Semiclassical methods and the summation of the scattering partial wave series

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The Hilb formula, an asymptotic continuous approximation in l for the Legendre polynomials that is valid from forward angles to nearly 180°, is tested numerically in semiclassical summations of scattering amplitudes. It works remarkably well even when the amplitude oscillates and falls by many orders of magnitude so long as the S matrix varies slowly with l. We generalize the Hilb formula to the associated Legendre functions.

Semiclassical methods are enjoying a great resurgence in physics. However, most widely used semiclassical approximations are only simply realized in one dimension and are hence not directly applicable to scattering problems. For the one dimensional or partial wave scattering problem, most methods, be they Feynman path integral[1] or stationary phase applied to transformation brackets,[2] lead to the traditional WKB answer for the scattering amplitude. In some applications, for example, heavy ion scattering, the partial wave sum for the full scattering amplitude is then summed numerically using these WKB amplitudes.[1] In other applications, for example, most studies of rainbow scattering,[3] an asymptotic expression for the Legendre polynomials is used with the WKB amplitudes to extract a particular scattering phenomenon. In this paper we emphasize the availability of a bridge between those approaches in terms of an integral form for the partial wave sum that is valid for all angles between 0° and nearly 180°. This form reproduces with remarkable accuracy cross sections evaluated by partial wave sum, even when the cross section oscillates and falls by many orders of magnitude over the angular range.

The formula is based on the Hilb form,[4] an asymptotic formula for the Legendre polynomials that, although known in the context of semiclassical scattering methods, particularly in atomic physics,[5] does not seem to be widely appreciated or used in nuclear physics.[6] The Hilb form for the Legendre polynomials is related to the better known asymptotic expression,[7] but has the great advantage of being valid for small angles. It also has an easy extension to the associated Legendre functions.[8]

The Hilb approximation for the Legendre polynomials of order l carries corrections of order $1/l^{3/2}$, but we find that when used to sum (as an integral) the partial wave series the corrections are exponentially smaller than that, so long as the scattering amplitudes are smooth in l. It is this remarkable ability of the Hilb approximation to provide an accurate closed form expression for the scattering amplitude in the semiclassical limit, that is the major message of this Brief Report.

An integral form for the full scattering amplitude not only provides an alternate way to do the partial wave sum but also gives a powerful starting point for approximate analytic methods and insights, for example, use of the method of stationary phase. The Hilb formula also gives promise of wide generalization of the data-to-data formulas[9] we have obtained in the context of the eikonal approximation. Many of these applications would not require an explicit analytic form for the partial wave scattering amplitudes, although in applications such as heavy ion physics, the WKB provides just such a form.

The semiclassical approximation assumes that contributions to the cross section come mainly from large angular momenta so that l can be treated as a continuous variable. It also assumes that the amplitudes are relatively smooth functions of l. Thus the derivation of the analytic form, which is the goal of this paper, requires a continuous representation of the S matrix and the Legendre polynomials, allowing the partial wave sum for the scattering amplitude

$$f(\theta)=\sum_l (2l+1)\frac{S_l-1}{2ik}P_l(\cos\theta) \ , \tag{1}$$

to be expressed as an integral. Here k is the wave number and S_l is related to the phase shift for the lth partial wave by $S_l=\exp(2i\delta_l)$.

We consider first the Legendre polynomials, taken as a special case of the associated Legendre functions. For $l >> |m|$ and $l >> 1$ the Legendre functions have asymptotic forms given by[10]

$$P_l^m(\cos\theta)\cong\left[\frac{2}{\pi\sin\theta}\right]^{1/2}\frac{\Gamma(l+m+1)}{\Gamma(l+\frac{3}{2})}\times\cos\left[(l+\tfrac{1}{2})\theta+\frac{m\pi}{2}-\frac{\pi}{4}\right] \ , \tag{2}$$

for

$$\epsilon\leqslant\theta\leqslant\pi-\epsilon, \quad l>>\frac{1}{\epsilon} \ ,$$

and

$$P_l^{-m}(\cos\theta)\cong\left[(l+\tfrac{1}{2})\cos\frac{\theta}{2}\right]^{-m}J_m\left((2l+1)\sin\frac{\theta}{2}\right) \ , \tag{3}$$

