Test of the reduced width amplitude distribution from proton resonance studies

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(Received 1 April 1985)

Measurements of width and amplitude correlations allow sensitive tests of the amplitude distribution. Experimental results show an apparent discrepancy from the expected multivariate Gaussian distribution. A test which shows that the disagreement cannot be explained solely by the limited sample sizes is presented.

I. INTRODUCTION

The use of low-energy, high-resolution beams to study compound nuclear resonances permits study of statistical properties such as the distributions of energy levels and of widths (see, e.g., Ref. 1). The statistical model of nuclear spectra[2] assumes that the reduced width amplitudes, $\gamma_{\lambda c}$, for the decay of resonances λ into channel c have a Gaussian distribution centered at zero; this assumption leads to the Porter-Thomas distribution[3] for reduced widths $\gamma_{\lambda c}^2$. Reduced widths for elastic scattering are well reproduced by the Porter-Thomas distribution for both neutron[4] and proton[5] resonances.

More sensitive tests of the statistical model are possible if one studies the amplitude distribution.[6] By studying inelastic proton scattering from resonances, Dittrich *et al.*[7] have shown that both the magnitudes and relative signs of the decay amplitudes can be determined. Study of a group of resonances then produces not only a value for the width correlation $\rho(\gamma_c^2,\gamma_d^2)$ but also a value for the amplitude correlation $\rho(\gamma_c,\gamma_d)$. Here, ρ is the linear correlation coefficient

$$\rho(x,y)=\sum_i (x_i-\bar{x})(y_i-\bar{y}) \Bigg/ \left[\sum_i (x_i-\bar{x})^2 \sum_i (y_i-\bar{y})^2\right]^{1/2} . \tag{1}$$

For the amplitude correlations $\bar{\gamma}_c=\bar{\gamma}_d=0$ is assumed.

A general treatment of the joint distribution of amplitudes in different channels[8] predicts that the reduced width amplitudes should have a multivariate Gaussian distribution. For a single channel this distribution reduces to a Gaussian. Any deviation from Gaussian behavior would be most interesting, since the derivation of Krieger and Porter[8] assumes only the rotational invariance of the Hamiltonian and the statistical independence of levels. Because the absolute phases of the amplitudes cannot be determined, a direct test of the distribution is not possible. However, if the amplitudes are indeed multivariate Gaussian with first moments all zero, then[8]

$$\rho^2(\gamma_c,\gamma_d)=\rho(\gamma_c^2,\gamma_d^2) . \tag{2}$$

Thus, a comparison of width and amplitude correlations provides a test of the joint probability distribution.

Data sufficient to determine correlations are available for four different nuclei (^{45}Sc, ^{49}V, ^{51}Mn, and ^{57}Co) and include resonances with $J^\pi=\frac{3}{2}^-$, $\frac{3}{2}^+$, and $\frac{5}{2}^+$.[6,9] For $\frac{3}{2}^-$ resonances, only two inelastic channels are considered, and each set of data provides one pair of correlations; for $\frac{3}{2}^+$ and $\frac{5}{2}^+$ resonances, the analysis includes three inelastic channels, and there are three different pairings available.

Comparisons between experimental values, denoted by $r(x,y)$, of width and amplitude correlations show that Eq. (2) appears to be violated in several cases;[6,9] however, interpretation of these results is complicated by the need to consider experimental errors in the reduced widths and reduced width amplitudes. Because the different inelastic channels are, in general, correlated, the description of errors for each resonance requires an $N\times N$ error matrix, where N is the number of inelastic channels. Since the expression for the correlation coefficient in Eq. (1) does not incorporate correlated errors, other methods for dealing with them must be considered. Hofmann *et al.*[10] used Monte Carlo techniques to simulate the effects of errors; they concluded that the inclusion of errors is indeed important and offers at least a partial explanation for deviations from Eq. (2), although some of their assumptions are not valid for these data. In the one case they consider in detail (the $\frac{5}{2}^+$ resonances in ^{45}Sc), the significance level of the discrepancy for one pair of amplitudes was decreased from 99% when errors were not included in the analysis to 90—95 % when they were considered.

