Gamow separable approximations for realistic N-N interactions: Single channel case

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A separable approximation for the $^1S_0$, $^1P_1$, $^3P_0$, and $^3P_1$ channels of the N-N Reid soft core and Argonne potentials, based on the use of Gamow states, is presented. The on- and off-shell scattering amplitudes obtained from these separable forms are compared with the corresponding exact ones. Excellent agreement is obtained, which makes the Gamow separable approximation a successful and unambiguous method to construct separable approximations for local realistic N-N interactions.

I. INTRODUCTION

Models for the nucleon systems are usually built from nonrelativistic protons and neutrons interacting through two-body potentials. These potentials are partially phenomenological in nature, since no first principle calculation of the N-N interaction is presently available. The parameters are then adjusted to reproduce the experimental phase shifts for the N-N scattering, as well as deuteron properties. In this sense they can be considered realistic. The Reid[1] and Argonne[2] potentials fall in this category. The Reid is the oldest realistic model which is able, like other realistic potentials, to reproduce the two-particle data, but the results for ^{3}H and ^{4}He as well as for the nuclear matter are not satisfactory. The Argonne potential is the most recently proposed, and is especially simple in form. In particular its velocity dependence is, in contrast to the Paris potential,[3] contained in the conventional terms involving the orbital angular momentum.

All the calculations done with the realistic potentials for few-body systems and nuclear matter are based on variational, perturbative approaches, or exact solutions of the Fadeev equations with few N-N channels. This is due to the complexity of the interaction which therefore prevents an exact treatment. It is then not clear if the poor results obtained are due to deficiencies of the interaction or of the method of calculation. To overcome this ambiguity one can construct a separable representation of the given potential, therefore solving the few-body equations exactly and getting also simple solutions for the nuclear matter problems. Furthermore an exact solution where all the N-N channels are taken into account could help in isolating the contributions coming from other degrees of freedom of quark origin which are present at short N-N distances, and which obviously cannot be described even by a covariant N-N potential.

Many attempts to construct separable approximations have been presented in the literature. For a review see Ref. 4, and Ref. 5 for a more recent discussion.

The most general separable interaction is of the form,

$$V_M=\sum_{n,m}^{M}|g_n\rangle\Delta_{nm}\langle g_m| \ .$$

Different methods to construct this expression correspond to different choices of Δ_{nm} and the basis vectors $|g_n\rangle$. The former can be energy dependent like in the Weinberg[6] procedure or fixed like in the general method of Adhikari and Sloan.[7] The basis vectors can be a general fixed set of states like in Haidenbauer and Plessas,[8] or a set of solutions of the Schrödinger equation for a given potential with well-defined boundary conditions like in the Ernst, Shakin, and Thaler (EST) or Fuller[9] approaches. In the recent work of Bund[9] a variant of the EST approach was used to get a separable expansion of the ^3S N-N interaction. Elements of the basis satisfy a linear boundary condition at a distance which is varied as a free parameter of the method.

In this work we present a separable representation of both Reid and Argonne potentials, which follows the general prescription of Ref. 7 and uses Gamow states to obtain the basis vectors. The details of the method are described in Sec. II. The applications to the chosen N-N interaction are presented in Sec. III. Section IV is devoted to the conclusions and prospects.

II. THE METHOD

Recently we have presented a method[10] to obtain a separable expansion V_M of rank M for a local potential V, namely

$$V_M=\sum_{n,m}^{M}V|f_n\rangle\Delta_{nm}\langle f_m|V \ , \tag{1}$$

where the vectors $|f_n\rangle$ are chosen as the bound and Gamow states associated with the original potential V, and $(\Delta^{-1})_{nm}=\langle f_n|V|f_m\rangle$. Such ansatz guarantees, according to Adhikari and Sloan,[7] that $V_M|f_n\rangle=V|f_n\rangle$ for all $n=1,\ldots,M$, therefore the first M bound and Gamow states of V_M coincide with those of V, and the corresponding S matrix has the first M poles at the correct positions.

This Gamow separable approximation (GSA) was applied to a schematic model and analyzed in detail in Ref. 10: The advantages of the GSA, in relation to other methods, rely on the possibility of constructing an approximate S matrix which satisfies unitarity, reproduces low-lying bound states and resonances, is free of spurious

zero width resonances unlike the EST procedure, independent of energy at variance with the Sturmian expansions,[11] and is simply manageable. Furthermore there are no free parameters involved in the calculation.

