ref. 11. In this subsection we shall rederive the BCS + BET, mostly as a preparation for the next subsection where the NCQP + BET is presented. In doing this, we employ the method of KT3, rather than that of KT2 used in Ref. 11. (See the end of this subsection for the difference in the results obtained by using these two methods.)

As is well known, application of the BCS theory begins by performing the Bogoliubov transformation of the single particle operators written as

$$a^{\dagger}_{jm}=u\chi^{\dagger}_{jm}+v\chi_{j\tilde{m}};\quad a_{j\tilde{m}}=u\chi_{j\tilde{m}}-v\chi^{\dagger}_{jm}\ . \tag{2.4}$$

In (2.4) $\chi^{\dagger}_{jm}$ and χ_{jm} are quasiparticle (QP) operators and the u and v factors are given as

$$u=[(\Omega-n)/\Omega]^{1/2};\quad v=(n/\Omega)^{1/2}\ . \tag{2.5}$$

In (2.5) $n=N/2$ is the number of particle pairs in the system. Note that, because of the degeneracy of the four orbits in the Ginocchio model, the u and v factors are independent of j.

Performing the Bogoliubov transformation for the pair operators we obtain

$$B^{\dagger}_{\lambda\mu}=u^2\bar{B}^{\dagger}_{\lambda\mu}-v^2\bar{B}_{\lambda\tilde{\mu}}-\frac{1}{\sqrt{2}}uv(\bar{C}^{\dagger}_{\lambda\mu}+\bar{C}_{\lambda\tilde{\mu}})+uv\sqrt{\Omega}\delta_{\lambda 0}\ , \tag{2.6}$$

$$C^{\dagger}_{kq}=\sqrt{2}uv(\bar{B}^{\dagger}_{kq}+\bar{B}_{k\tilde{q}})+u^2\bar{C}^{\dagger}_{kq}-v^2\bar{C}_{\kappa\tilde{q}}-v^2\sqrt{2\Omega}\delta_{k0}\ .$$

On the rhs of (2.6), $\bar{B}_{\lambda\mu}$, etc., are QP pair operators. In the Ginocchio model they are given by

$$\bar{B}^{\dagger}_{\lambda\mu}=\sum_{jj'}\frac{1}{\sqrt{\Omega}}(-)^{l+j'+3/2}\hat{j}\hat{j}'W(jj'\tfrac{3}{2}\tfrac{3}{2};\lambda l)[\chi^{\dagger}_{j}\chi^{\dagger}_{j'}]_{\lambda\mu}\ , \tag{2.7a}$$

$$\bar{B}_{\lambda\mu}=(\bar{B}^{\dagger}_{\lambda\mu})^{\dagger}\ , \tag{2.7b}$$

$$\bar{C}^{\dagger}_{kq}=\sum_{jj'}\sqrt{2/\Omega}(-)^{k+l+j'+3/2}\hat{j}\hat{j}'W(jj'\tfrac{3}{2}\tfrac{3}{2},kl)[\chi^{\dagger}_{j}\tilde{\chi}_{j'}]_{kq}\ . \tag{2.7c}$$

Clearly the relations in (2.7) are the same as those given in (2.1), except that the QP operators $\chi^{\dagger}_{j}$ and χ_j have now replaced the real particle operators $a^{\dagger}_{j}$ and a_j. The BCS states $|n;\alpha\rangle\rangle$ are constructed by superimposing the QP pair operators as

$$|n_q;\alpha\rangle\rangle=\bar{B}^{\dagger}_{2\mu_1}\bar{B}^{\dagger}_{2\mu_2}\cdots\bar{B}^{\dagger}_{2\mu_{n_q/2}}|0;\text{BCS}\rangle\ . \tag{2.8}$$

In (2.8), n_q stands for the QP number, α stands for any additional quantum number needed to specify the state, and $|0;\text{BCS}\rangle$ denotes the BCS vacuum.

The QP description so constructed is a convenient basis for transcribing the fermion problem into boson language. With this basis, we need only to expand the images of the QP pair operators $\bar{B}^{\dagger}_{2\mu}$ and $\bar{C}^{\dagger}_{2\mu}$ in terms of quadrupole bosons. To the third order, the boson image for $\bar{B}^{\dagger}_{2\mu}$ is

$$(\bar{B}^{\dagger}_{2\mu})_B=d^{\dagger}_{\mu}-\frac{1}{\Omega}y\,d^{\dagger}_{\mu}\hat{n}_d+\frac{1}{2\Omega}zK^{\dagger}d_{\tilde{\mu}}\ , \tag{2.9a}$$

with

$$K^{\dagger}=\sqrt{5}[d^{\dagger}d^{\dagger}]_{00}\ . \tag{2.9b}$$

The z and y coefficients in (2.9a) are defined by

$$y=\Omega\left[1-\left[1-\frac{2}{\Omega}\right]^{1/2}\right]\ ,$$
$$z=\frac{2\Omega}{5}\left[1-\frac{2}{\Omega}\right]^{1/2}\left[\left[1+\frac{5}{\Omega-2}\right]^{1/2}-1\right]\ . \tag{2.9c}$$

In (2.9) $d^{\dagger}$ is the quadrupole boson, and $\hat{n}_d$ is the boson number operator. The boson image of the scattering operator $\bar{C}^{\dagger}_{2\mu}$ to the second order vanishes accidentally due to the fact that $W(22\tfrac{3}{2}\tfrac{3}{2};2\tfrac{3}{2})=0$.

Having thus obtained the boson images of the basic QP pair operators, we can immediately write the boson images of the real particle pair operators (2.6), and then derive the boson images of the Hamiltonian and of the quadrupole transition operator. They are given as

$$\begin{aligned}H_{\text{BET}}=&(2h_{11}+A\Omega)\hat{n}_d-2yA\hat{n}_d(\hat{n}_d-1)+zAK^{\dagger}K\\&+B(\Omega-y+\tfrac{5}{2}z)(K^{\dagger}+K)\\&+B(2y-z)(K^{\dagger}\hat{n}_d+\hat{n}_dK)\end{aligned} \tag{2.10}$$

and

$$(Q_{2\mu})_{\text{BET}}=\frac{[2n(\Omega-n)]^{1/2}}{\Omega}\left[d^{\dagger}_{\mu}-\frac{1}{\Omega}yd^{\dagger}_{\mu}\hat{n}_d+\frac{1}{2\Omega}zK^{\dagger}d_{\tilde{\mu}}\right]+\text{H.c.} \tag{2.11}$$

The constants h_{11}, A, and B in (2.10) are defined by

$$\begin{aligned}h_{11}&=-\tfrac{1}{2}G_0\Omega-5(G_2v^4+b_2u^2v^2)\ ,\\A&=G_2(u^4+v^4)+2b_2u^2v^2\ ,\\B&=(b_2-G_2)u^2v^2\ .\end{aligned} \tag{2.12}$$

(Note that, as in Ref. 11, we do not include the dipole and octupole terms of the Hamiltonian.)

