Separable potentials from Gamow states

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A separable approximation to a two-body interaction is constructed from Gamow states. It was found possible to reproduce on- and off-shell quantities of a solvable model with good accuracy and without the occurrence of spurious zero with resonances. No free parameters were introduced to achieve these results.

I. INTRODUCTION

The description of realistic two-body interactions in few-body calculations is a difficult task to achieve. One usually tries to approximate them by a separable form which ideally should reproduce well both the on-shell and off-shell behavior of the scattering amplitudes.

Many attempts in this direction have been pursued with a view to either directly obtain a good fit to phase shifts and other observables[1] or to approximate given nonseparable interactions.[2–4] For the latter purpose the Ernst-Shakin-Thaler (EST) method has been used extensively.[4,5] However, this method, besides being ambiguous as a prescription, also introduces quite often spurious results, as can be seen from the discussion in Ref. 6.

In this work we discuss an ansatz for the separable approximation based on the use of Gamow states.[7] This procedure not only appears to be suitable for reproducing the phase shifts well, it also gives a good description of the off-shell T matrix and is free of spurious anomalies such as poles on the real energy axis which are responsible for "oscillations" or jumps in the phase shifts as functions of energy.

II. The Method

Gamow states were first introduced[7] to describe alpha decay. They are defined as solutions of the Schrödinger equation with purely outgoing wave boundary conditions. These states correspond in general to complex energies, and are solutions of the equation[8]

$$[1-G_0(k)V]\,|\psi\rangle=0\,, \tag{1}$$

where $G_0(k)$ is the analytic continuation in the complex k plane of the free propagator. With each Gamow state ψ, a resonance vector $|\varphi\rangle=V|\psi\rangle$ can be associated, obviously satisfying

$$[1-VG_0(k)]\,|\varphi\rangle=0\,. \tag{2}$$

While the Gamow states grow exponentially for large r, one sees from (2) that the corresponding resonance vectors belong to L^2, provided V is of compact support, or more generally that it decreases exponentially; see Refs. 8 and 9 for the specific case of the nucleon-nucleon interaction. For negative energies Eq. (1) is solved by the bound states, while for complex momenta, with negative imaginary part, we consider the general vector to describe resonant phenomena. The momenta associated with the bound and Gamow states mark the poles of the S matrix in the complex k plane.

Our method consists in constructing a separable rank N potential whose first N bound or Gamow states coincide with the corresponding N states of the original potential. This procedure reproduces quite well the phase shifts for all energies, provided N is large enough.

According to Adhikari and Sloan,[2] the simplest way to build up a separable expansion V_N of rank N for a potential V, such that the action of V_N and V on a set of linearly independent suitably chosen functions $|f_m\rangle$ is the same, namely, $V_N|f_m\rangle=V|f_m\rangle$ for all $m=1,\ldots,N$, has the explicit form,

$$V_N=\sum_{n,m} V|f_n\rangle\Delta_{nm}\langle f_m|V\,, \tag{3}$$

with $(\Delta^{-1})_{nm}=\langle f_n|V|f_m\rangle$. The functions $|f_m\rangle$ are then chosen as the bound and Gamow states, respectively, associated with the potential V. From this ansatz the separable potential of Eq. (3) shares these states and their energies with the potential V, and the corresponding S matrix has the first N poles at the correct positions. In fact, the T matrix for the potential (3) can be written[2] as

$$t_N(k)=\sum_{n,m}|\varphi_n\rangle D_{nm}(k)\langle\varphi_m|\,, \tag{4}$$

where

$$(D^{-1}(k))_{nm}=\langle f_n|(V-VG_0(k))V|f_m\rangle\,.$$

Note that we obtain Hermitian V_N without involving "anti-Gamow vectors," that is, vectors with negative real part of k, in contrast to the approach of Fuller.[10] Actually one verifies that the potential V_N produces not only the first Gamow vector but also automatically the anti-Gamow state with k interchanged by $-k^*$. Calculations support the idea that better fits are obtained by including a larger number of Gamow vectors instead of following Fuller's procedure.

