Method of continued fractions for on- and off-shell t matrix of local and nonlocal potentials

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(Received 11 March 1985)

The method of continued fractions recently proposed by the authors is generalized to an off-shell t-matrix calculation for any nonlocal nonsymmetric interaction. The efficiency of the method is demonstrated for some examples in nuclear physics. The method is not only very efficient, but yields very accurate results when it is combined with the Romberg extrapolation method. A new separable approximation of a potential and the off-shell t matrix is proposed in connection with the method of continued fractions.

I. INTRODUCTION

In various scattering problems which appear in nuclear physics, we have to calculate on- or off-shell scattering matrix (t-matrix) elements for local or nonlocal potentials. In particular, for handling three-body problems in momentum space, calculations of off-shell t-matrix elements are required. In the present paper, we propose a very efficient method with a simple algorithm for calculation. The method is an off-shell extension of the method of continued fractions (MCF), proposed by us for atomic physics.[1,2] This method is very efficient for obtaining a convergent value. Also, we can handle the scattering problem for a nonlocal potential with the same simplicity and the same rapidity of convergence as for a local potential. Examples in the present paper show that this property is not lost by the extension to off shell. Thus we expect the method will be very useful for handling nuclear scattering problems such as three-body scattering, a cluster model, or nuclear reactions.

Since Refs. 1 and 2 were aimed at use in atomic physics, here we recapitulate the MCF for the on-shell t matrix and give some examples in nuclear physics. A self-contained description is given in Sec. II. In Sec. III, the method is extended to the off-shell t matrix. In Sec. IV, we propose a new separable expansion of the t matrix. This expansion is closely related to the MCF. The MCF is not only very efficient, but attains a very accurate result if it is combined with the Romberg extrapolation technique, which is discussed in the Appendix.

II. METHOD OF CONTINUED FRACTIONS ON ENERGY SHELL

Let us denote the potential by V and the initial plane wave state by $|k\rangle$, where the momentum k is related to the energy E of the system by $E=(\hbar^2/2m)k^2$. The Lippmann-Schwinger equation that we have to solve is given by

$$|\phi\rangle=|k\rangle+G_0(E)V|\phi\rangle \ . \tag{1}$$

If we define the wave matrix $\Omega(E)$ by

$$|\phi\rangle=\Omega(E)|k\rangle \ , \tag{2}$$

this matrix satisfies the equation

$$\Omega(E)=1+G_0(E)V\Omega(E) \ . \tag{3}$$

As usual, the scattering matrix (t matrix) is defined by

$$t(E)=V\Omega(E) \ . \tag{4}$$

Hereafter, we omit E for simplicity, unless there is fear of confusion. To solve Eq. (3), first we define a potential V_1 by

$$V_1=V-\frac{V|k\rangle\langle k|V}{\langle k|V|k\rangle} \ . \tag{5}$$

We define functions $|k_1\rangle$ and $|\phi_1\rangle$, and the wave matrix Ω_1 by equations similar to Eqs. (1) and (3),

$$|k_1\rangle=G_0V|k\rangle \ , \tag{6}$$

$$|\phi_1\rangle=\Omega_1|k_1\rangle=|k_1\rangle+G_0V_1|\phi_1\rangle \ , \tag{7}$$

$$\Omega_1=1+G_0V_1\Omega_1=\frac{1}{1-G_0V_1} \ . \tag{8}$$

If we put Eq. (5) into Eq. (3), it is expressed in terms of these functions as

$$\Omega=\Omega_1+\Omega_1|k_1\rangle\frac{1}{\langle k|V|k\rangle-\langle k|V|\phi_1\rangle}\langle k|V\Omega_1 \ . \tag{9}$$