More recently, Harney[11,12] examined the "finite-range-of-data" (FRD) error inherent in cases involving a finite number of resonances. He concluded that the FRD error is so large as to preclude a test of Eq. (2) with any of the present sets of data, but that the combination of all data[12] yields results which can be reproduced by the multivariate Gaussian distribution. The present paper presents a different evaluation of the FRD error, following the spirit of Harney but choosing a different statistic to test Eq. (2). This statistic produces vastly different results than does Harney's and implies that the apparent deviations from a Gaussian distribution cannot be explained solely as reflecting the limited sample sizes.

Section II examines briefly the formalism of error propagation and discusses the basic difference between the current method of choosing a statistic and Harney's. Section III presents tests of the Porter-Thomas distribution using both methods, while Sec. IV examines the question of the amplitude distribution. A summary of the tests, methods, and results is given in Sec. V.

II. THE FINITE-RANGE-OF-DATA ERROR

The question to be examined in this paper is as follows: If the amplitudes are multivariate Gaussian, how much variation from Eq. (2) should one expect to find as a consequence of the finite number of resonances studied? It should be noted that this error is the minimum value to be anticipated in such a test, since no experimental errors are included. Nonetheless, it is of considerable importance and can in some cases (see, e.g., Sec. IV) be of such a magnitude as to make sensitive comparisons impossible.

The necessary equations for evaluating the FRD error have been given by Harney[11] and will only briefly be reviewed here. If $v(y_1,y_2,\ldots,y_n)$ is a function of statistical variables $\{y_i\}$, then for small fluctuations of $\{y_i\}$,

$$v(y)\approx v(\bar{y})+\sum_{i=1}^{n}R_i(y_i-\bar{y_i}) , \tag{3}$$

where

$$R_i\equiv \left.\frac{\partial v}{\partial y_i}\right|_{\{y_i\}=\{\bar{y_i}\}} . \tag{4}$$

It is also convenient to define

$$(K_y)_{ij}\equiv\overline{(y_i-\bar{y_i})(y_j-\bar{y_j})} . \tag{5}$$

In all cases the overbar denotes an ensemble average. The variance of v is then given by

$$\sigma_v^2=\overline{v^2(y)}-\overline{v(y)}^{\,2}=RK_yR^T , \tag{6}$$

where R^T is the column vector with elements R_i. The procedure for evaluating the FRD error for any statistic is thus as follows: Specify the $\{y_i\}$, evaluate R and K_y from Eqs. (4) and (5), and then evaluate the variance from Eq. (6).

Of course, from the experimental data one cannot determine the ensemble averages; therefore, they must be replaced by the finite averages $\langle y_i\rangle$ obtained from the measurement. The FRD error represents the precision with which one can expect a relation [e.g., Eq. (2)] to hold when expressed in terms of the estimates $\langle y_i\rangle$.

Given an equality $a=b$ there are two methods one can choose to define a simple statistic (with known expectation value) to test this relation. The ratio a/b has expectation value 1, while the difference $a-b$ has expectation value 0. In Secs. III and IV I will examine the two methods for two different cases.

TABLE I. Tests of the Porter-Thomas distribution using both ratio and difference methods. Data are inelastic proton reduced widths (Refs. 6 and 9). Errors quoted in v and z are one standard deviation.