III. RESULTS AND DISCUSSION

A. Model calculation of the phase shifts

In order to apply the method described above, a model for the potential V has to be selected. We have chosen an attractive spherical rectangular well and a barrier of the form

$$\begin{aligned} V(r) &= -V_1 \quad r < a \\ &= V_2 \quad a \le r \le b \\ &= 0 \quad r > b \ , \end{aligned} \tag{5}$$

which allows for an easy comparison with exact results.

The s-wave Gamow states for this potential are obtained looking for the solution of the Schrödinger equation for $r > b$ with purely outgoing boundary conditions. This imposes a restriction on the complex momentum k, according to the equation

$$\left[\cosh x - \chi \frac{\sinh x}{ik}\right] \frac{\tanh y}{k_1} + \frac{\sinh x}{\chi} - \frac{\cosh x}{ik} = 0 \ , \tag{6}$$

with $x = (b-a)\chi$, $y = ak_1$, $\chi = (V_2 - k^2)^{1/2}$, and $k_1 = (V_1 + k^2)^{1/2}$.

The solution of Eq. (6) was obtained by making use of the Newton method for a complex variable. In Table I the first few Gamow momenta for the quoted potential parameters are reported. Figure 1 shows different rank approximations compared with the exact result. The rank three, i.e., the approximation which includes the first three Gamow states quoted in Table I, already reproduces quite well the phase shifts up to $ka \approx 5.5$. For larger k, the phase shifts become negligibly small and it is not necessary to include further Gamow vectors in the separable expansion. The position of the corresponding Gamow momenta in the complex plane is displayed in Fig. 2 for the exact and rank $N=3$ calculations.

We can also see an interesting phenomenon which occurs going from rank one to rank two. The asymptotic value of the phase shift jumps from $-\pi$ to zero. In view of Levinson's theorem, this means that the rank-one approximation introduces a spurious bound state in spite of providing a good description of the resonance behavior around $ka \approx 1.51$. This can easily be understood since the first Gamow state $|\psi_1\rangle$ is well inside the attractive well, so that the matrix element $\langle \psi_1 | V | \psi_1 \rangle$ is quite large and negative, playing the role of an effective coupling strength for the separable potential. Therefore, the rank-one approximation is overbinding with respect to the original local potential. This deficiency is cured by the rank-two approximation, i.e., when the second Gamow vector is also included. Similar behavior is expected to occur whenever an attractive potential produces a sharp resonance at low energies.

The rank-two approximation already gives a reasonable description of the phase shift, but it is necessary to also include the third Gamow vector to get a fit, especially in the resonance region at $ka \sim 1.51$. It should be noticed that both the second and the third Gamow state do not correspond to real resonances. This implies that resonance poles can be relevant in the description of the S matrix even if they do not lie close to the real axis. We have also verified that the Gamow separable approximation works as well for a pure repulsive potential.

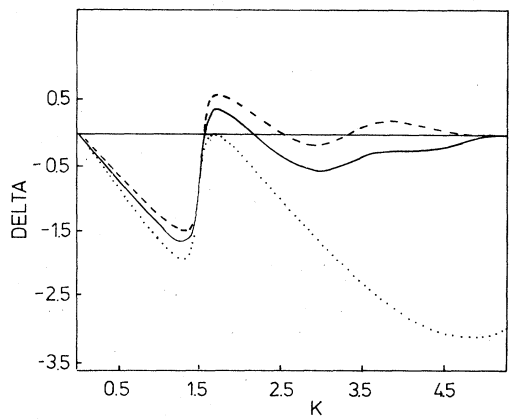

FIG. 1. Phase shifts in radians for different rank approximations for the spherical well potential described in the text, with parameters $\hbar^2/2m = 1$, $V_1 = 2.25$, $V_2 = 4$, $a = 1$, $b = 2$. The full line corresponds to the exact phase shifts, the dotted line to the rank one approximation, and the dashed line to rank two. The rank $N=3$ curve coincides with the exact one within graphical accuracy. Note that for fm as the unit of length $V_1 = 46.0$ MeV.