A note is in order here regarding the BCS + BET formalism presented by Arima in Ref. 11. The boson Hamiltonian given by Arima in his Eq. (11) agrees, up to the terms $O(1/\Omega)$, with our Hamiltonian of (2.10). However, the procedure used in Ref. 11 is quite different from the one we used above. As discussed at length in Ref. 17, Arima used the KT1 formalism, which is valid when all the components are retained, to obtain the expansion given in his Eq. (9), and then suppressed in it the s component. In our procedure, which was set forth in KT3, we first truncated the fermion space to a pure quadrupole component, and then performed the bosonization. [Note that the KT2 version of the expansion can be obtained by setting $y=z=1$ in (2.9) and (2.10).] In the case of Ginocchio's model, the two procedures give rise to practically the same final boson formulas. However, in general, and even in the $1j$ shell model case,[17] the above two methods give rise to quite different expansions both formally and numerically. As explained in Ref. 17, if the KT1 method is used, no truncation of the components should be done after the bosonization is performed, because this generally results in a large error in the commutation relations, and thus in any final numerical results.

C. NCQP + BET

As mentioned in the Introduction, the NCQP approach[3] was developed to cure the number nonconservation and spurious components problems in the BCS theory. As will be briefly discussed below, the NCQP formalism needed for such a purpose can be constructed in a few steps starting from the BCS representation. Details on how to combine the NCQP method with the BET can be found in Refs. 4 and 10. Here we explain only the basic philosophy of the method and give the relevant formulas for Ginocchio's models.

The first step of the NCQP approach is to replace the BCS states of (2.8) by

$$|n_q;\alpha\rangle=N_{\alpha}^{-1}\hat{P}\,|n_q,\alpha\rangle\rangle\ , \tag{2.13}$$

where N_{α} is an appropriate normalization factor. The operator $\hat{P}$ in (2.13) is constructed so as to knock out all the spurious components in $|n_q,\alpha\rangle\rangle$. The states $|n_q;\alpha\rangle$ thus contain only *physical* components, and may be called *physical* QP states. The explicit form for $\hat{P}$ has been given in Ref. 3. In the case of Ginocchio's SO(8) model, it is straightforward to show that the states $|n_q;\alpha\rangle$ are annihilated by the QP pairing operator $\overline{B}_{00}$, i.e., that they have the highest seniority nature.

The second step in the NCQP procedure is to introduce a projection operator, called Λ_N, which, when operating upon $|n_q;\alpha\rangle$, projects out and renormalizes the component with the correct particle number N. Formally, we may write

$$\Lambda_n |n_q;\alpha\rangle = |N;n_q,\alpha\rangle \qquad (2.14)$$

In Eq. (2.14) $|N;n_q,\alpha\rangle$ is a normalized number-projected state with good particle number N extracted from the physical QP state $|n_q;\alpha\rangle$.

We want now to incorporate the effect of Λ_N into the real particle operators, so as to obtain corresponding *effective* operators that can be used with the physical QP states. Let $O_k^\dagger$ be an operator that creates k real particles. The corresponding effective operator $(O_k^\dagger)_N$ is then constructed as

$$(O_k^\dagger)_N = \Lambda_{N+k}^\dagger O_k^\dagger \Lambda_N = (O_k)_N^\dagger \ . \qquad (2.15a)$$

From (2.14) and (2.15a) it immediately follows that

$$\langle n_q;\alpha | (O_k^\dagger)_N | n_q';\alpha'\rangle = \langle N+k;n_q,\alpha | O_k^\dagger | N;n_q';\alpha\rangle \ . \qquad (2.15b)$$

It is clear from (2.15b) that, once $(O_k^\dagger)_N$ is constructed, we can perform number-conserving calculations in the QP space, thus preserving the merits of the QP description.

The final and the most crucial step of the NCQP method is to construct $(O_k^\dagger)_N$ explicitly. This is done by using in a non-Hermitian way a new projection operator T_N, so that the following equality holds:

$$\langle N+k;n_q,\alpha | O_k^\dagger | N,n_q',\alpha'\rangle = (\langle n_q,\alpha | \overleftarrow{\mathbf{T}}_{N+k} O_k^\dagger | n_q';\alpha'\rangle \langle n_q;\alpha | O_k^\dagger \overrightarrow{\mathbf{T}}_N | n_q';\alpha'\rangle)^{1/2} \ . \qquad (2.16)$$

Namely, the number projection is done either on the bra or ket QP state, but not on both states at the same time. The operator T_N is expressed as a power series of the real particle number operator. The power of this technique shows itself in the fact that, e.g., for $O_k^\dagger = a_{jm}^\dagger$, the first two terms of T_N are enough to obtain, in general, a very accurate result, or, as in Ginocchio's model, even an exact result. Furthermore, since any operator $O_k^\dagger$ can be constructed as a product of $a_{jm}^\dagger$ and its Hermitian conjugate a_{jm}, the effective forms of these two-single particle operators are all we need to derive the *effective form* of any particle operator, as we shall illustrate below. The use of (2.16) also permits one to avoid calculating the norm of $|N;n_q;\alpha\rangle$ explicitly, a task that can become very involved in realistic (nondegenerate) many-j cases. See Ref. 3 for details on these points.

When the single-particle operator $a_{jm}^\dagger$ is taken as $O_k^\dagger$, we find[3] that (2.15) and (2.16) give rise to

$$(a_{jm}^\dagger)_N = \Lambda_{N+1}^\dagger a_{jm}^\dagger \Lambda_N = \dot{\chi}_{jm}^\dagger U(N,\dot{n}) + V(N+1,\dot{n})\dot{\chi}_{j\tilde{m}} \ . \qquad (2.17a)$$

Here

$$\dot{\chi}_{jm}^\dagger = (\dot{\chi}_{jm})^\dagger = \hat{\mathbf{P}}\chi_{jm}^\dagger \hat{\mathbf{P}}$$

and (2.17b)

$$\dot{n} = \sum_{jm} \dot{\chi}_{jm}^\dagger \dot{\chi}_{jm'}$$

and the U and V factors are given by

$$U(N,\dot{n}) = \{(2\Omega - N - \dot{n})/[2(\Omega - \dot{n})]\}^{1/2} \ ,$$
$$V(N,\dot{n}) = \{(N-\dot{n})/[2(\Omega-\dot{n})]\}^{1/2} \ . \qquad (2.17c)$$

As in Ref. 3, as well as in Ref. 18, we call $\dot{\chi}_{jm}^\dagger$ and $\dot{\chi}_{jm}$ "ideal QP operators." The anticommutation relations for the ideal operators are given by

$$\{\dot{\chi}_{jm}, \dot{\chi}_{j'm'}^\dagger\} = \delta_{jj'}\delta_{mm'} - \dot{\chi}_{jm}^\dagger \frac{1}{\Omega - \dot{n}} \dot{\chi}_{j'\tilde{m}'} \ , \qquad (2.17d)$$

$$\{\dot{\chi}_{jm}^\dagger, \dot{\chi}_{j'm'}^\dagger\} = \{\dot{\chi}_{jm}, \dot{\chi}_{j'm'}\} = 0 \ . \qquad (2.17e)$$

An important feature of the ideal QP operators is that $\dot{\chi}_{jm}^\dagger$ ($\dot{\chi}_{jm}$) creates (destroys) exactly one unit of seniority, and thus $\dot{n}$ preserves seniority. Since the U and V factors are now operators, Eq. (2.17a) may be called a *quantized* Bogoliubov transformation.