Compound nucleus	J^π	N	Channel l',s'	Ratio v	Difference z keV^2
^{45}Sc	$\frac{3}{2}^-$	37	$1,\frac{3}{2}$	0.79±0.27	−0.04±0.05
			$1,\frac{5}{2}$	0.55±0.27	−0.25±0.15
	$\frac{5}{2}^+$	53	$0,\frac{5}{2}$	0.62±0.22	−0.08±0.05
			$2,\frac{3}{2}$	1.08±0.22	0.03±0.09
			$2,\frac{5}{2}$	0.72±0.22	−0.05±0.04
^{49}V	$\frac{3}{2}^-$	70	$1,\frac{3}{2}$	0.87±0.20	−0.08±0.13
			$1,\frac{5}{2}$	1.04±0.20	0.06±0.30
	$\frac{3}{2}^+$	30	$0,\frac{3}{2}$	0.86±0.30	−0.03±0.07
			$2,\frac{3}{2}$	0.64±0.30	−0.11±0.09
			$2,\frac{5}{2}$	0.86±0.30	−0.15±0.34
	$\frac{5}{2}^+$	45	$0,\frac{5}{2}$	0.61±0.24	−0.14±0.09
			$2,\frac{3}{2}$	0.64±0.24	−0.03±0.02
			$2,\frac{5}{2}$	0.65±0.24	−0.05±0.03
^{51}Mn	$\frac{3}{2}^-$	24	$1,\frac{3}{2}$	0.78±0.33	−0.09±0.14
			$1,\frac{5}{2}$	0.89±0.33	−0.06±0.19
	$\frac{5}{2}^+$	38	$0,\frac{5}{2}$	1.03±0.26	0.03±0.21
			$2,\frac{3}{2}$	1.28±0.26	0.35±0.34
			$2,\frac{5}{2}$	0.87±0.26	−0.11±0.22
^{57}Co	$\frac{5}{2}^+$	77	$0,\frac{5}{2}$	1.16±0.19	0.12±0.14
			$2,\frac{3}{2}$	1.13±0.19	0.14±0.19
			$2,\frac{5}{2}$	1.66±0.19	0.83±0.24

III. TESTS OF THE PORTER-THOMAS DISTRIBUTION

Consider first the Porter-Thomas distribution:[3] The variables $\{\gamma_c\}$ are expected to be Gaussian. It is a general property of the Gaussian distribution that

$$\overline{\gamma_c^{2k}}=(2k-1)!!\overline{\gamma_c^2}^{\,k} , \tag{7}$$

and, in particular,

$$\overline{\gamma_c^4}=3\overline{\gamma_c^2}^{\,2} . \tag{8}$$

Harney[11] defines $y_1\equiv\langle\gamma_c^4\rangle$ and $y_2\equiv\langle\gamma_c^2\rangle$, constructs the ratio

$$v(y_1,y_2)\equiv y_1/3y_2^2 , \tag{9}$$

and shows that the FRD error is

$$\sigma_v^2=8/3N , \tag{10}$$

where N is the number of resonances. The experimental values of v and σ_v are given in Table I for the inelastic proton widths of Refs. 6 and 9.

For the difference method, define

$$z(y_1,y_2)\equiv y_1-3y_2^2 . \tag{11}$$

Then

$$K_y=\frac{2\overline{\gamma_c^2}^{\,2}}{N}\begin{bmatrix}48\overline{\gamma_c^2}^{\,2} & 6\overline{\gamma_c^2}\\ 6\overline{\gamma_c^2} & 1\end{bmatrix} ,$$

$$R=(1,-6\overline{\gamma_c^2}) , \tag{12}$$

$$\sigma_z^2=24\overline{\gamma_c^2}^{\,4}/N .$$

Experimental values of z and σ_z are also included in Table I.

Comparison of the two methods reveals several interesting features. The ratio is of course dimensionless, whereas the difference is not. Also, in this case σ_z^2 depends on an experimental parameter $(\overline{\gamma_c^2})$, whereas σ_v^2 does not. While these two factors certainly make the ratio more aesthetically pleasing, the content of the two methods appears to be the same: For every case the ratio of the FRD error to the deviation from the expected value is essentially independent of the choice of statistic. The one case where the Porter-Thomas distribution appears to disagree badly with the data ($l'=0$, $s'=\frac{5}{2}$ in ^{57}Co) shows equally poor results for the two methods. For this case the conclusion is that the two statistics provide the same information.