If we note that $\langle k|V\Omega_1$ is expressed as

$$\langle k|V\Omega_1=\langle k|V+\langle\phi_1|V_1 \ , \tag{10}$$

where

$$\langle\phi_1|=\langle k_1|\Omega_1 \ , \tag{11}$$

Eq. (9) reads

$$\Omega=\Omega_1+|\phi_1\rangle\frac{1}{\langle k|V|k\rangle-\langle k|V|\phi_1\rangle}(\langle k|V+\langle\phi_1|V_1) \tag{12}$$

We use this equation together with Eq. (5) on the right-hand side of Eq. (4) to obtain

$$t=t_1+\frac{(V\mid k\rangle+V_1\mid\phi_1\rangle)(\langle\phi_1\mid V_1+\langle k\mid V)}{\langle k\mid V\mid k\rangle-\langle k\mid V\mid\phi_1\rangle}\ , \tag{13}$$

where t_1 is defined by

$$t_1=V_1\Omega_1\ . \tag{14}$$

Equation (7) takes the form of Eq. (1), provided that $\mid k\rangle$ and V in Eq. (1) are replaced by $\mid k_1\rangle$ and V_1 in Eq. (7). Therefore, if we define a set of potentials V_i, the wave functions $\mid k_i\rangle$ and $\mid\phi_i\rangle$, and the matrix t_i for $i=0,1,2,\ldots$ by

$$V_{i+1}=V_i-\frac{V_i\mid k_i\rangle\langle k_i\mid V_i}{\langle k_i\mid V_i\mid k_i\rangle}\ , \tag{15}$$

$$\mid k_{i+1}\rangle=G_0V_i\mid k_i\rangle\ , \tag{16}$$

$$\mid\phi_i\rangle=\Omega_i\mid k_i\rangle\ , \tag{17}$$

and

$$t_i=V_i\Omega_i\ , \tag{18}$$

we obtain the following equations:

$$\Omega_i=\Omega_{i+1}+\Omega_{i+1}\mid k_{i+1}\rangle\frac{1}{\langle k_i\mid V_i\mid k_i\rangle-\langle k_i\mid V_i\mid\phi_{i+1}\rangle}\times\langle k_i\mid V_i\Omega_{i+1} \tag{19}$$

and

$$t_i=t_{i+1}+\frac{(V_i\mid k_i\rangle+V_{i+1}\mid\phi_{i+1}\rangle)(\langle\phi_{i+1}\mid V_{i+1}+\langle k_i\mid V_i)}{\langle k_i\mid V_i\mid k_i\rangle-\langle k_i\mid V_i\mid\phi_{i+1}\rangle}. \tag{20}$$

As we see from Eq. (5), V_1 and $\mid k\rangle$ have an orthogonality relationship

$$V_1\mid k\rangle=\langle k\mid V_1=0\ , \tag{21}$$

and, as a result, we have

$$\Omega_1\mid k\rangle=\mid k\rangle\ . \tag{22}$$

This property is a special case of more general properties for $n\leqq i$,

$$V_{i+1}\mid k_n\rangle=\langle k_n\mid V_{i+1}=0\ , \tag{23}$$

and

$$\Omega_{i+1}\mid k_n\rangle=\mid k_n\rangle\ . \tag{24}$$

Operating Ω_i, given by Eq. (19), to the left of the function $\mid k_i\rangle$, and using Eqs. (23) and (24), we obtain an equation for the function $\mid\phi_i\rangle$,

$$\mid\phi_i\rangle\equiv\mid k_i\rangle+\mid\phi_{i+1}\rangle\frac{1}{\langle k_i\mid V_i\mid k_i\rangle-\langle k_i\mid V_i\mid\phi_{i+1}\rangle}\times\langle k_i\mid V_i\mid k_i\rangle\ . \tag{25}$$

The on-shell t-matrix element $\langle k\mid t(E)\mid k\rangle$ is obtained from Eqs. (13) and (21) with the result

$$\langle k\mid t(E)\mid k\rangle=\langle k\mid V\mid k\rangle\times\frac{1}{\langle k\mid V\mid k\rangle-\langle k\mid V\mid\phi_1\rangle}\langle k\mid V\mid k\rangle\ . \tag{26}$$