for

$$m\geqslant 0, \quad l\theta<<1 \ .$$

We note that the cosine function which appears in Eq. (2) is part of the asymptotic expansion of the Bessel function of argument $(l+1/2)\theta$ and order $(-m)$. Exploiting this connection to Bessel functions, using large l approximations for the gamma functions and the small angle approximation in Eq. (3) we can combine the two forms to obtain

$$P_l^m(\cos\theta)\cong(l+\tfrac{1}{2})^m\left[\frac{\theta}{\sin\theta}\right]^{1/2}J_{-m}[(l+\tfrac{1}{2})\theta] \ , \tag{4}$$

which is valid for all m and all angles except when $\pi-\theta$

$\leq 1/l$. The asymptotic form Eq. (4), for the special case $m=0$ was first derived by Hilb,[4] who showed that corrections to it are of order $(1/l)^{3/2}$. In fact, it is remarkably accurate for all $l>1$. For arbitrary m, Eq. (4) is given in Szego.[8]

The scattering amplitude can now be written as an integral over impact parameter b, using the Hilb formula and making the replacement $(l+1/2)\rightarrow kb$,

$$f(\theta)\cong -ik\left[\frac{\theta}{\sin\theta}\right]^{1/2}\int_0^\infty b\,db\,J_0(kb\theta)[S(kb)-1] \ . \qquad (5)$$

This is the formula we test for the semiclassical scattering amplitude.

Tests of Eq. (5) require some continuous approximation for the S matrix. As a first example we choose an eikonal form for $S(b)$, an approximation frequently used in medium energy nuclear physics,

$$S(kb)=\exp\left[\frac{-im}{k}\int_{-\infty}^{\infty} V[(b^2+z^2)^{1/2}]dz\right] , \qquad (6)$$

with $k=3.777\ \mathrm{fm}^{-1}$ (corresponding to a 300 MeV proton incident on an $A=150$ target). For V we take a Wood-Saxon form $V=V_0(1+e^{(r-c)/\beta})^{-1}$ with $c=6$ fm, $\beta=0.5$ fm, and a complex potential strength $V_0=-(20+40i)$ MeV. The grazing angular momentum is about $l=22$. We emphasize that we are not implying that the eikonal form (6) is valid in physical situations to all angles, but rather that it is a convenient analytic form with which to check the relative performance of the partial wave sum (1) and the Hilb integral (5) for the scattering amplitude. We find that for large angles where the cross section falls by 17 orders of magnitude, the numerical integration is very delicate, and requires careful evaluation of the Bessel function. The partial wave sum, of course, has a corresponding delicate cancellation of terms for large angles.

For the eikonal form (6) the Hilb integral can also be done by the stationary phase method of Amado, Dedonder, and Lenz[11] with only a trivial kinematic replacement. Since that method is a large momentum transfer approximation, it is not valid at small angles but can be used to check and stabilize the large angle integration.

The results comparing the partial wave sum and the Hilb integral for the eikonal S matrix are shown in Fig. 1. The Hilb form agrees with the partial wave sum out to 170°. This is remarkable. A detailed examination of the numbers reveals that the agreement is at the 5% level or better for the large angles ($>90°$) and considerably better than that at smaller angles. In the first 20° agreement is better than 0.4%. Thus the Hilb summation is not simply qualitatively correct, which would in itself be noteworthy given the oscillation and enormous range of exponential fall off, but is in fact quantitatively correct. Note that to achieve this accuracy, cancellation in the integral of Eq. (5) must be correctly reproducing the corresponding cancellations in Eq. (1) to one part in 10^7 or 10^8. (The cross section falls by 16 orders of magnitude, between zero and 170°; the amplitude must decrease by 10^7 or 10^8.) However, the Hilb formula for a given l is supposed to be correct to one part in $(1/l)^{+3/2}$ or for the grazing of l of 22 to one part in 103. In fact the Hilb formula is capable of doing five orders of magnitude better than that.