Equation (2.17) contains all the information we need to perform the NCQP transformation of any fermion operator. Thus, consider, e.g., products of single-particle operators, and first note that the following relations hold:

$$(a_{jm}^\dagger a_{j'm'}^\dagger)_N = (a_{jm}^\dagger)_{N+1}(a_{j'm'}^\dagger)_N \ ,$$
$$(a_{jm}^\dagger a_{j'm'})_N = (a_{jm}^\dagger)_{N-1}(a_{j'm'})_{N-1} \ , \qquad (2.18)$$

By combining (2.17) and (2.18), we find that

$$(B_{\lambda\mu}^\dagger)_N = \dot{B}_{\lambda\mu}^\dagger \hat{u}_{1,1}\hat{u}_{0,0} - \dot{B}_{\lambda\tilde{\mu}}\hat{v}_{2,-2}\hat{v}_{1,-1} - \left[\frac{1}{\sqrt{2}}(\dot{C}_{\lambda\mu}^\dagger + \dot{C}_{\lambda\tilde{\mu}}) - \sqrt{\Omega}\delta_{\lambda 0}\right]\hat{u}_{0,0}\hat{v}_{2,0} \ , \qquad (2.19a)$$

$$(C_{kq}^\dagger)_N = \sqrt{2}\dot{B}_{kq}^\dagger \hat{u}_{-1,1}\hat{v}_{0,0} + \sqrt{2}\dot{B}_{k\tilde{q}}\hat{u}_{-1,-1}\hat{v}_{0,-2} + \dot{C}_{kq}^\dagger \hat{u}_{0,0}^2 - \dot{C}_{k\tilde{q}}\hat{v}_{0,0}^2 + \sqrt{2\Omega}\delta_{k0}\hat{v}_{0,0}^2 \ . \qquad (2.19b)$$

In (2.19) we have introduced the ideal QP pair operators defined as

$$\dot{B}_{\lambda\mu}^\dagger \equiv \hat{P}\overline{B}_{\lambda\mu}^\dagger \hat{P}, \quad \dot{C}_{kq}^\dagger \equiv \hat{P}\overline{C}_{kq}^\dagger \hat{P} \ . \qquad (2.20)$$

[The explicit definitions of $\dot{B}_{\lambda\mu}^\dagger$ and $\dot{C}_{kq}^\dagger$ can be read off from Eq. (2.7) by replacing the QP operators $\chi^\dagger$ and χ on the rhs by the ideal QP operators $\dot{\chi}^\dagger$ and $\dot{\chi}$, respectively.] We have also introduced the factors $\hat{u}_{i,j}$ and $\hat{v}_{i,j}$ which are defined as

$$\hat{u}_{i,j} \equiv U(N+i,\dot{n}+j)$$

and (2.21)

$$\hat{v}_{i,j} \equiv V(N+i,\dot{n}+j) \ ,$$

A repeated use of (2.18) allows us to obtain the NCQP Hamiltonian $(H)_N$, corresponding to Ginocchio's Hamiltonian (2.3), as

$$(H)_N = G_0\Omega(B^\dagger_{00})_{N-2}(B_{00})_{N-2} + G_2\Omega\sum_\mu (B^\dagger_{2\mu})_{N-2}(B_{2\mu})_{N-2} + \tfrac{1}{2}b_2\Omega\sum_q (C^\dagger_{2q})_N(C_{2\bar{q}})_N \ . \tag{2.22}$$

By using (2.19), this can further be rewritten as

$$(H)_N = (\Omega-\dot{n})[\hat{v}^{\,2}_{0,0}G_0(\Omega-\dot{n})\hat{u}_{-2,0}+5R_1] + \Omega\sum_\mu \dot{B}^\dagger_{2\mu}\dot{B}_{2\mu}\left[R_2+R_3\frac{(\Omega-\dot{n}+3)}{(\Omega-\dot{n}+2)}\right] - \left[\Omega\sum_\mu \dot{B}^\dagger_{2\mu}\dot{B}^\dagger_{2\bar{\mu}}R_4+\text{H.c.}\right] , \tag{2.23}$$

where

$$\begin{aligned} R_1 &= v^2_{0,0}(G_2\hat{v}^{\,2}_{-1,1}+b_2\hat{u}^{\,2}_{-1,1}) , \\ R_2 &= \hat{u}^{\,2}_{-1,-1}(G_2\hat{u}^{\,2}_{-2,-2}+b_2\hat{v}^{\,2}_{0,-2}) , \\ R_3 &= \hat{v}^{\,2}_{-1,-1}(G_2\hat{v}^{\,2}_{-1,1}+b_2\hat{u}^{\,2}_{-1,1}) , \\ R_4 &= \hat{u}_{-1,3}\hat{v}_{0,0}(G_2\hat{u}_{-2,1}\hat{v}_{-1,1}-b_2\hat{u}_{-1,1}\hat{v}_{0,2}) . \end{aligned} \tag{2.24}$$

The derivation of (2.19) and (2.23) completes the transcription of the original Ginocchio problem into the NCQP form. This transcription is exact. This came about because the four orbits in the SO(8) model are completely degenerate, and thus the u and v factors in the original Bogoliubov transformation are independent of j, as remarked below Eq. (2.5). This condition is sufficient to guarantee that the NCQP approach is exact.[3]

The basis states onto which the operators in (2.19) and (2.23) are applied are the physical QP states $|n_q;\alpha\rangle$ of (2.13). By using the ideal QP operators $\dot{B}^\dagger_{2\mu}$ defined in (2.20), such states can be written explicitly as

$$|n_q;\alpha\rangle = N_\alpha^{-1}\dot{B}^\dagger_{2\mu_1}\dot{B}^\dagger_{2\mu_2}\cdots\dot{B}^\dagger_{2\mu_{n_q/2}}|0;\text{BCS}\rangle , \tag{2.25}$$

N_α being an appropriate normalization factor.

With $|n_q;\alpha\rangle$ in the form of (2.25), it is easy to show that $\dot{C}^\dagger_{2\mu}|n_q;\alpha\rangle=0$. [To prove this, the fact that $W(22\frac{3}{2}\frac{3}{2};2\frac{3}{2})=0$ is used.] This is the reason why $\dot{C}^\dagger_{2\mu}$ does not appear in (2.23). Also, $\dot{B}^\dagger_{00}$ does not appear either in (2.23) since $\dot{B}^\dagger_{00}=0$ as a consequence of the commutation relation (2.17d).

We shall now consider the bosonization of the NCQP results obtained above. As shown in Refs. 8 and 9, the first step of this procedure is to evaluate the matrix elements of the operators in (2.19) and (2.23) in the states of (2.25). The contribution to these matrix elements from the $\hat{u}_{ij}$ and $\hat{v}_{ij}$ factors can be easily calculated by replacing $\dot{n}$ in them with the appropriate QP number n_q. Correspondingly, the boson images of these two factors are obtained by replacing $\dot{n}$ by the boson number operator $2\hat{n}_d$ [see Eq. (2.29) below].