TABLE I. First seven Gamow vector momenta for the model potential described in the text with parameters given by $V_1 = 2.25$, $V_2 = 4$. $a = 1$, $b = 2$.

Rek	Imk
1.51	−0.06
3.30	−0.54
4.68	−0.69
6.31	−0.95
7.82	−0.98
9.42	−1.15
10.97	−1.16

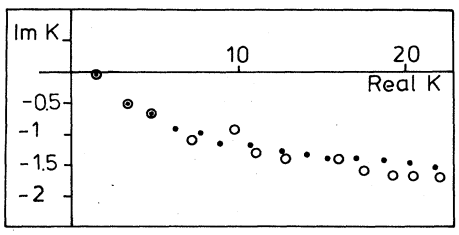

FIG. 2. Complex plane representation of Gamow momenta for exact and rank $N=3$ calculations. The dots represent the exact solution and the circles the rank $N=3$ case.

B. The problem of zero-wave resonances in the EST method

It is instructive to compare our procedure with the usual EST method which is simply a special case of the Adhikari and Sloan ansatz (3) with scattering states calculated at certain energies, called the reference energies, for f_n. It was found that, if one chooses the real part of the Gamow state energies as the selected energies at which the reaction matrix has to be reproduced according to the EST method, the phase shifts still compare very well with the exact one; see Fig. 3. However, if in the EST procedure one does not properly choose the reference energies, the resulting phase shift can behave quite differently from the exact ones. To demonstrate this point we display in Fig. 4 a conventional EST calculation with rank three. While Gamow momenta or even their real parts only would lead to a perfect fit of the phase shift, a small displacement of the reference momenta to slightly larger real values leads to quite inacceptable results (except, of course, for the immediate neighborhoods of the three reference momenta). This indicates that the quality of the approximation profits dramatically from the introduction of Gamow vectors and shows that the conventional EST procedure is quite unstable with respect to variations in the reference energies. Moreover, the conventional EST procedure is beset with unphysical zero-width resonances. Recently, Haidenbauer and Plessas[6] discussed the application of the EST method performed by Pieper to obtain a finite rank approximation of the Reid soft core potential. They found that the phase shifts produced by Pieper's potential can be totally wrong due to the appearance of unphysical jumps in the phase shifts. This is attributed to the presence of unphysical poles very near the real energy axis of the matrix D_{nm} appearing in Eq. (4) and was previously misinterpreted as a spurious small oscillation.

To our knowledge no simple prescription has been presented in the literature to cure this anomaly. The only attempt seems to be the one of Oryu[3] who suggested a method for a separable expansion that is able to shift the unphysical poles to higher energies by increasing the rank of the separable potential. However, it cannot be guaranteed that these spurious poles are removed except in the infinite rank limit. The separable expansions of the type discussed here guarantee for a wide class of potentials that the finite rank expansion cannot produce poles on the real axis for large energies. Asymptotically all the poles of the S matrix move away from the real k axis, since by a simple application of the Riemann-Lebesgue lemma $D^{-1}(k)$ of Eq. (4) approaches the constant matrix Δ of Eq. (3), i.e., the Born approximation is correct.