As a second example of the accuracy of the Hilb integral we chose for our S matrix a form obtained from the WKB approximation. We use the same kinematics as in the

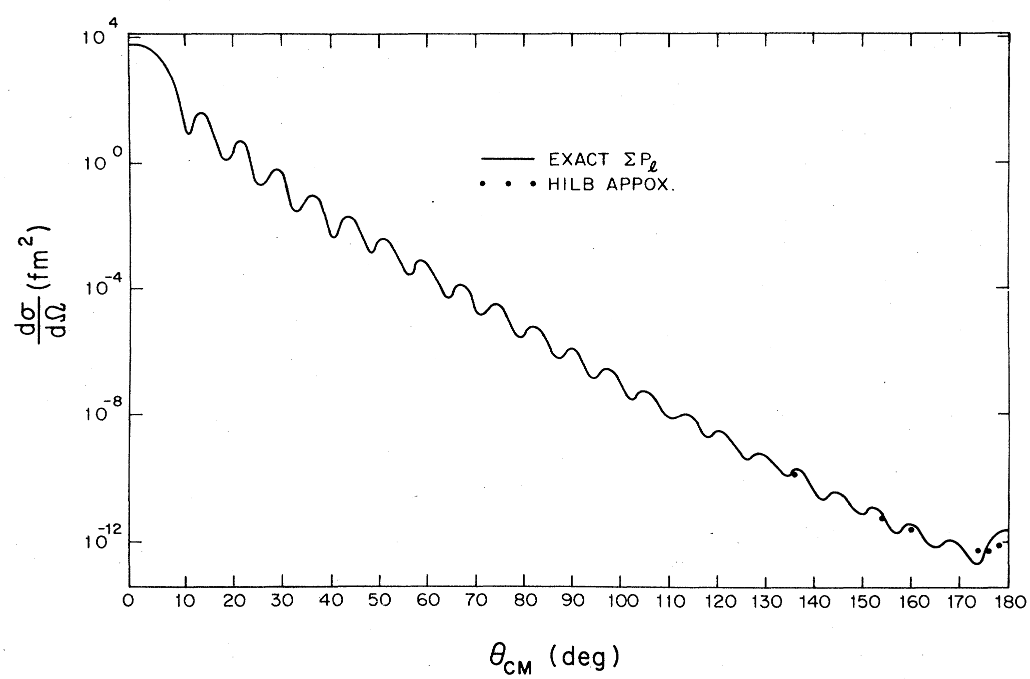

FIG. 1. Comparison of partial wave sum and Hilb integral for an eikonal S matrix [Eq. (6)]. The parameters for the potential are given in the text. Note the 16 orders of magnitude of change in cross section. Where not explicitly shown, the Hilb results coincide with the partial wave form. On the scale of this graph, differences between the Hilb and partial wave forms are only apparent at the largest angles.