The Gamow vectors are the solutions of the Schrödinger equation with purely outgoing wave boundary conditions, and correspond in general to complex energies. For general potentials the computation of these vectors can be done numerically in configuration space[12] integrating the Schrödinger equation from the origin up to some cutoff radius $R_{\max}$. At this radius the Gamow vector is matched to a pure outgoing free wave with complex momentum k ($\mathrm{Im}k<0$). This is equivalent to assuming that the potential V vanishes for r larger than $R_{\max}$, that is, V is approximated by a potential of compact support. This is a reasonable approximation for short-range potentials and sufficiently large $R_{\max}$. In agreement with this fact the cut-off dependence of the GSA is negligible for on-shell as well as off-shell amplitudes. The theoretical aspects of this will be discussed elsewhere. It should be noted that even if the Gamow vectors are exponentially growing at infinity the product $|\psi_n\rangle = V|f_n\rangle$ belongs to L^2 for a compact support potential V.

The T matrix for the potential of Eq. (1) can be written as

$$t_n^{\mathrm{GSA}}(k)=\sum_{n,m}^{M}|\psi_n\rangle D_{nm}(k)\langle\psi_m| \ , \tag{2}$$

where

$$[D^{-1}(k)]_{nm}=\langle f_n|V-VG_0(k)V|f_m\rangle \ ,$$

and G_0 is the free propagator.

The matrix elements of the operator of Eq. (2) can be explicitly written for a given partial wave l,

$$t_l^{\mathrm{GSA}}(k,k';k''^2)=\sum_{n,m}^{M} g_n^l(k)D_{nm}^l(k'')g_m^l(k') \ , \tag{3}$$

with

$$g_n^l(k)=\frac{2\mu}{\hbar^2}\int_0^\infty u_l(k,r)V_l(r)f_n^l(r)r^2dr \tag{4}$$

and

$$(D^{l^{-1}})_{nm}=\int_0^\infty f_n^l(r)V_l(r)f_m^l(r)r^2dr+\left[\frac{2\mu}{\hbar^2}\right]ik''\int_0^\infty r'^2dr'\int_0^\infty r^2dr\, f_n^l(r')V_l(r')u_l(k'',r_<)v_l(k'',r_>)V_l(r)f_m^l(r) \ , \tag{5}$$

where $f_m^l(r)$ and V_l are the radial part of the Gamow vector and the potential for the partial wave l. The functions u_l and v_l are the regular and irregular solutions of the free problem. The quantity μ is the reduced mass of the system.

For $k=k'=k''$ in Eq. (3) we get the on-shell t matrix, which can be related to the phase shifts δ_l by the relation

$$t_l(k,k;k^2)=-\frac{1}{k}e^{i\delta_l}\sin\delta_l \ . \tag{6}$$

Both Eqs. (3) and (6) satisfy automatically the off- and on-shell two-body elastic unitarity.

III. APPLICATION TO THE N-N INTERACTION

An interesting application of the GSA is an approximation to realistic nucleon-nucleon interaction potentials where both an attractive part and a repulsive core are present.

Our first choice for such interaction is the Reid soft core potential[1] (RSC) since it was one of the most widely used and modeled to reproduce the Yale and Livermore phase shifts as well as the low-energy data. This local potential contains central, tensor, and spin orbit components, with channel dependent parameters.

The second choice for the N-N interaction is the recently devised Argonne v_{14} potential (Av_{14})[2], similar in form to the Urbana model.[13] It contains 14 different operator components describing the long-range part (OPE), intermediate range, and short-range contribution to the nucleon-nucleon interaction. It fits the recent data of Arndt and Roper[14] being comparable in the quality of the fit of phase shifts and low energy observables to the Paris potential.[3] It is worth mentioning that the Reid and Argonne potentials have a quite different short distance behavior.

The model described in Sec. II was then applied to these interactions for the uncoupled S and P channels, calculating the phases and off-shell amplitudes. In order to achieve these, Gamow vectors for both interactions in these specific channels were calculated using the method of Ref. 12. The radial equation is integrated numerically using the Fox-Goodwin method[15] with trial initial values for the Gamow energy. Two solutions are calculated. One is regular at the origin and the other satisfies the out-

TABLE I. First five Gamow vector momenta in fm^{-1} for the RSC potential calculated with $R_{\max}=6.5$ fm.