Therefore, the quantity that we have to calculate is $\langle k\mid V\mid\phi_1\rangle$. We define

$$\tau_i=\langle k_{i-1}\mid V_{i-1}\mid\phi_i\rangle,\ \text{for } i=1,2,\ldots\ . \tag{27}$$

We put Eq. (25) into Eq. (27) using an identity

$$\langle k_{i-1}\mid V_{i-1}\mid\phi_{i+1}\rangle=\langle k_i\mid V_i\mid k_i\rangle\ , \tag{28}$$

which is obtained from Eqs. (16), (23), and (25); we get τ_i in a form of continued fractions,

$$\tau_i=\langle k_{i-1}\mid V_{i-1}\mid k_i\rangle+\frac{\langle k_i\mid V_i\mid k_i\rangle^2}{\langle k_i\mid V_i\mid k_i\rangle-\tau_{i+1}}\ . \tag{29}$$

If we use Eqs. (26)—(28), we can calculate the on-shell t-matrix element. Also, we obtain the wave function from Eqs. (2), (9), (19), and (28). Here, we should note that this method is applicable to local as well as to nonlocal potentials.

As an example, Table I shows the on-shell t-matrix element of the Reid soft core potential[3] for the ^1S state at 12 MeV. We see a quick convergence of this method. To account for the rapidity of the convergence, the relevance of this method to the Neumann series as well as to the Schwinger variational principle was demonstrated in Refs. 1 and 2.

The method is especially useful for treating a nonlocal potential. As an example, we take the n-d elastic scattering, neglecting the breakup effect.[4] Due to the exchange of particles, this three-body problem is equivalent to a two-body scattering from a nonlocal nonsymmetric potential $U(y_1;y_2)$,

TABLE I. The on-shell matrix element $\langle k\mid t(E+i\epsilon)\mid k\rangle$ for the Reid ^1S soft-core potential at $E=12$ MeV. N denotes the number of iterations.

N	Re$\langle k\mid t\mid k\rangle$	Im$\langle k\mid t\mid k\rangle$
1	0.0098	9.6×10^{-5}
2	−0.1755	0.0167
3	−0.9259	1.011
4	−0.9262	1.007
5	−0.9187	1.070
6	−0.9186	1.071

TABLE II. The phase shift for the scattering from the nonlocal, nonsymmetric potential (30). The phase shift for the s wave is related to the on-shell K-matrix element by $\tan\delta=-(1/k)\langle k \mid K(E) \mid k \rangle$, where $\mid k \rangle = \sin kr$.

N \ E_{lab} (MeV)	0.25	1.0	5.0	10.0	50.0	100.0
1	32.02	57.48	91.35	99.87	62.01	20.56
2	32.18	57.76	91.42	99.58	61.95	20.65
3			91.41	99.57	61.99	20.72
Exact	32.1812	57.7597	91.4119	99.5748	62.0006	20.7192

$$U(y_1,y_2)=\frac{8\pi}{3}\int_{(4/3)\mid y_1-y_2/2\mid}^{(4/3)(y_2+y_1/2)} x_1 dx_1 \phi(x_1) V(x_1) \phi\left(\sqrt{\mid x_1^2+(4/3)(y_1^2-y_2^2)\mid}\right), \tag{30}$$

where ϕ denotes the wave function of the deuteron and $V(x)$ the nucleon-nucleon potential. Here we take a regularized potential fitted to $^3S_1$ scattering data,

$$V(x)=\hbar c\left[-p_1 e^{-p_2 x}+p_3 e^{-2p_2 x}+(p_1-p_2)e^{-5p_2 x}\right]/x, \tag{31}$$

with $\hbar c=197$ MeV fm, $p_1=3.1344$, $p_2=1.5502$ fm^{-1}, and $p_3=7.4616$. Table II shows the phase shift calculated from the K matrix, for which PG_0 (P the principal value of Cauchy) is taken in place of G_0 which appeared throughout this section. The convergence is remarkable.