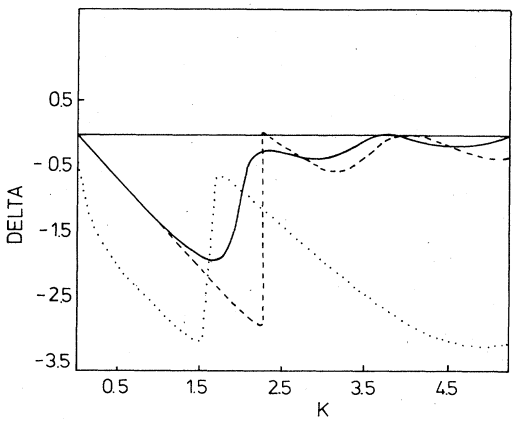

FIG. 3. Phase shifts in radians for rank-one potentials. The dashed line corresponds to the EST method with reference momentum chosen at the real part of the resonance pole. The full line is the Gamow separable approximation constructed from the second Gamow vector. The dotted line is again EST with reference momentum $K=1.58$.

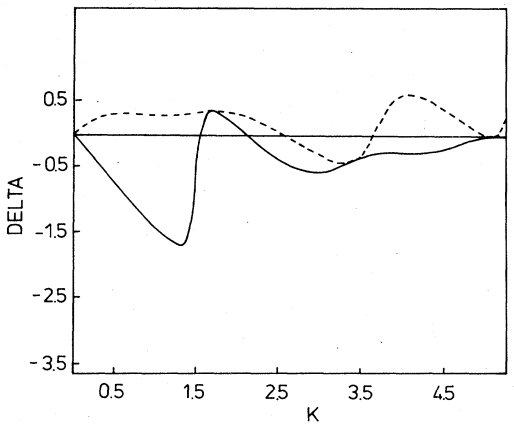

FIG. 4. Phase shifts in radians. The full and dashed lines correspond, respectively, to the exact phase shift and to the EST method with reference momenta $k=1.7$, 3.4, 5.1. Note that for fm as the unit of length these K values are obviously in fm^{-1}.

An example of a spurious "wiggle" appearing in the EST method is shown in Fig. 4 for a rank-one approximation. This "wiggle" is due to the presence of a node in the Fourier-Bessel transform of the resonance wave function $\varphi=V\psi$. In fact, the denominator in Eq. (4) can be written in this case as

$$D^{-1}(k)=\langle\psi\,|\,V\,|\,\psi\rangle-\langle\psi\,|\,VG_0(k)V\,|\,\psi\rangle$$

and

$$\langle\psi\,|\,VG_0(k)V\,|\,\psi\rangle=\mathrm{P}\int d^3p\frac{|\,g(p)\,|^2}{k^2-p^2}-i\pi\,|\,g(k)\,|^2\,, \quad (7)$$

where for simplicity, $\hbar^2/2m=1$, and $g(k)$ is the Fourier-Bessel transform of the resonance wave function $\varphi(r)=\langle r\,|\,V\,|\,\psi\rangle$. The on-shell t matrix is given by

$$t(p)=|\,g(k)\,|^2D^{-1}(k)\,; \quad (8)$$

thus if the real part of $D^{-1}(k)$ vanishes very near a node of $g(k)$, an extremely sharp peak in $t(k)$ can be produced. All our calculations indicate that the use of Gamow states prevents such phenomena; see, for example, Fig. 1. For example, fitting the scattering state at the real part of the second Gamow energy the corresponding rank-one potential given by the EST produces a wiggle at a slightly lower energy, but the wiggle is automatically removed by using a rank-one approximation with the Gamow state at the true complex energy, as shown in Fig. 3.

Unfortunately we were not able to find a rigorous proof that the method avoids in general these anomalies at low energy for all rank approximations. We can argue, however, on physical grounds. In fact, according to our method if the lowest energy resonances are generated at energies with a proper finite imaginary part, the higher energy resonances are expected to have a width at least as large, consequently moving away from the real axis even further. This argument is essentially based on semiclassical considerations. Moreover, one can show that asymptotically for large energies all the poles of a finite rank approximation have an increasing imaginary part. Let us suppose for simplicity the potential to be of compact support and consider the rank-one approximation.