Having thus taken care of the $\hat{u}_{ij}$ and $\hat{v}_{ij}$ factors, we can now concentrate on the matrix elements of the two pair operators $\dot{B}^\dagger_{2\mu}$ and $\dot{C}^\dagger_{00}$ in the space of (2.25), and then obtain their boson images. Up to the third order terms, they are given as

$$\begin{aligned} (\dot{B}^\dagger_{2\mu})_B &= d^\dagger_\mu - \frac{1}{\Omega}pd^\dagger_\mu\hat{n}_d + \frac{1}{2\Omega}qK^\dagger d_{\bar{\mu}} , \\ (\dot{C}^\dagger_{00})_B &= (2/\Omega)\hat{n}_d , \end{aligned} \tag{2.26}$$

with

$$\begin{aligned} p &= \Omega\left[1-\left[1-\frac{2}{\Omega}\right]^{1/2}\right] , \\ q &= \frac{2\Omega}{5}\left[1-\frac{2}{\Omega}\right]^{1/2}\left[\left[1+\frac{5}{\Omega-1}\right]^{1/2}-1\right] . \end{aligned} \tag{2.27}$$

Note that the coefficient p is the same as y in (2.9c). On the other hand, q is slightly different from z in (2.9c). This difference is the result of the presence of the projection operator $\hat{P}$ in NCQP. By expanding the square roots in (2.9c) and (2.27), and comparing the results, one can easily see that the effect of the projection operator is of the order of $O(1/\Omega^2)$.

By inserting (2.26) into (2.23), the bosonized Ginocchio Hamiltonian is finally obtained as

$$(H)_B = G_0(n-\hat{n}_d)(\Omega-n-\hat{n}_d+1)+5(\Omega-2\hat{n}_d)R_1+\{\hat{n}_d[\Omega-2p(\hat{n}_d-1)]+qK^\dagger K\}\left[R_2+R_3\frac{(\Omega-2\hat{n}_d+3)}{(\Omega-2\hat{n}_d+2)}\right] - \{K^\dagger[\Omega-(2\hat{n}_d+1)p+(\hat{n}_d+\tfrac{5}{2})q]R_4+\text{H.c.}\} . \tag{2.28}$$

The coefficients R_i $(i=1,2,3,4)$ are defined by (2.24) with the $\hat{u}_{i,j}$ and $\hat{v}_{i,j}$ factors now being given by

$$\begin{aligned} \hat{u}_{i,j} &= \left[\frac{[\Omega-n-\hat{n}_d-(i+j)/2]}{(\Omega-2\hat{n}_d-j)}\right]^{1/2} , \\ \hat{v}_{i,j} &= \left[\frac{[n-\hat{n}_d+(i-j)/2]}{(\Omega-2\hat{n}_d-j)}\right] . \end{aligned} \tag{2.29}$$

Finally, the boson form of the quadrupole transition operator of (2.19b) is

$$(Q^\dagger_{2\mu})_B = \sqrt{2}\left[d^\dagger_\mu - \frac{1}{\Omega}p\,d^\dagger_\mu\hat{n}_d + \frac{1}{2\Omega}qK^\dagger d_{\bar{\mu}}\right] \times\hat{u}_{-1,1}\hat{v}_{0,0}+\text{H.c} . \tag{2.30}$$

For a comparison with the boson expansion given by

Arima[11] in his Eq. (14), we note that the coefficient in the square brackets in the third term of Eq. (2.28) can also be written as

$$\left[G_2+\frac{1}{(\Omega-2\hat{n}_d+2)}b_2 + \frac{2(\Omega-2\hat{n}_d)}{(\Omega-2\hat{n}_d+2)}(b_2-G_2)\hat{v}^{\,2}_{0,0}\hat{u}^{\,2}_{-1,1}\right] .$$

III. NUMERICAL RESULTS

As stated in the Introduction, one of the primary purposes of this paper is to test the NCQP + BET against the exact fermion results in SO(5)×SU(2), SO(6), and SO(7) limits of Ginnocchio's SO(8) model. [For simplicity we shall henceforth refer to the first case as the SO(5) limit.] The interaction strengths used are (in keV): $G_0=-40$, $G_2=b_2=-10$ for the SO(5) case, $G_0=G_2=-20$ and $b_2=-40$ for the SO(6) case, and $G_0=b_2=-40$ and $G_2=-10$ for the SO(7) case. The calculations were performed for a system with 16 nucleons and $\Omega=22$. The system and the strengths chosen for the three limiting cases are thus the same as in Arima's paper.[11]

We have calculated the complete energy spectra and the $B(E2)$'s for the above three cases and the results are presented in Tables I–III and in Figs. 1 and 2. In Tables I–III we list the energies of all the states involved for the SO(5), SO(6), and SO(7) limits, respectively. (For the meaning of the quantum numbers labeling the states see Ref. 5.) In Figs. 1(a)–(c) we give the plot of the yrast states and a few low-lying states with spin (I) equal to 0 and 2. In Figs. 2(a)–(c) the plot is given for the $B(E2;I\rightarrow I-2)$ values between the yrast states again for the above three limits. These tables and figures contain the exact fermion results, which the various boson results can be compared directly against. In subsection A we concentrate on the BCS + BET and NCQP + BET while the discussion of other boson theories is given in subsection B.

TABLE I. Energies in the SO(5)×SU(2) limit. [$G_0=-0.04$, $G_2=-0.01$, $b_2=-0.01$ (MeV), $N_f=16$, $(n_d)_{\max}=8$, $\Omega=22$.]

k	τ	Exact	OAI BET	OAIT BET	BZM BET	BCS + BET	NCQP + BET	Spin
0	0	0.00	0.00	0.00	0.00	0.00	0.00	0
1	1	0.70	0.70	0.70	0.69	0.70	0.70	2
2	2	1.36	1.36	1.36	1.35	1.43	1.36	2,4
	0	1.26	1.26	1.26	1.28	1.33	1.26	0
3	3	1.98	1.98	1.98	1.98	2.21	1.98	0,3,4,6
	1	1.84	1.84	1.84	1.88	2.07	1.84	2
4	4	2.56	2.56	2.56	2.58	3.03	2.57	2,4,5,6,8
	2	2.38	2.38	2.38	2.45	2.85	2.39	2,4
	0	2.28	2.28	2.28	2.38	2.75	2.28	0
5	5	3.10	3.10	3.10	3.14	3.89	3.11	2,4,5,6,7,8,10
	3	2.88	2.88	2.88	2.98	3.67	2.89	0,3,4,6
	1	2.74	2.74	2.74	2.88	3.53	2.75	2
6	6	3.60	3.60	3.60	3.67	4.79	3.62	0,3,4,6^2,7,8,9,10,12
	4	3.34	3.34	3.34	3.47	4.54	3.35	2,4,5,6,8
	2	3.16	3.16	3.16	3.33	4.36	3.16	2,4
	0	3.06	3.06	3.06	3.26	4.26	3.06	0
7	7	4.06	4.06	4.06	4.16	5.73	4.08	2,4,5,6,7,8^2,9,10,11,12,14
	5	3.76	3.76	3.76	3.92	5.44	3.77	2,4,5,6,7,8,10
	3	3.54	3.54	3.54	3.74	5.22	3.54	0,3,4,6
	1	3.40	3.40	3.40	3.63	5.08	3.40	2
8	8	4.48	4.48	4.48	4.58	6.71	4.51	2,4,5,6,7,8^2,9,10^2,11,12,13,14,16
	6	4.14	4.14	4.14	4.27	6.38	4.15	0,3,4,6^2,7,8,9,10,12
	4	3.88	3.88	3.88	4.03	6.12	3.87	2,4,5,6,8
	2	3.70	3.70	3.70	3.87	5.96	3.68	2,4
	0	3.60	3.60	3.60	3.78	5.85	3.57	0

TABLE II. Energies in the SO(6) limit. [$G_0=-0.02$, $G_2=-0.02$, $b_2=-0.04$ (MeV); $N_f=16$, $(n_d)_{max}=8$, $\Omega=22$.]