III. EXTENSION TO OFF-ENERGY SHELL

We extend the method of continued fractions written in Sec. II to the off-shell t matrix $\langle p \mid t(E) \mid q \rangle$. In place of Eq. (1), we must solve the off-shell Lippmann-Schwinger equation

$$\mid \phi \rangle = \mid q \rangle + G_0(E) V \mid \phi \rangle . \tag{32}$$

The potential V_{i+1} and the iterated states are defined by

$$V_{i+1}=V_i-\frac{V_i \mid q_i \rangle \langle p_i \mid V_i}{\langle p_i \mid V_i \mid q_i \rangle} , \tag{33}$$

$$\begin{aligned} &\mid q_{i+1} \rangle = G_0(E) V_i \mid q_i \rangle , \\ &\langle p_{i+1} \mid = \langle p_i \mid V_i G_0(E) , \end{aligned} \tag{34}$$

with $\mid p_0 \rangle = \mid p \rangle$, $\langle q_0 \mid = \langle q \mid$. In the same way that led to Eq. (26), we get the off-shell t-matrix element in a form

$$\langle p \mid t(E) \mid q \rangle = \langle p \mid V \mid \phi \rangle = \frac{\langle p \mid V \mid q \rangle^2}{\langle p \mid V \mid q \rangle - \langle p \mid V \mid \phi_1 \rangle} . \tag{35}$$

In analogy to Eq. (29), the off-shell quantity defined by

$$\tau_i = \langle p_{i-1} \mid V_{i-1} \mid \phi_i \rangle \tag{36}$$

is expressed as continued fractions,

$$\tau_i = \langle p_{i-1} \mid V_{i-1} \mid q_i \rangle + \frac{\langle p_i \mid V_i \mid q_i \rangle^2}{\langle p_i \mid V_i \mid q_i \rangle - \tau_{i+1}} . \tag{37}$$

Equations (36) and (37) are used in Eq. (35) to obtain the off-shell t-matrix element.

Tables III and IV show the off-shell t- (K-)matrix element for the Reid soft core potential of the ^3S state and the nonlocal potential given by Eq. (30), respectively, for various negative (positive) energies E, with $p=0.153$ fm^{-1} and $q=0.567$ fm^{-1}. Again, we see very rapid convergence.

In concluding this section, we remark that the wave function may be expressed as

$$\begin{aligned} \mid \phi \rangle = &\mid q \rangle + \Gamma_1(p,q;k) \mid q_1 \rangle \\ &+ \Gamma_1(p,q;k)\Gamma_2(p,q;k) \mid q_2 \rangle + \cdots , \end{aligned} \tag{38}$$

TABLE III. The off-shell t- (K-)matrix element $\langle p \mid t(E) \mid q \rangle$ ($\langle p \mid K(E) \mid q \rangle$) for the Reid soft-core potential for negative (positive) energies, with $p=0.153$ fm^{-1} and $q=0.567$ fm^{-1}. N denotes the number of iterations.

N \ E	−72 MeV	−12 MeV	12 MeV	72 MeV
1	0.006 277	0.004 985	0.007 286	0.003 253
2	−0.104 087	−0.147 002	−0.187 083	−0.190 969
3	−0.932 045	−1.295 101	−1.433 601	−0.710 423
4	−0.981 165	−1.358 811	−1.382 999	−0.725 580
5	−0.980 558	−1.367 944	−1.521 227	−0.726 555
6	−0.982 401	−1.369 785	−1.522 469	−0.727 039
7	−0.982 407	−1.369 794	−1.522 477	−0.727 038
8	−0.982 407	−1.369 794	−1.522 482	−0.727 042
9	−0.982 407	−1.369 797	−1.522 482	−0.727 042

TABLE IV. The off-shell t- (K-)matrix element $\langle p \mid t(E) \mid q \rangle$ ($\langle p \mid K(E) \mid q \rangle$) for the nonlocal, non-symmetric potential (30) for negative (positive) energies E, with $p=0.153$ fm^{-1} and $q=0.567$ fm^{-1}. N denotes the number of iterations.