Real k	Im k	Real k	Im k
$^1S_0$		$^1P_1$	
0.0	−0.056	0.5670	−0.574
0.466	−0.720	1.157	−0.726
1.080	−0.809	1.720	−0.777
1.646	−0.854	2.262	−0.808
2.192	−0.885	2.792	−0.831
$^3P_0$		$^3P_1$	
0.810	−0.635	0.621	−0.659
1.356	−0.747	1.175	−0.741
1.910	−0.808	1.707	−0.785
2.457	−0.848	2.225	−0.815
2.999	−0.877	2.739	−0.839

TABLE II. The same as in Table I but for the Av_{14} potential.

Real k	Im k	Real k	Im k
$^1S_0$		$^1P_1$	
0.0	−0.053	0.657	−0.578
0.472	−0.714	1.172	−0.662
1.081	−0.801	1.703	−0.731
1.646	−0.849	2.237	−0.777
2.194	−0.881	2.767	−0.809
$^3P_0$		$^3P_1$	
0.805	−0.633	0.645	−0.646
1.360	−0.741	1.174	−0.720
1.911	−0.797	1.701	−0.779
2.453	−0.832	2.228	−0.820
2.989	−0.856	2.750	−0.851

going wave boundary condition at R_{max}. At some intermediate radius the two solutions are matched and the corresponding logarithmic derivatives compared. From the difference of the two derivatives the next trial energy is computed until convergence is reached according to the specified accuracy. The normalization is properly calculated; however, the separable potential of Eq. (1) does not depend on this normalization. The first antibound and resonance poles in the complex k plane are shown in

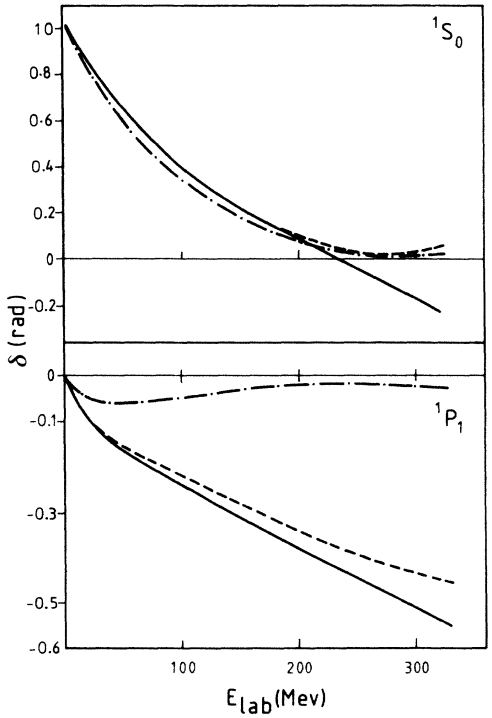

FIG. 1. $T=1$, $S=0$ and $T=0$, $S=0$ phase shifts for different rank approximations to the Av_{14} potential. The full, dashed, and dash-dotted lines correspond to the rank $M=5$, 3, and 1, respectively.