IV. TESTS OF THE AMPLITUDE DISTRIBUTION

Next, consider the amplitude distribution and, in particular, tests of Eq. (2). Following Harney, define

$$\begin{aligned} y_1&\equiv\langle\gamma_c\gamma_d\rangle ,\\ y_2&\equiv\langle\gamma_c^2\gamma_d^2\rangle ,\\ y_3&\equiv\langle\gamma_c^2\rangle ,\\ y_4&\equiv\langle\gamma_d^2\rangle ,\\ y_5&\equiv\langle\gamma_c^4\rangle ,\\ y_6&\equiv\langle\gamma_d^4\rangle . \end{aligned} \tag{13}$$

To test Eq. (2) using a ratio, Harney utilizes the statistic

$$v\equiv r^2(\gamma_c,\gamma_d)/r(\gamma_c^2,\gamma_d^2)=[y_1^2(y_5-y_3^2)^{1/2}(y_6-y_4^2)^{1/2}]/[y_3y_4(y_2-y_3y_4)] \tag{14}$$

and shows that the FRD error is

$$\sigma_v^2=\frac{1}{N}g^2(\rho) , \tag{15}$$

with

$$g^2(\rho)\equiv\rho^{-4}+4\rho^{-2}-8+3\rho^4 . \tag{16}$$

Here ρ is the linear correlation coefficient between γ_c and γ_d (evaluated with ensemble averages).

To use a difference method, define

$$z\equiv r^2(\gamma_c,\gamma_d)-r(\gamma_c^2,\gamma_d^2)=y_1^2/y_3y_4-[y_2-y_3y_4]/[(y_5-y_3^2)(y_6-y_4^2)]^{1/2} , \tag{17}$$

where the $\{y_i\}$ are again those from Eq. (13). Elements of the K_y matrix are given in Ref. 11; if one takes

$$C\equiv\overline{\gamma_c^2},\ D\equiv\overline{\gamma_d^2} ,$$

and

$$K\equiv\overline{\gamma_c\gamma_d} ,$$

then the gradient vector is

$$R=(2K/CD,-1/2CD,(CD-3K^2)/2C^2D,(CD-3K^2)/2CD^2,K^2/4C^3D,K^2/4CD^3) , \tag{18}$$

and

$$\sigma_z^2 = \frac{1}{N} f^2(\rho) , \tag{19}$$

where

$$f^2(\rho) \equiv 3\rho^8 - 8\rho^4 + 4\rho^2 + 1 . \tag{20}$$

The functions $g(\rho)$ and $f(\rho)$ are compared in Fig. 1; note the singularity of $g(\rho)$ at $\rho = 0$ which causes the FRD error to be very large for small values of ρ. It is this fact which led Harney to conclude that an effective test of Eq. (2) was not possible with the present data.[11] However, no such singularity exists for the difference method, and σ_z^2 is a well-defined quantity having a maximum value of $1.51/N$ at $\rho \approx 0.51$.

The effects of the different methods are exhibited in Table II, where the statistics v and z are tabulated for the available data. There is ambiguity in the experimental estimate for ρ, since one could use either $r(\gamma_c, \gamma_d)$ or $r(\gamma_c^2, \gamma_d^2)^{1/2}$; here the average of the two values has been used [setting $r(\gamma_c^2, \gamma_d^2) = 0$ in the one case where it is negative experimentally]. For the ratio many of the standard deviations are quite large due to the singularity at $\rho = 0$. This is not so for the differences. The statistic v has four values which differ by more than two standard deviations from the expected value of 1; no values differ by as much as three standard deviations. The difference statistic, on the other hand, has five values (out of 18) which differ by three or more standard deviations from the value 0. Clearly, the two statistics do show major differences in this case.