N \ E (MeV)	−72	−12	12	72
1	−11.36	106.9	−2.438	−5.369
2	−11.36	83.87	−0.4984	−5.331
3	−11.38	83.65	0.3404	−5.331
4	−11.38	83.65	0.3405	−5.331

with

$$\Gamma_{i+1}(p,q;k)=\frac{\langle p_i \mid V_i \mid q_i \rangle}{\langle p_i \mid V_i \mid q_i \rangle-\langle p_i \mid V_i \mid \phi_{i+1} \rangle} . \quad (39)$$

$$t(E)=\sum_{i=0}^{N-1} \frac{(V_i \mid k_i \rangle+V_{i+1} \mid \phi_{i+1} \rangle)(\langle \phi_{i+1} \mid V_{i+1}+\langle k_i \mid V_i)}{\langle k_i \mid V_i \mid k_i \rangle-\langle k_i \mid V_i \mid \phi_{i+1} \rangle}+t_N . \quad (40)$$

Equation (40) has the following properties. For $N=1$, it is equal to Eq. (13). If we use Eqs. (21) and (22), we obtain

$$\langle p \mid t_1 \mid k \rangle=\langle p \mid V_1\Omega_1 \mid k \rangle=\langle p \mid V_1 \mid k \rangle=0 \quad (41)$$

and

$$\langle k \mid t_1 \mid q \rangle=0 . \quad (42)$$

From these, we see in Eq. (13) that firstly the separable term gives the correct result for the half-off-shell t matrices $\langle p \mid t(E) \mid k \rangle$ and $\langle k \mid t(E) \mid q \rangle$, and secondly, since the denominator of Eq. (13) is equal to the denominator of

$$t(E)=\sum_{i=0}^{N-1} \frac{(V_i \mid k_i \rangle+V_{i+1} \mid \phi_{i+1} \rangle)(\langle \phi_{i+1} \mid V_{i+1}+\langle k_i \mid V_i)}{\langle k_i \mid V_i \mid k_i \rangle-\langle k_i \mid V_i \mid \phi_{i+1} \rangle} ,$$

with

$$V_N=0 . \quad (43)$$

Equation (43) is the separable expansion that we propose. We believe that Eq. (43) takes the simplest form among various separable approximations of the off-shell t matrix.[5] By virtue of Eqs. (23) and (24), we have the equations

$$\langle p \mid t_i \mid k \rangle=\langle p \mid V_i\Omega_i \mid k \rangle=\langle p \mid V_i \mid k \rangle=0, \text{ for } i \geq 1 . \quad (44)$$

Therefore, the separable expansion (43) yields the correct half-off-shell behavior. [This means that the expansion (43) is not actually separable at half-off-shell.]

Finally, we remark that the potential V is approximated by a separable form as

IV. SEPARABLE APPROXIMATION TO THE t MATRIX

If we sum up Eqs. (13) and (20) for $i=1,2,\ldots,N-1$, we get the t matrix in a form

the on-shell t matrix, all poles of the t matrix are involved in the separable term of Eq. (13), namely, in the first term on the right-hand side of Eq. (40).