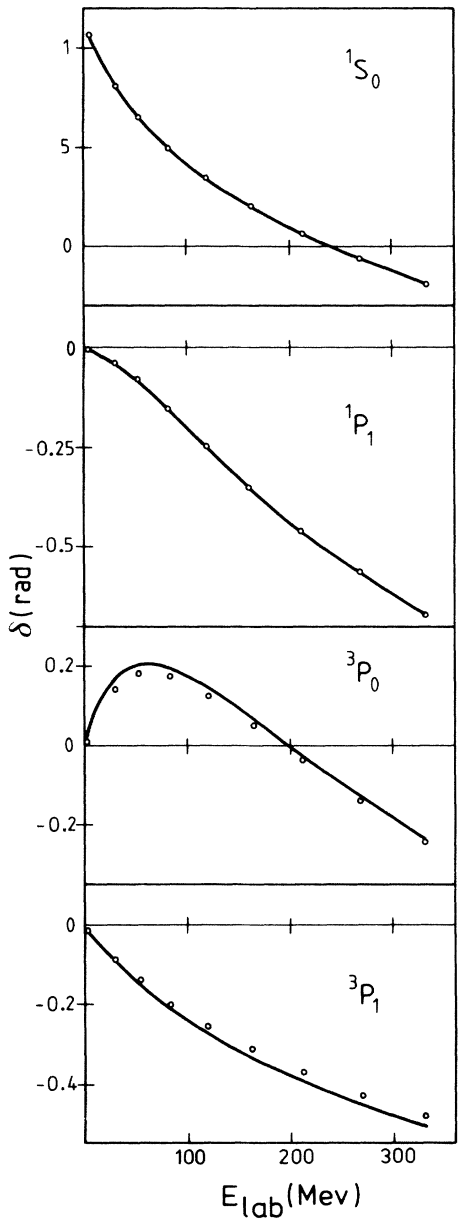

FIG. 2. $T=1$, $S=0$; $T=0$, $S=0$; and $T=1$, $S=1$ phase shifts for the RSC potential. The solid line represents the exact phases. The circles represent the rank $M=5$ approximation.

Tables I and II calculated with $R_{max}=6.5$ fm. It was found that at this cut-off radius the phase shifts for the Reid and Argonne v_{14} potentials have stabilized to one part in a thousand. The Gamow vectors were then used in Eq. (3) in order to obtain the GSA t matrix. For comparison the Lippmann-Schwinger equation was solved in order to obtain the t matrix for the corresponding local potentials, also using the same cut-off radius. Though Gamow vectors can be calculated if the Coulomb interaction is present using the code of Ref. 12, in all the calculations presented here the Coulomb interaction was neglected for simplicity.

A. The GSA phase shifts

A necessary requirement for a separable expansion to a realistic N-N potential should be the capacity to reproduce the corresponding phase shifts up to the pion production threshold. This implies a minimal rank M for the GSA which is able to reproduce such phases. It was found that for the uncoupled S and P wave channels of the RSC and Av_{14} potentials this rank is essentially $M=5$. This fact is illustrated in Fig. 1 for the $^1S_0$ and $^1P_1$ waves of the Av_{14} interaction, where we compare the results for $M=1$, 3, and 5.

For the $^1S_0$ the first Gamow momentum corresponds to the well-known virtual state occurring for the RSC and Av_{14} at -0.129 and -0.115 MeV, respectively. The main structure at low energies of the $^1S_0$ phase shift is already well reproduced by the $M=1$ expansion, therefore described by this state only. A good fit for higher energies, and in particular the change of sign around $E_{lab}=240$ MeV, is only obtained with the inclusion of the other Gamow vectors up to the rank $M=5$. It should be noted however that the t matrix around this energy is very small and probably negligible in the context of few-body calculations.

Similar results are obtained for the other channels $^3P_0$ and $^3P_1$ for both potentials as illustrated in Figs. 2 and 3. In the $^1P_1$ channel the rank $M=4$ already reproduces the correct results.

It should be noted that in contrast to most of the separable approximations available no free parameters are introduced by the GSA. This fact makes it particularly interesting to investigate how well the approximation works for off-shell momenta.

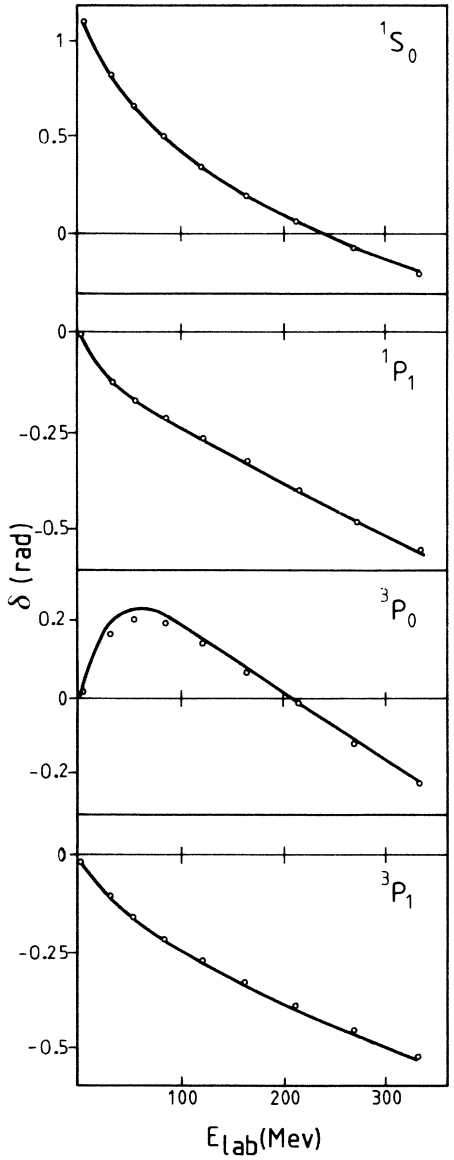

FIG. 3. As in Fig. 2 but for the Av_{14} potential.