σ	τ	Exact	OAI BET	OAIT BET	BZM BET	BCS + BET	NCQP + BET	Spin
8	0	0.00	0.00	0.00	0.00	0.00	0.00	0
	1	0.16	0.15	0.15	0.16	0.21	0.16	2
	2	0.40	0.37	0.39	0.40	0.41	0.39	2,4
	3	0.72	0.66	0.69	0.72	0.80	0.71	0,3,4,6
	4	1.12	1.01	1.07	1.12	1.17	1.11	2,4,5,6,8
	5	1.60	1.45	1.53	1.60	1.78	1.60	2,4,5,6,7,8,10
	6	2.16	1.91	2.06	2.16	2.34	2.16	0,3,4,6^2,8,9,10,12
	7	2.80	2.48	2.64	2.80	3.26	2.84	2,4,5,6,7,8^2,9,10,11,12,14
	8	3.52	3.00	3.36	3.52	4.02	3.56	2,4,5,6,7,8^2,9,10^2,11,12,13,14,16
6	0	0.72	0.38	0.61	0.72	0.67	0.71	0
	1	0.88	0.58	0.76	0.88	0.93	0.87	2
	2	1.12	0.72	1.00	1.12	1.09	1.11	2,4
	3	1.44	1.09	1.29	1.44	1.55	1.44	0,3,4,6
	4	1.84	1.31	1.68	1.84	1.92	1.84	2,4,5,6,8
	5	2.32	1.86	2.07	2.32	2.74	2.35	2,4,5,6,7,8,10
	6	2.88	2.16	2.70	2.88	3.34	2.90	0,3,4,6^2,7,8,9,10,12
4	0	1.28	0.66	1.11	1.28	1.25	1.26	0
	1	1.44	1.01	1.21	1.44	1.51	1.43	2
	2	1.68	1.02	1.50	1.68	1.69	1.66	2,4
	3	2.00	1.54	1.70	2.00	2.33	2.02	0,3,4,6
	4	2.40	1.66	2.19	2.40	2.80	2.39	2,4,5,6,8
2	0	1.68	1.08	1.52	1.68	1.66	1.65	0
	1	1.84	1.48	1.55	1.84	2.09	1.84	2
	2	2.08	1.48	1.83	2.08	2.42	2.05	2,4
0	0	1.92	1.53	1.64	1.92	2.22	1.89	0

A. BCS + BET and NCQP + BET

By looking at Fig. 1, one immediately sees the dramatic improvement of NCQP + BET over BCS + BET. In fact, the NCQP method yields virtually exact results in the SO(5) and SO(7) cases. In the SO(6) limit the fit is also very good, with only the $I=14$ and 16 states appearing slightly too high. As discussed in Sec. II, the NCQP approach removes the BCS problems, and this explains the difference between the BCS + BET and NCQP + BET results. (The boson expansion method used is essentially the same for the two approaches; see the remarks given in Sec. II.)

The violation of number conservation is the major source of BCS error, which is $O(1/\Omega)$, while the presence of spurious components gives a smaller error, typically of $O(1/\Omega^2)$. In any event, it is worthwhile to emphasize here the fact that, while the BCS error is significant for high-lying states (high n_q), it is reasonably small for low-lying states, reconfirming the well-known fact that the BCS theory is rather accurate for states with a small n_q.

As remarked above, Tables I—III give the energies of all the states. As one can see, the NCQP + BET fits very well not only the yrast states, but also all the other states.

In Figs. 2(a)—(c), the results for the $B(E2;I\to I-2)$ values for yrast states are plotted. We see that BCS + BET generally overestimates the $B(E2)$'s between high spin states. In the SO(5) and SO(7) limits, the transitions between low-lying states are reproduced well, while in the SO(6) case, the BCS + BET also overestimates these transitions. The various BCS troubles are cured very well by the NCQP method. In fact, the NCQP + BET results are very close to the exact values almost everywhere.

As an aside, we want to note here that our BCS + BET results are somewhat different from those given by Arima.[11] The relatively small differences in the spectra may be accounted for by the fact that Arima used a sixth order Hamiltonian, while we stopped at the fourth order. [Note that, because of the truncation to the sole quadrupole component in BCS + BET, the addition of the sixth order terms does not always improve the results. This is, in particular, the case for the energies of the 0_2 and 2_3 states in the SO(6) limit.]

A major discrepancy was noticed between our and Arima's BCS + BET results for the $B(E2)$'s in the SO(6) case. The results obtained by Arima are larger than ours by 10—30 units. We were unable to find the cause of this difference. (It is unlikely that it can be explained as an effect of higher order terms in the Hamiltonian and quadru-

TABLE III. Energies in the SO(7) limit. [$G_0=-0.04$, $G_2=-0.01$, $b_2=-0.04$ (MeV); $N_f=16$, $(n_d)_{max}=8$, $\Omega=22$.]

k	τ	Exact	OAI BET	OAIT BET	BZM BET	BCS + BET	NCQP + BET	Spin
0	0	0.00	0.00	0.00	0.00	0.00	0.00	0
1	1	0.40	0.45	0.44	0.41	0.43	0.40	2
2	2	0.88	0.97	0.95	0.92	0.97	0.87	2.4
	0	0.78	0.89	0.87	0.91	0.86	0.78	0
3	3	1.44	1.54	1.52	1.51	1.64	1.43	0,3,4,6
	1	1.30	1.38	1.39	1.45	1.44	1.30	2
4	4	2.08	2.16	2.16	2.18	2.40	2.07	2,4,5,6,8
	2	1.90	1.91	1.97	2.07	2.11	1.89	2,4
	0	1.80	1.78	1.86	2.01	1.97	1.80	0
5	5	2.80	2.82	2.87	2.92	3.32	2.79	2,4,5,6,7,8,10
	3	2.58	2.47	2.61	2.76	2.97	2.58	0,3,4,6
	1	2.44	2.26	2.45	2.66	2.76	2.43	2
6	6	3.60	3.50	3.63	3.74	4.28	3.59	0,3,4,6^2,7,8,9,10,12
	4	3.34	3.04	3.31	3.53	3.83	3.34	2,4,5,6,8
	2	3.16	2.72	3.08	3.38	3.52	3.15	2,4
	0	3.06	2.55	2.96	3.30	3.36	3.04	0
7	7	4.48	4.19	4.45	4.63	5.62	4.50	2,4,5,6,7,8^2,9,10,11,12,14
	5	4.18	3.62	4.06	4.36	5.21	4.21	2,4,5,6,7,8,10
	3	3.96	3.22	3.76	4.16	4.89	3.97	0,3,4,6
	1	3.82	2.98	3.57	4.04	4.68	3.81	2
8	8	5.44	4.85	5.41	5.59	6.76	5.44	2,4,5,6,8^2,9,10^2,11,12,13,14,16
	6	5.10	4.12	5.07	5.26	6.24	5.10	0,3,4,6^2,7,8,9,10,12
	4	4.84	3.57	4.80	5.01	5.83	4.82	2,4,5,6,8
	2	4.66	3.20	4.61	4.84	5.54	4.62	2,4
	0	4.56	3.01	4.51	4.74	5.38	4.50	0

pole operator.) Other minor differences, which exist between our $B(E2)$'s and Arima's, concern only the high spin states, and could be explained as higher order effects.