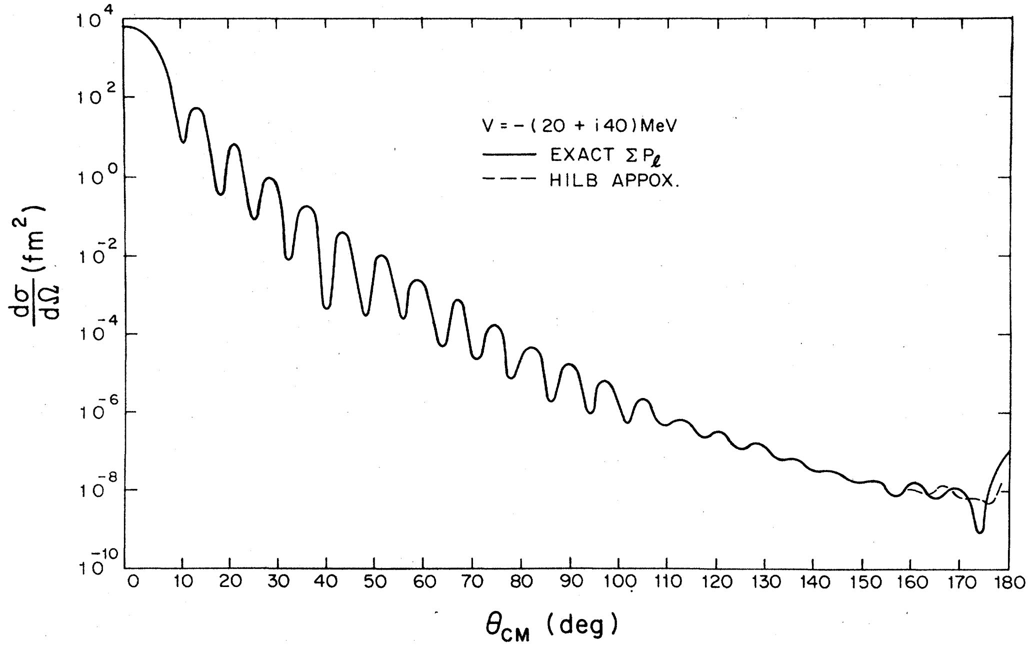

FIG. 2. Comparison of partial wave and Hilb integral for the WKB S matrix [Eq. (7)]. The potential shape parameters are given in the text. The strength is $-(20+40i)$ MeV. Where not explicitly shown, the Hilb result coincides with the partial wave form. Differences are only apparent at the largest angles.

eikonal case and a Wood-Saxon potential of the same geometry and strength. The WKB S matrix is given by[1]

$$S(b) = \exp\left\{ik\left\{b\pi - 2r_0 + 2\int_{r_0}^{\infty}\left[\left(1-\frac{b^2}{r^2}-\frac{2mV(r)}{k^2}\right)^{1/2}-1\right]dr\right\}\right\}, \quad (7)$$

where r_0 is the classical turning point given by

$$1-\frac{b^2}{r_0^2}-\frac{2mV(r_0)}{k^2}=0. \quad (8)$$

The cross sections obtained from this amplitude using the partial wave sum (1) and the Hilb integral (5) are compared

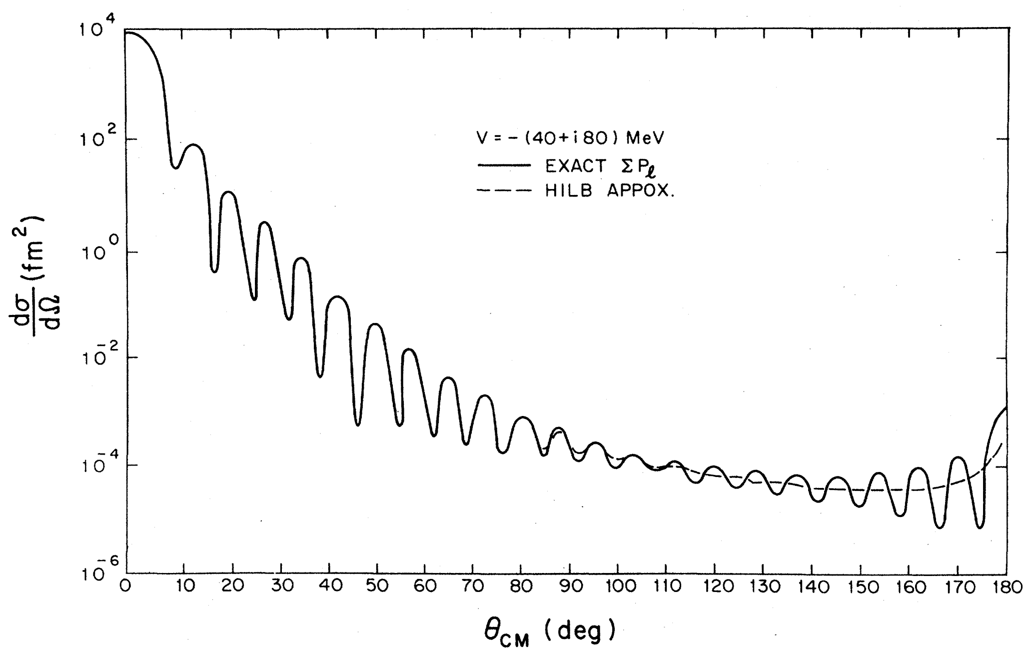

FIG. 3. Same as Fig. 2 but with the potential strength doubled. The Hilb and partial wave results now disagree beyond 80°.