B. The GSA in the off-shell regime

Off-shell extensions of the t matrix are of major importance for the description of nuclear structure and in the scattering formalism for few-body systems, such as ^{3}H and ^{3}He and n-d scattering. Hence an approximation of

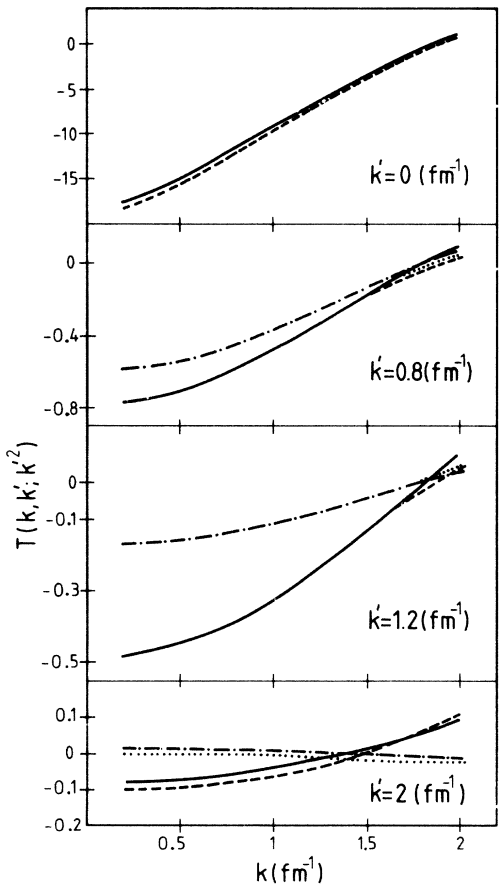

FIG. 4. Real and imaginary part of the half-off-shell scattering amplitude, t, for the ^1S exact and rank $M=5$ approximation to the Av_{14} potential. (——) real t (exact); (— — —) real t (rank $M=5$); (—·—·—·) Im t (exact); (· · ·) Im t (rank $M=5$). The curves for rank $M=5$ are not drawn since they coincide with the exact ones within graphical accuracy.

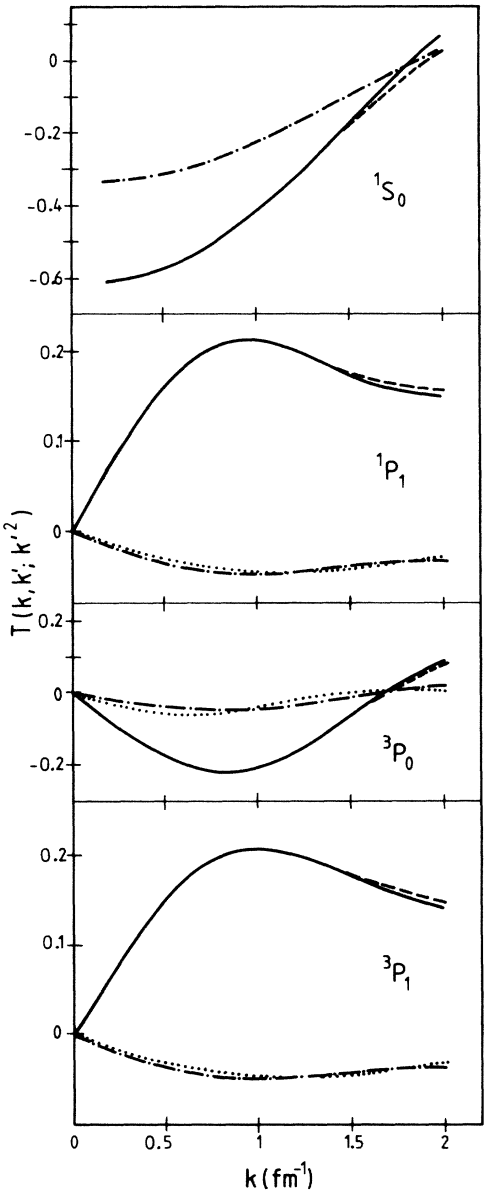

FIG. 5. Like in Fig. 4 with $k'=1$, 0 fm^{-1} but for different channels of the Av_{14} potential.