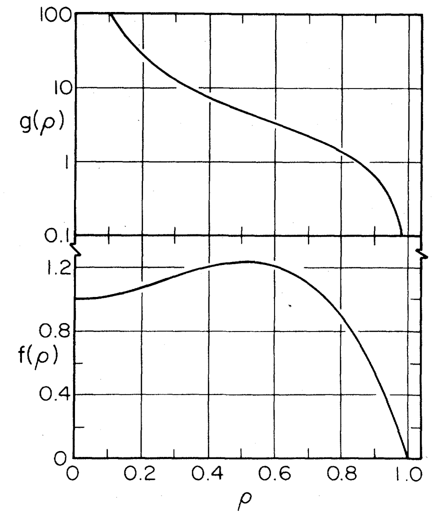

FIG. 1. The functions $g(\rho)$ and $f(\rho)$ obtained in evaluating the FRD error for the amplitude distribution for ratio and difference methods, respectively. Note the logarithmic scale for $g(\rho)$.

A further test of Eq. (2) (Ref. 12) can be performed by calculating weighted means of all 18 values from Table II for each statistic. Using the inverse variance as the weighting factor gives

TABLE II. Tests of the multivariate Gaussian distribution using both ratio and difference methods. Data are taken from Refs. 6 and 9. The value of ρ used to evaluate σ_v and σ_z is the average of $r(\gamma_c, \gamma_d)$ and $r(\gamma_c^2, \gamma_d^2)^{1/2}$.

Compound nucleus	J^π	N	Channel c l', s'	Channel d l', s'	$r(\gamma_c, \gamma_d)$	$r(\gamma_c^2, \gamma_d^2)$	Ratio v	Difference z
^{45}Sc	$\frac{3}{2}^-$	37	$1, \frac{3}{2}$	$1, \frac{5}{2}$	−0.66	0.23	1.89±0.62	0.21±0.20
	$\frac{5}{2}^+$	53	$0, \frac{5}{2}$	$2, \frac{3}{2}$	0.22	0.67	0.07±0.62	−0.62±0.17
			$0, \frac{5}{2}$	$2, \frac{5}{2}$	0.72	0.27	1.92±0.43	0.25±0.16
			$2, \frac{3}{2}$	$2, \frac{5}{2}$	0.06	−0.08[a]	−0.04±153	0.08±0.14
^{49}V	$\frac{3}{2}^-$	70	$1, \frac{3}{2}$	$1, \frac{5}{2}$	−0.09	0.51	0.02±0.90	−0.50±0.14
	$\frac{3}{2}^+$	30	$0, \frac{3}{2}$	$2, \frac{3}{2}$	0.84	0.43	1.64±0.33	0.28±0.19
			$0, \frac{3}{2}$	$2, \frac{5}{2}$	−0.51	0.85	0.31±0.38	−0.60±0.20
			$2, \frac{3}{2}$	$2, \frac{5}{2}$	−0.65	0.15	2.82±0.83	0.27±0.22
	$\frac{5}{2}^+$	45	$0, \frac{5}{2}$	$2, \frac{3}{2}$	−0.06	0.45	0.01±1.29	−0.44±0.18
			$0, \frac{5}{2}$	$2, \frac{5}{2}$	0.88	0.71	1.09±0.14	0.06±0.10
			$2, \frac{3}{2}$	$2, \frac{5}{2}$	0.01	0.28	0.00±2.29	−0.28±0.17
^{51}Mn	$\frac{3}{2}^-$	24	$1, \frac{3}{2}$	$1, \frac{5}{2}$	0.34	0.04	2.89±3.13	0.08±0.23
	$\frac{5}{2}^+$	38	$0, \frac{5}{2}$	$2, \frac{3}{2}$	0.62	0.90	0.43±0.25	−0.51±0.15
			$0, \frac{5}{2}$	$2, \frac{5}{2}$	0.58	0.56	0.60±0.43	−0.21±0.19
			$2, \frac{3}{2}$	$2, \frac{5}{2}$	0.55	0.33	0.92±0.63	−0.03±0.20
^{57}Co	$\frac{5}{2}^+$	77	$0, \frac{5}{2}$	$2, \frac{3}{2}$	0.38	0.79	0.18±0.34	−0.65±0.13
			$0, \frac{5}{2}$	$2, \frac{5}{2}$	0.62	0.51	0.75±0.29	−0.13±0.13
			$2, \frac{3}{2}$	$2, \frac{5}{2}$	0.38	0.27	0.53±0.69	−0.13±0.14