$D^{-1}(k)$ can be written as

$$D^{-1}(k)=\Delta^{-1}-\int d^3x \int d^3y\, \psi^*(x)V(x) \times \frac{e^{ik|x-y|}}{4\pi|x-y|}V(y)\psi(y) . \quad (9)$$

For momenta k in the lower half plane the integral will not decrease as $|k|\to\infty$, hence $D(k)$ may have zeros there, corresponding to resonance poles of the t matrix. More detailed information is obtained if $V(x)$ is of compact support with $V\sim V_0(R-r)^\sigma$, $\sigma>0$ as r approaches R from below, and zero for $r>R$. The leading contribution to D for large $|k|$ comes from the region of largest $|x-y|$, i.e., from $|x|, |y| \lesssim R$ and a straightforward application of Watson's lemma[11] yields

$$D(k)=\Delta^{-1}-A(\sigma)\frac{e^{2ikR}}{(ik)^{2\sigma+3}}(1+0(|k|^{-1})) , \quad (10)$$

with

$$A(\sigma)=2\pi\Gamma(2\sigma+3)B(\sigma+1,\sigma)RV_0^2|\psi(R)|^2/(\sigma+1) ,$$

B and Γ being the usual beta and gamma functions. The derivation makes use of the fact that Gamow vectors do not have nodes,[8] i.e., the behavior of the integrand near R is given by that of V.

The imaginary and real parts of this asymptotic expression yield the asymptotic distributions of resonance poles $k=k_r-i\gamma$:

$$k_r=\frac{\pi}{2R}(n+\sigma+\tfrac{3}{2})+0\left[\frac{\ln n}{n}\right] ,$$

$$\gamma=\frac{1}{2R}\left\{(2\sigma+3)\ln(n+\sigma+\tfrac{3}{2}) +\ln\left[\frac{\Delta^{-1}}{A(\sigma)}\left[\frac{\pi}{2R}\right]^{2\sigma+3}\right]\right\}+0(n^{-1}) , \quad (11)$$

where n is even (odd) according to the positive (negative) sign of Δ.

This infinite sequence of resonances is distributed much like the resonances of the local potential V which we are approximating and for which it is known[12] that

$$k_r=\frac{\pi}{2R}n+0(1) ,$$
$$\gamma=\frac{\sigma+2}{2R}\ln n+0(1) . \quad (12)$$

These (weaker) estimates are in agreement with ours except for the coefficient $\sigma+2$ instead of $2\sigma+3$ for the imaginary part of the Gamow momenta. It is amusing to note that they would agree for $\sigma=-1$. This is the scaling dimension for the δ-shell potential and indeed describes