the two-body interaction is only acceptable if it reproduces well the off-shell t matrix. The GSA calculations of the half-off-shell t matrix, $t_l^{\mathrm{GSA}}(k,k';k'^2)$, according to Eq. (3) are displayed in Figs. 4 and 5 for the different channels and interactions. The real and imaginary parts of the t_l^{GSA} calculated with rank $M=5$ are compared with those obtained directly from the local potentials. Its behavior for momenta $k \lesssim 2$ fm^{-1} seems to govern the results for the three-body system.[4] The present study was therefore restricted to this range of momenta. The overall agreement is quite good in the whole range of momenta considered. Similar to the on-shell case a lower rank approximation is probably acceptable for the purpose of three-body calculations as is indicated by Fig. 6.

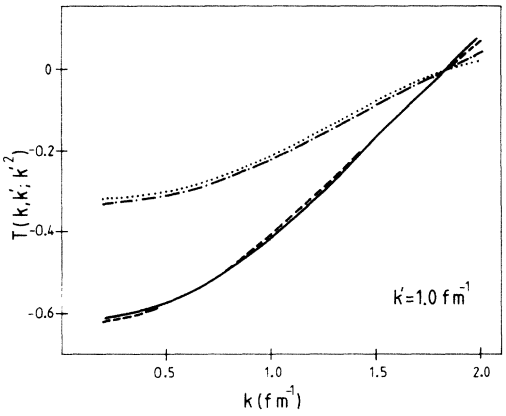

FIG. 6. Different rank approximations to the half-off-shell t matrix in the $^1S_0$ channels of the Av_{14} potential. (——) real t (rank $M=5$); (— — —) real t (rank $M=3$); (—·—·—·) Im t (rank $M=5$); (· · ·) Im t (rank $M=3$).

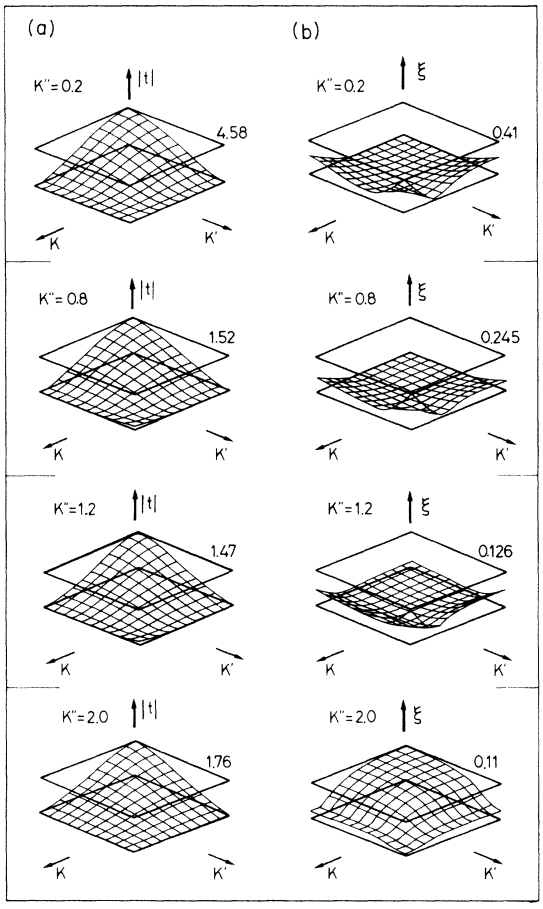

FIG. 7. Off-shell behavior for the $^1S_0$ amplitude of the Av_{14} interaction. (a) Absolute value of the exact t matrix in fm. (b) Absolute deviation defined by Eq. (7); on both axis $0<K<2$ fm^{-1}. The superimposed rectangles correspond to $|t|$ or $\xi=0$ fm and the indicated values.