B. Other boson approaches

Besides the BCS + BET and NCQP + BET calculations, we have also performed the calculations by using the formalisms given by Otsuka, Arima, and Iachello (OAI),[13] by Otsuka, Arima, Iachello, and Talmi (OAIT),[14] and by Belyaev, Zelevinsky, and Marshalek (BZM).[15] The results for the last two approaches were also presented in Ref. 11, where, however, Arima improperly referred to the second method as OAI, instead of OAIT.

As discussed at length in Ref. 4, OAIT and OAI are different theories, although the technique used in deriving them is very similar. The discriminating factor is that the coefficients in the boson Hamiltonian are constants in OAI, but they depend on the boson number in OAIT. This sometimes causes a noticeable difference in the numerical results obtained, as we have already experienced in the $1j$-SM case,[4] and as we shall see below for the Ginocchio model. The reason for stressing the difference between OAI and OAIT is that both these theories have been constructed with the purpose of providing a microscopic foundation of the interacting boson approximation (IBA).[19] However, IBA is characterized by having constant Hamiltonian coefficients, and thus, strictly speaking, only OAI can be regarded as a microscopic version of IBA, while OAIT cannot.

From Figs. 1 and 2, one can see that the OAIT theory is exact for the SO(5) case, works well for the SO(7) case, while it gets somewhat poorer in the SO(6) case. It predicts exact $B(E2)$'s, again for the SO(5) case, but tends to underestimate the $B(E2)$ values in both the other two cases. Overall, we may say that the OAIT approach for the yrast states works reasonably well, with only a tendency [apart from the SO(5) case, where OAIT happens to be exact] to underestimate the energies. For the nonyrast states such a tendency becomes slightly more noticeable,

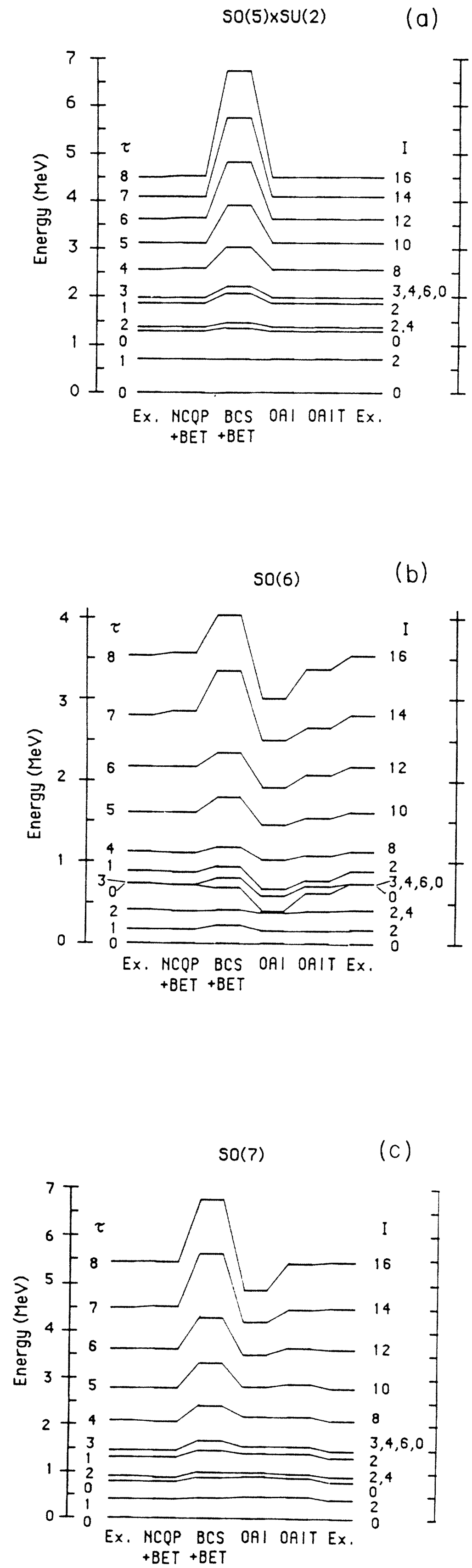

FIG. 1. Part of the energy spectra [(a) for the SO(5)×SU(2) case, (b) for the SO(6) case, and (c) for the SO(7) case].

as one can see from Tables II and III.

In the same tables, we also give the result obtained by using the OAI (Ref. 13) and BZM (Ref. 15) formalisms. (The OAI Hamiltonian for the Ginocchio model was not given anywhere previously. We give it in the Appendix.) For the BZM method we give only the results pertaining to the energies. [For the $B(E2)$'s obtained with BZM see Fig. 7 of Ref. 11.] The OAI formalism is exact for the SO(5) case (as OAIT was), while BZM is exact for the SO(6) case. As one can see from Tables I and II, BZM tends to overestimate the energies in both the SO(5) and SO(7) cases. This tendency is only slightly noticeable in the yrast states, but becomes more conspicuous for the nonyrast states. In the OAI results for the SO(6) cases, the tendency to underestimate the energy, already exhibited by OAIT, is accentuated, especially in the nonyrast states. As for the $B(E2)$'s, OAI is again exact for the SO(5) case, but underestimates the fermion results in both the other two cases, particularly in the SO(6) limit.

Summarizing, we may say that, in view of the poor fit of the nonyrast states and high-lying yrast states in the SO(6) and SO(7) cases, OAI performance is not satisfactory. OAIT, on the other hand, works rather well in all three cases, with only a tendency to underestimate the nonyrast and high-lying yrast states in the SO(6) case. Nevertheless, apart from the SO(5) case, the overall performance of OAIT is not as good as that of NCQP + BET. As one can see from the Tables I–III, as well as from the Figs. 1 and 2, NCQP + BET works consistently well for the yrast and the nonyrast states in all three cases. The difference in the performance of OAIT and NCQP + BET may be, at least in part, explained by the following argument.

As shown in Sec. II C, we derived the NCQP + BET by first bosonizing the pair operators, and then using the results to construct the boson Hamiltonian. OAIT (and OAI), on the other hand, calculates the coefficients of the boson Hamiltonian directly by equating the boson and fermion matrix elements with $v=0$, 2, and 4. With our method, the accuracy (or the error due to the termination of the boson expansion at the fourth order) is the same for all the terms in the boson Hamiltonian. With the OAIT (and OAI) methods this is not the case. When the diagonal terms are derived with fourth order accuracy, the off-diagonal terms can maintain only a second order accuracy. This fact may be the explanation of why OAIT performs, overall, not as well as the NCQP + BET does. (In order to derive the nondiagonal term with a fourth order accuracy in OAIT, the states with $v=6$ should also be included. We confirmed that the numerical results for the energies obtained with such an extra term did improve the OAIT results.)