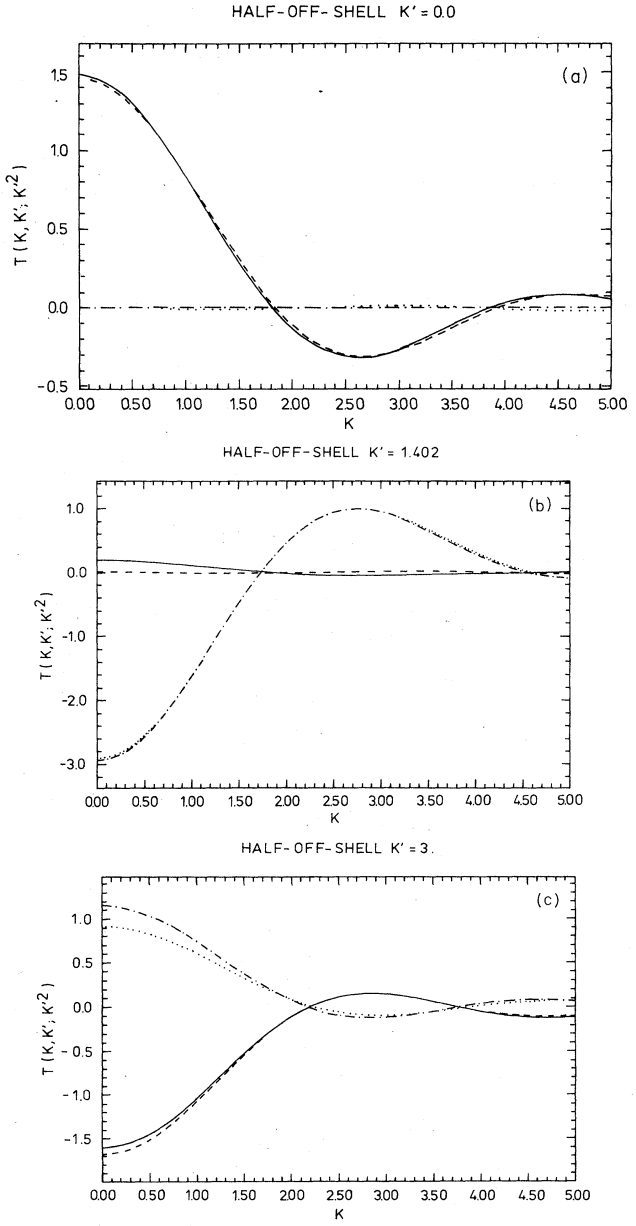

FIG. 5. (a) Half-shell exact and rank $n=3$. The T matrix $T(k,k';k'^2)$ is shown as a function of k for $k'=0$. ——, ReT (exact); — — —, ReT (rank $n=3$); —·—·—·, ImT (exact); · · ·, ImT (rank $n=3$). (b) The same as in (a) for $k'=1.402$. (c) The same as in (a) for $k'=3$.

the asymptotic distribution of the resonance poles for this interaction which is at the same time local and separable.

To discuss the case of rank $N>1$ we must find the asymptotic distribution of zeros for the determinant of the matrix D, given by Eq. (10) where now Δ and A are $N\times N$ matrices. The matrix A_{ij} is defined as in Eq. (11), with $|\psi(R)|^2$ replaced by $\psi_i^*(R)\psi_j(k)$, $i,j=1,\ldots,N$. Note that A is a matrix of rank one, its range the multiples of the vector $(\psi_i(R))_{i=1,\ldots,N}$.

As a result we find

$$\det D \sim \det\Delta^{-1} - \text{const}\frac{e^{2ikR}}{(ik)^{2\sigma+3}}\ , \tag{13}$$

which is of the same form as Eq. (10) of the rank one case. We can see from Fig. 2 that the asymptotic limit is smoothly approached at values of ReK of the order of 20.

C. Model calculation of the half-off-shell T matrix

Off-the-energy extensions are of major importance in the description of nuclear structure and scattering quantities in light nuclei, such as the ^{2}H, n-^{2}H, ^{3}He, and ^{3}H systems. Hence an approximation for the two-body interaction is only suitable if it reproduces well the half-off-shell t matrix.[5] In order to verify this point the half-off-shell t matrix $t(k,k';k'^2)$ was evaluated for the exact solution of the interaction model and for the separable approximation. The results are displayed in Figs. 5(a)—(c) for different on-shell values.

The agreement between the exact and approximate real and imaginary parts of the t matrices is excellent, up to an energy roughly four times the height of the barrier V_2. It should be noted that a strongly varying set of functions is reproduced well without any free parameters.

IV. CONCLUSIONS

From the above results and discussion one may conclude that Gamow state vectors can be successfully used to construct separable approximations of local potentials. The method presented here provides an approximate S matrix which satisfies unitarity, is simply manageable, free of spurious zero width resonances, and displays the correct on- and off-shell behavior. Furthermore, the prescription to obtain it is unambiguous; it contains no free parameters.

Summarizing, the comparison of the exact solution and the Gamow separable approximation for our model interaction appears rather promising.

Equally good results were obtained in preliminary calculations for "realistic potentials." A full account of these results will be presented in a forthcoming publication.[9]

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