When the contribution from V_N is negligibly small, the continued fractions (29) practically terminate at τ_{N-1}, which reads

$$\tau_{N-1}=\langle k_{N-2} \mid V_{N-2} \mid k_{N-1} \rangle+\frac{\langle k_{N-1} \mid V_{N-1} \mid k_{N-1} \rangle^2}{\langle k_{N-1} \mid V_{N-1} \mid k_{N-1} \rangle-\langle k_{N-1} \mid V_{N-1} \mid k_N \rangle} .$$

Therefore, if V_N is small, Eq. (40) is approximated by a sum of separable terms as

$$V=\frac{V \mid k \rangle\langle k \mid V}{\langle k \mid V \mid k \rangle}+\frac{V_1 \mid k_1 \rangle\langle k_1 \mid V_1}{\langle k_1 \mid V_1 \mid k_1 \rangle}+\cdots+\frac{V_{N-1} \mid k_{N-1} \rangle\langle k_{N-1} \mid V_{N-1}}{\langle k_{N-1} \mid V_{N-1} \mid k_{N-1} \rangle} . \quad (45)$$

This expression is obtained by summing up Eq. (15) for $i=0,1,2,\ldots,N-1$, and setting $V_N=0$.

Table V shows the result of calculations of the off-shell

TABLE V. The off-shell K matrix calculated by the separable approximation (43) of the square well potential given in the text (Ref. 6) with $p=0.153$ fm^{-1} and $q=0.567$ fm^{-1}. N denotes the number of separable terms.

N \ E (MeV)	0	12	24
1	2.949 00	−5.955 95	−2.015 55
2	2.948 36	−5.956 04	−2.031 09
Exact	2.948 35	−5.956 02	−2.031 09

TABLE VI. The off-shell K matrix calculated by the separable approximation (43) of the square well potential given in the text (Ref. 6) with $p=0.567$ fm^{-1} and $q=1.405$ fm^{-1}. N denotes the number of separable terms.

N \ E (MeV)	0	12	24
1	1.081 05	−2.642 26	−0.841 970
2	1.050 12	−2.645 28	−0.819 085
3	1.050 04	−2.645 28	−0.819 116
Exact	1.050 04	−2.645 27	−0.819 113

K-matrix element $\langle p \mid K(E) \mid q \rangle$, with the separable expansion (43) for the K matrix. We used a square well potential fitted to low energy triplet N-N scattering data; $V=-V_0$, $r<R$; $V=0$, $r>R$; $V_0=31.28$ MeV, $R=2.205$ fm.[6] Here we chose the values $p=0.153$ fm^{-1}, $q=0.567$ fm^{-1} for a few different values of E. The first term of the expansion, which is equal to the Kowalski-Noyes approximation,[7] is already a very good approximation to the exact $\langle p \mid t(E) \mid q \rangle$. The addition of the second separable term brings the result almost to the exact value. The two-term separable expansion yields the results which are correct up to five or six significant figures. This remains true, even if we move into a more off-shell region. Table VI shows the result for $p=0.567$ fm^{-1} and $q=1.405$ fm^{-1}.

With an increasing number of separable terms, and for more singular potentials like the Reid potential, roundoff errors start to appear, which result in poor convergence. Nevertheless, the first two or three terms approximate the t matrix with high accuracy.

V. CONCLUSION

The method of continued fractions for calculations of off-shell (and, of course, on-shell) t-matrix elements for local and nonlocal potentials as well as the related separable expansion yields very rapid convergence with simple algorithms compared with other methods. If the method of continued fractions is combined with the Romberg extrapolation technique, very high accuracy is easily attained. This is demonstrated in the Appendix.

APPENDIX: EXTRAPOLATION TECHNIQUE

In the text, we have demonstrated that we can calculate the off-shell t- (or K-)matrix elements very quickly by the method of continued fractions. All matrix elements $\langle p_i \mid V_i \mid q_i \rangle$ and $\langle p_i \mid V_i \mid q_{i+1} \rangle$ are integrals involving the Green's function G_0 (or PG_0). This Green's function is continuous, but its derivative is discontinuous at $r=r'$. Numerical experiences show that the best method for treating such an integral is the simple trapezoidal rule,[8]

$$\int_a^b f(x)dx=\frac{h}{2}[f(a)+\sum_{i=1}^{M-1} f(a+h)+f(b)]+F_M(f)\ , \tag{A1}$$