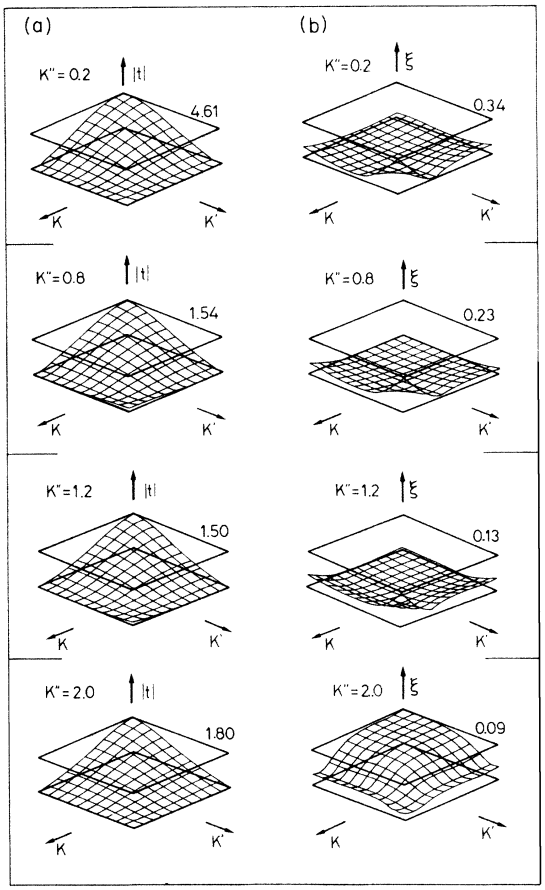

FIG. 8. As in Fig. 7 but for the approximation to the RSC.

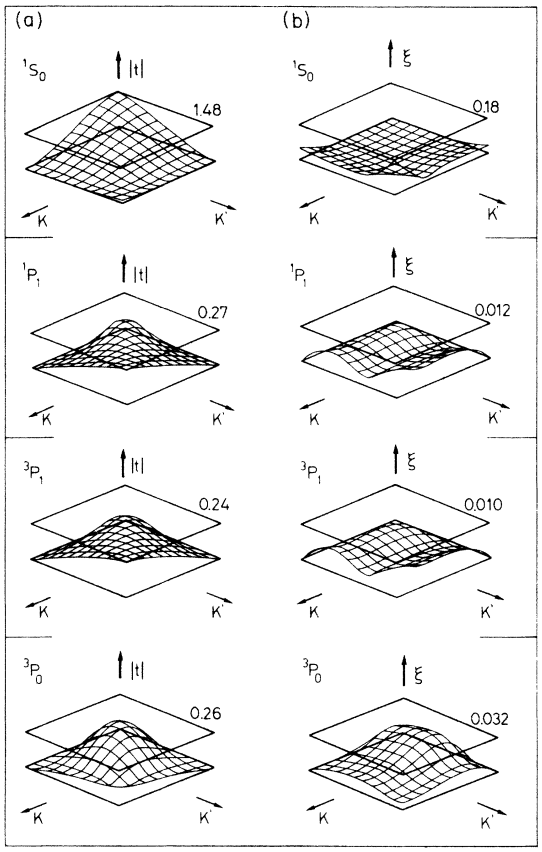

FIG. 9. As in Fig. 7 but for different channels of the Av_{14} interaction with $k''=1$ fm^{-1}.

A three-dimensional overview of the absolute deviation of the full off-shell GSA t matrix from its local potential counterpart, given by

$$\xi_l(k,k';k'') = |\, t_l^{\mathrm{GSA}}(k,k';k''^2) - t_l(k,k';k''^2)\,| \ , \qquad (7)$$

is shown in Figs. 7–9, for different on-shell reference momenta k''. The agreement is good for all the different k'' values of both the RSC and Av_{14} potentials in the various channels.

A strongly varying set of functions, as the on- and off-shell t matrix, was therefore well reproduced without any free parameters.

IV. CONCLUSIONS

For the first time the GSA method was applied to obtain a separable expansion of two realistic N-N potentials, namely the RSC and the Av_{14}.

The analysis has been restricted to the uncoupled S and P wave channels. Excellent results for the phase shifts, as well as for the off-shell scattering matrix, have been obtained with a rank $M=5$ expansion. As a remark it can be pointed out that the remaining deviations could be cured by perturbative approaches. Thus the GSA appears to be a successful and unambiguous method to construct separable approximations for local realistic N-N interactions.

The two chosen potentials have a rather different behavior at short distances, but the method is flexible enough to give equally good approximations to both of them.

All these results give confidence in applying these separable potentials to few-body calculations. In this context a lower rank approximation could be accurate enough, since a lower rank GSA expansion already gives a reasonably good approximation for the on- and off-shell scattering amplitudes, at least in the range of momenta where they are appreciably different from zero. The other N-N channels, in particular the coupled $^3S_1$-$^3D_1$ ones, are under study. We plan to apply the different separable GSA to three-body nuclear systems as well as to nuclear matter.

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