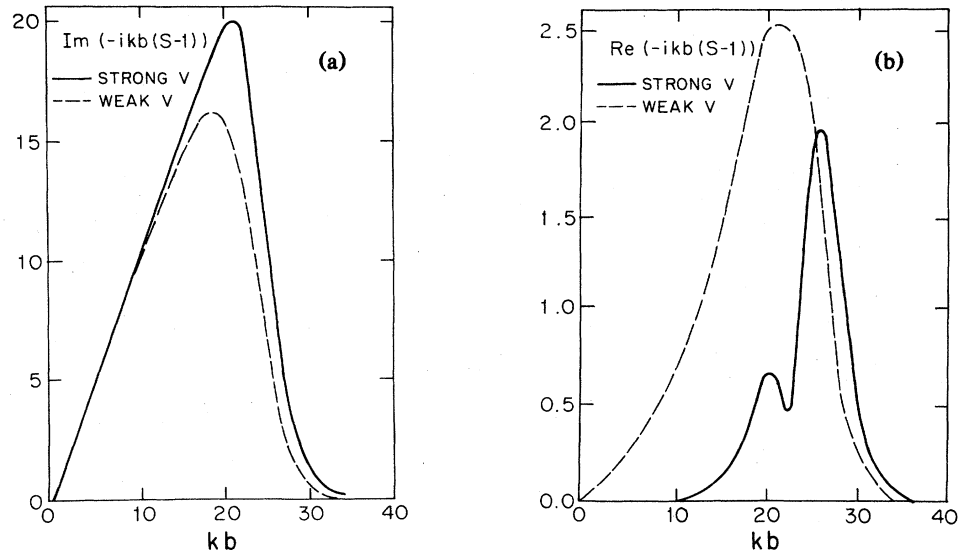

FIG. 4. (a) The imaginary, and (b) real parts of $ikb[S(b)-1]$ for the "weak" $[-(20+40i)$ MeV] and "strong" $[-(40+80i)$ MeV] Wood-Saxon potentials for a WKB S matrix. Note the change in scale between (a) and (b).

in Fig. 2. Again one sees the typical pattern of diffraction oscillations in an exponential envelope, this time falling by 12 orders of magnitude over the physical range of angles. Again the Hilb and partial wave results agree to beyond 165°. Again this is far better than $(1/l)^{3/2}$ would have us expect.

The potential we have used in the previous two examples is somewhat weak by medium energy nuclear physics standards. Typical nuclear potential would be about twice as strong. In Fig. 3 we show the WKB cross sections calculated using $V_0=-(40+80i)$ MeV with the same geometry and kinematics. Note that beyond 80° there are significant differences, with the partial wave summed cross section oscillating about the Hilb value. The reason for this discrepancy can be seen in Fig. 4 where we compare the real and imaginary parts of the WKB integrand $-ikb[S(b)-1]$ for the two potential strengths. We see that while the imaginary part dominates for both cases and is of the same shape in both cases, the stronger potential has a marked oscillation in the real part. This is not an artifact of the WKB. Qualitatively the same shape appears in the exact amplitude. This oscillation means that the S matrix has a region of rapid variation in b (or l). This rapid variation over very few l's vitiates the semiclassical approximation as well as spoiling the close agreement of the Hilb and partial wave sums. Even here it should be stressed that one has to go to 80° and fall by more than 10^3 in the amplitude before the discrepancy is visible and then the Hilb form is still correct on average. The details of that variation spoil the close agreement of the Hilb and partial wave sums.

We have seen that by using the Hilb formula for the Legendre polynomials we can convert the partial wave sum for a scattering amplitude to an integral. The Hilb form is valid for all scattering angles from 0° to nearly 180° and yields cross sections in remarkable agreement with the direct sum of partial waves, provided the partial wave amplitudes vary slowly with angular momentum l.

The integral form should permit new approximate expressions for the scattering amplitude when the partial wave amplitudes are explicitly known, as, for example, from the method of stationary phase. In other cases where relations among scattering processes are important, the Hilb form should serve as a useful starting point. We are presently exploring generalizations of the data-to-data relations[9] we obtained for medium energy nuclear physics observables using the eikonal approximation, to these processes for which the small angles or relatively weak potential of the eikonal form are not appropriate. In cases such as heavy ion physics, where the WKB partial wave amplitudes are a very good approximation, the Hilb formula should be particularly useful.

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