The above argument is further corroborated by the fact that OAI and OAIT are exact in the SO(5) limit. Although all three limits of the SO(8) model are peculiar in some sense, the SO(5) case is very special in that the nondiagonal terms in the boson Hamiltonian vanish identically, since they are multiplied by the factor $b_2-G_2=0$. [See the Appendix of the present paper, and Eq. (14) of Ref. 11.] Thus the exact fit given by OAIT and OAI for the SO(5) case can hardly be taken as an indication that

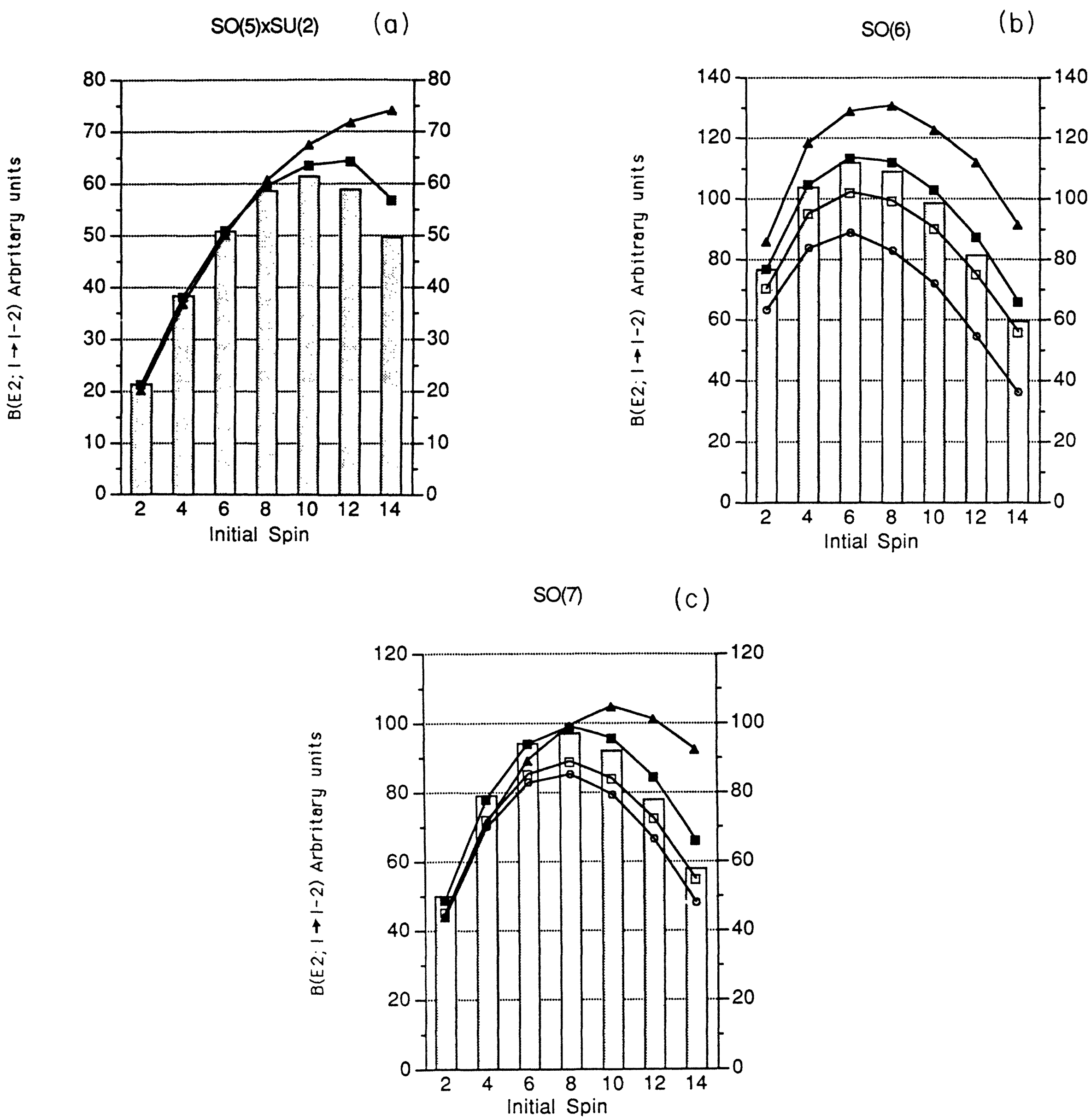

FIG. 2. $B(E2;I \rightarrow I-2)$'s between yrast states [(a) for the SO(5)×SU(2) case, (b) for the SO(6) case, and (c) for the SO(7) case]. Legend: bar=exact fermion values; ■=NCQP + BET; ▲=BCS + BET; □=OAIT; and ○ =OAI. In (a) OAI and OAIT results are exact. Effective charge $e_{\text{eff}}=\sqrt{2\Omega}$.

they will perform as well in more general situations. In fact, even within the Ginocchio model, their performance in the SO(6) and SO(7) cases, where the nondiagonal term does not vanish, is not as good as for the SO(5) limit.

IV. SUMMARY AND CONCLUSIONS

In the present paper we derived a number conserving boson expansion theory, NCQP + BET, for the Ginocchio SO(8) model. We performed numerical calculations for the SO(5)×SU(2), SO(6), and SO(7) limits of this model. The results, presented in Tables I—III and Figs. 1 and 2, show that the NCQP + BET works consistently very well in all the three cases. It is thus confirmed that this theory provides a valid and sound solution to the number conservation problem in the usual quasiparticle description. This confirmation is particularly important, because the NCQP + BET can be extended to realistic many-j cases in a straightforward manner. As is obvious from our results, the improvement of NCQP + BET over BCS + BET is remarkable, especially for high-lying states. For the low-lying states the improvement is less dramatic, because BCS + BET already gives rather accurate results. The calculations for Ginocchio's model reconfirmed what we already knew, namely that BCS is rather accurate for low-lying states, for which the number of quasiparticles is small compared with the number of particles.

We have also presented the results pertaining to two other boson approaches, given earlier in Ref. 11, namely OAIT, BZM, as well as the results obtained by using OAI, which, to our knowledge, have not been reported anywhere else previously for this model. In our discussion of the OAIT and OAI theories, we concluded that OAIT performs quite well, overall, while OAI performs rather poorly. The exception is the SO(5) case, for which both OAI and OAIT are exact. However, this particular limit is too special a case to give any indication on the performance of a theory in general, as is proven by the results given by these two theories in the SO(6) and SO(7) cases. (See the end of Sec. III.)

The OAIT and OAI results pertaining to Ginocchio's model confirmed our assessment of these theories given in Ref. 4 for the $1j$-SM. In particular, these two theories give rise to sometimes quite different numerical results, and both need higher order terms to achieve an overall good accuracy. (For a more detailed discussion of these and related aspects concerning OAI and OAIT we refer the reader to Ref. 4.)

In conclusion, the results of the present paper, together with those of Ref. 5 for the $1j$-SM, showed that NCQP + BET can provide us with a rather powerful method for the investigation of nuclear collective motions. The work on the extension of NCQP + BET to realistic many-j cases is in progress.

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APPENDIX

We shall give here the boson Hamiltonian and the quadrupole operator used to calculate the results presented under the OAI heading in Tables I—III, and Figs. 1 and 2. The OAI Hamiltonian can be written as

$$H_{\mathrm{OAI}}=h_0+n\epsilon_s+(\epsilon_d-\epsilon_s)\hat{n}_d+\tfrac{1}{2}c_2\hat{n}_d(\hat{n}_d-1)+\tfrac{1}{10}(c_0-c_2)K^\dagger K+\tfrac{1}{2}v_0(K^\dagger s^\dagger s+s^\dagger sK)\ , \tag{A1}$$

where $n=N/2$ is the particle pair number, and h_0, c_0, c_2, and v_0 are constants to be fixed by equating the matrix elements between boson states ($n_d=0$, $I=0$; $n_d=1$, $n_d=2$, $I=0$ and 2) to the corresponding exact elements between fermion states with seniority $v=0$, 2, and 4.