[a] This value has been assumed to be zero for the evaluation of errors.

$$V \equiv \bar{v} = \left[\sum_i v_i / \sigma_{v_i}^2 \right] \Big/ \left[\sum_i 1/\sigma_{v_i}^2 \right], \tag{21}$$

$$\sigma_V^2 = \left[\sum_i 1/\sigma_{v_i}^2 \right]^{-1}. \tag{22}$$

The result is $V=0.91\pm0.09$. This result differs slightly from Harney's because the errors have been evaluated with a different correlation, the weighting factors are different, and all correlations have been treated as independent. Nonetheless, the result is essentially the same, and one concludes that overall the data are in agreement with the value $V=1$ expected from a multivariate Gaussian distribution.

For the difference statistic, the weighted mean is $Z=-0.18\pm0.04$, clearly in disagreement with the expected value $Z=0$. The two methods will be compared further in Sec. V.

V. CONCLUSIONS

For the Porter-Thomas distribution, an approach employing a difference to test equality gives basically the same result as one employing a ratio. In light of this, it appears that the two methods in general are equally good and that the choice of one or the other should be dictated by the specific problem. For the Porter-Thomas case the ratio is probably preferable since, as discussed earlier, one then deals with a dimensionless quantity and the FRD error is independent of any experimental measurement. However, for the amplitude distribution neither of these arguments holds: Both methods yield a dimensionless quantity, and each requires an experimental estimate of ρ to evaluate the FRD error.

The major difference between these two different ways of evaluating the amplitude data then becomes the singularity in $g(\rho)$. The form of $g(\rho)$ practically guarantees that a significant deviation from Eq. (2) cannot be observed except for large values of ρ. The singularity also ensures that measurements with small values of ρ are given relatively little weight when computing the weighted mean. Utilizing a difference removes the singularity at $\rho=0$ and leads to a more equal weighting of all experiments. However, this brings into question the validity of the assumption of Gaussian errors which is inherent in Eqs. (21) and (22); Monte Carlo calculations[10,11] have suggested, for example, that in some cases (particularly those with small ρ) the ratio v has an asymmetric distribution. If this is so, then Eq. (22) will underestimate the variance of the weighted mean. Presumably the difference statistic also shows asymmetric distributions, and in this case the problem may be more serious, since the measurements with smaller ρ are given relatively greater weight. However, it seems unlikely that this effect is sufficient to account for the discrepancy of more than four standard deviations.

One must remember that the FRD error discussed in this paper is only the minimum error in a quantity such as z, since experimental errors in the widths and amplitudes used to determine z are not included. Therefore, it does not seem reasonable to say that this analysis disproves the Gaussian assumption. I only wish to assert that it does not appear that the observed deviations can be the result of the limited sample sizes.

Finally, I wish to point out one other method that can be used to consider the effects of errors and limited sample sizes in these measurements, namely the bootstrap method of Efron.[13] This approach provides a fairly direct incorporation of the effect of a finite sample size. It also has the advantage that no assumption is made about the underlying distributions (unlike the analytical results of Sec. II). With this method, confidence intervals for any desired percentage can be generated. The 99% intervals for the difference statistic are included in Ref. 6 for all data discussed here except those for ^{57}Co; those intervals are $[-0.81,-0.17]$, $[-0.49,+0.16]$, and $[-0.44,+0.35]$ (corresponding to the values -0.65, -0.13, and -0.13, respectively, from Table II). With this test, three (of 18) sets of data appear to violate the Gaussian distribution at the 99% level of confidence. More detailed studies on the effects of experimental error on the determination of the correlation coefficients appear to be necessary to resolve the Gaussian question.

ACKNOWLEDGMENTS

I wish to thank G. E. Mitchell and H. L. Harney for helpful discussions. This work was supported in part by USDOE Contract No. DE-AC02-76ER03074.

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