where M denotes the number of mesh points. The error term $E_M(f)$ is expressed as[9]

$$E_M(f)=\frac{\alpha_2}{M^2}+\frac{\alpha_4}{M^4}+\frac{\alpha_6}{M^6}+\cdots\ , \tag{A2}$$

where the constants α_{2i} depend on the function $f(x)$. Since our expression for the t matrix Eq. (35) with Eq. (37) and the truncation by $\mid q_{N+1} \rangle$ for $\mid \phi_{N+1} \rangle$ at a suitable number N,

$$\tau_{N+1}=\langle p_N \mid V_N \mid q_{N+1} \rangle\ , \tag{A3}$$

is a finite rational function of such integrals, the same error term must be applied to the truncated t matrix. We denote by $\langle p \mid t(E) \mid q \rangle_M$ an off-shell t-matrix element for which M mesh-point calculations are performed for all integrals $\langle p_i \mid V_i \mid q_i \rangle$ and $\langle p_i \mid V_i \mid q_{i+1} \rangle$. Then, the t-matrix element $\langle p \mid t(E) \mid q \rangle$ is expressed as

$$\langle p \mid t(E) \mid q \rangle=\langle p \mid t(E) \mid q \rangle_M+\frac{a_2}{M^2}+\frac{a_4}{M^4}+\cdots\ . \tag{A4}$$

Since calculations of many such off-shell t matrices are needed, for instance, in a three-body problem in momentum space, we are forced to save time in calculating each off-shell t-matrix element. On the other hand, we are required to perform very accurate calculations of these matrix elements. These contradicting requirements are satisfied if we use the Romberg extrapolation technique.[9] The idea of this technique is very simple. We calculate the t matrix twice: First, the calculation is performed for M mesh points, and then the same calculation is performed but with $2M$ mesh points. With these two calculations, we get

TABLE VII. The off-shell K matrix element of the exponential, Yukawa, and Reid potentials, for $p=0.153$ fm^{-1}, $q=0.567$ fm^{-1}, and $E=1.736$ fm^{-2} (72 MeV). M denotes the maximal number of mesh points used in each calculation. The values are obtained by using Eq. (A4) for $M=29$, and Eq. (A5) for $M=57$, 113, 225, and 449.

M	Exponential	Yukawa	Reid
29	−1.419 127 07	−1.245 452 31	−0.739 180 697
57	−1.411 246 78	−1.235 308 11	−0.726 925 287
113	−1.411 269 80	−1.235 390 64	−0.727 042 226
225	−1.411 269 76	−1.235 390 43	−0.727 041 766
449	−1.411 269 76	−1.235 390 43	−0.727 041 766

$$\langle p \mid t(E) \mid q \rangle = \tfrac{1}{3}[4\langle p \mid t(E) \mid q \rangle_{2M} - \langle p \mid t(E) \mid q \rangle_{M}] + \frac{b_4}{M^4} + \frac{b_6}{M^6} + \cdots . \quad \text{(A5)}$$

In Eq. (A5), the leading term of the error behaves as $0(M^{-4})$, contrary to $0(M^{-2})$ in Eq. (A4). If we repeat this procedure, we obtain very precise results with a small number of calculations.

The efficiency of this technique is manifested in Table VII, where the K-matrix element $\langle p \mid K(E) \mid q \rangle$ is calculated for the exponential,[10] Yukawa,[11] and Reid soft core potentials for $E=12$ MeV, $p=0.153$ fm^{-1}, and $q=0.567$ fm^{-1}. We see that 113 mesh points are sufficient to get results which are accurate up to eight (seven, six) digits for the exponential (Yukawa, Reid soft-core) potential. The difference between these results are due to the fact that the exponential (Yukawa, Reid soft-core) potential is a monotonic (monotonic, nonmonotonic) function which is nonsingular (singular, singular) at the origin.

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