We derived the OAI coefficients in two ways. One was to use the OAIT Hamiltonian [Ref. 11, Eq. (14)] and equate the OAIT boson matrix elements between the boson states specified above to the OAI matrix elements calculated by using (A1). The second method was to use the NCQP fermion Hamiltonian (2.23), which is exact in the Ginocchio model, and follow the standard OAI procedure outlined above.

The methods gave rise, not surprisingly, to the same results. However, the first procedure is helpful in elucidating both the difference and the similarity between the OAI and OAIT bosonization methods. The similarity lies in the technique used to fix the Hamiltonian coefficients, the difference lies in the functional form assumed for the same coefficients. In OAI they are assumed to be constants, while in OAIT they are assumed to be n_d dependent. The OAIT formalism reduces to the OAI formalism, if such dependence is eliminated.

The OAI coefficients thus obtained are given as

$$\begin{aligned}
&\epsilon_s=5G_2,\quad h_0=NG_0(\Omega-n+1)+5A_0(b_2-G_2)\ ,\\
&\epsilon_d=A_1-A_0+\tfrac{1}{10}\Omega B_1-\Omega G_0\ ,\\
&c_2=A_0-2A_1+A_2+\tfrac{1}{5}(\Omega-2)B_2-\tfrac{1}{5}\Omega B_1+2G_0\ ,\\
&c_0=c_2+5\frac{\Omega-2}{\Omega-1}B_2\ ,\\
&v_0=(b_2-G_2)\frac{2}{\Omega-1}\left[\frac{(\Omega+4)(\Omega-N)(\Omega-n-1)}{\Omega-3}\right]^{1/2}\ ,
\end{aligned} \tag{A2}$$

where

$$\begin{aligned}
A_i&=(b_2-G_2)\frac{(n-i)(\Omega-n-i)}{\Omega-2i-1}\ ,\\
B_i&=10\frac{\Omega-2i+3}{\Omega-2i+2}G_2+\frac{10}{(\Omega-2i+1)(\Omega-2i+2)}\\
&\quad\times\left[(n-i+1)(\Omega-n-i+1)+(n-i)(\Omega-n-i)\frac{\Omega-2i+3}{\Omega-2i-1}\right](G_2-b_2)\ .
\end{aligned}$$

The OAI quadrupole operator is given as

$$(Q_{2\mu}^\dagger)_{\mathrm{OAI}}=\sqrt{2/\Omega}[g_1d_\mu^\dagger+g_2d_\mu^\dagger\hat{n}_d+\tfrac{1}{5}(g_0-g_2)K^\dagger d_{\tilde{\mu}}]s+\mathrm{H.c.}\ , \tag{A3}$$

where

$$\begin{aligned}
g_1&=\left[\frac{\Omega-n}{\Omega-1}\right]^{1/2}\ ,\\
g_2&=\left[\frac{\Omega-n-1}{\Omega-3}\right]^{1/2}-g_1\ ,\\
g_0&=\left[\frac{(\Omega-n-1)(\Omega+4)}{(\Omega-1)(\Omega-3)}\right]^{1/2}-g_1\ .
\end{aligned}$$

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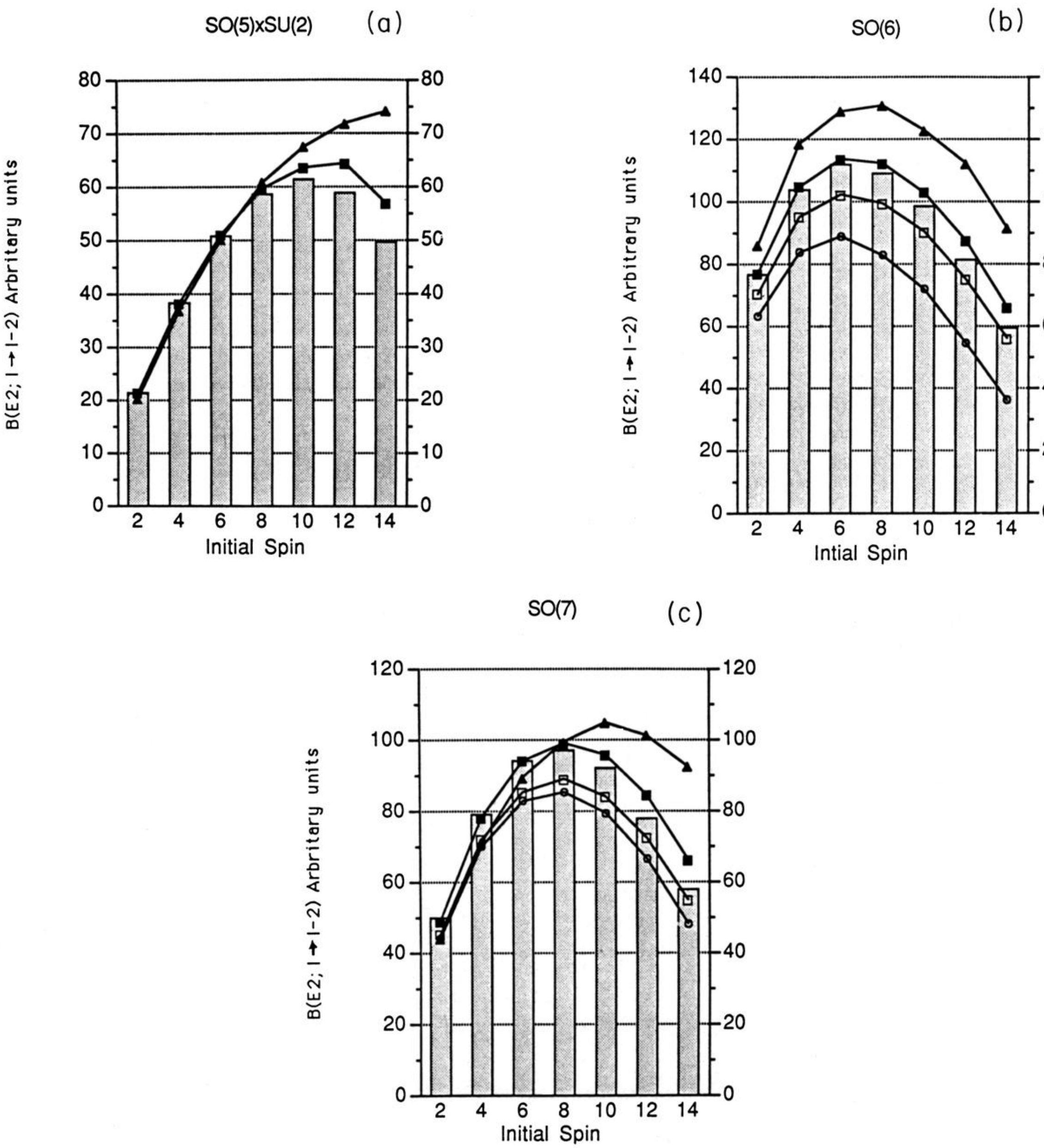

FIG. 2. $B(E2;I\rightarrow I-2)$'s between yrast states [(a) for the SO(5)×SU(2) case, (b) for the SO(6) case, and (c) for the SO(7) case]. Legend: bar=exact fermion values; ■=NCQP + BET; ▲=BCS + BET; □=OAIT; and ○=OAI. In (a) OAI and OAIT results are exact. Effective charge $e_{\rm eff}=\sqrt{2\Omega}$.