Equivalent local potentials from nonlocal separable ones

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A method is presented for the explicit construction of a family of equivalent local potentials from a given nonlocal separable potential. This procedure can be also applied in the case when a Coulomb potential is added to the separable short-range interaction.

The construction of equivalent local potentials (ELP) from nonlocal separable ones has been of considerable interest ever since the first work of Fiedeldey appeared on this subject.[1] In this context, the expression "local potential" does not mean a potential diagonal in the coordinates, as usually, but only a potential whose radial Schrödinger equation is a differential equation; two potentials are called "equivalent" when they produce the same phase shifts. Of particular importance has been the determination of ELP potentials when the nonlocal potential is of the separable type, as the latter kind of potential has been successfully employed for the description of nucleon-nucleon, nucleon-nucleus, and nucleus-nucleus scattering.[2-6] It was shown in this journal by Husain and one of us (S.A.)[7] that following the general prescription of Fiedeldey for the definition of the ELP, the latter can be constructed with impressive simplicity in terms only of the form factors of the separable interaction. Later, our attention was brought to a critique[8] of this work, also in this journal, in which it was pointed out that some of the integrals which would be involved in the ELP would not be convergent when the Coulomb interaction is present. In the present Brief Report we address ourselves to this problem and show that an alternative formulation, completely different from following the prescription of Fiedeldey, can be made, in which such difficulties do not appear. In order to see the origin of the problem and find the possible solution, one could start from the nonlocal Schrödinger equation, with the usual boundary conditions appropriate to scattering,

$$\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}\right]u_l(r)=\int_0^\infty K_l(r,r')u_l(r')dr' \, , \quad (1)$$

in usual notations; the equivalent local potential $U_l(k,r)$ could be defined by the equation

$$\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}\right]v_l(r)=U_l(k,r)v_l(r) \quad , \qquad (2)$$

the equivalence being guaranteed by the fact that $v_l(r)$ and $u_l(r)$ would be asymptotically the same and could therefore be related to each other through

$$u_l(r)=f_l(r)v_l(r) \, , \qquad (3)$$

with the property that $\lim_{r\to\infty}f_l(r)=1$. If Eq. (1) is solved for the regular and irregular wave functions $\mu_l(r)$ and $\nu_l(r)$, then following the Fiedeldey method, one obtains for $U_l(r)$

$$U_l(r)=-\frac{f_l''(r)}{f_l(r)}+2\left[\frac{f_l'(r)}{f_l(r)}\right]^2+\frac{1}{f_l^2}\int_0^\infty K_l(r,r')[\mu_l'(r)\nu_l(r')-\nu_l'(r)\mu_l(r')]dr' \, , \qquad (4)$$

with $f_l^2=\mu_l'\nu_l-\nu_l'\mu_l$. Husain and Ali showed that for a nonlocal separable potential of the form

$$K_l(r,r')=\lambda_l q_l(r)q_l(r') \qquad (5)$$

the calculation of (4) becomes straightforward; as in this case, μ and ν can be readily obtained from the solution of (1). The assumption of rank-1 potential was only for the sake of simplicity. However, their calculations of the ELP involved integrals like

$$\int_0^\infty kr\eta_l(kr)q_l(r)dr \quad ,$$

where η_l is the Neumann function. In the case in which the two-body interaction contains a Coulomb potential besides the separable one, the function $kr\eta_l(kr)$ would be replaced by the irregular Coulomb wave function $G_l(k,r)$, and it was pointed out by Shah Jahan that now in the calculation of the ELP, one has to deal with integrals of the type

$$\int_0^\infty G_l(kr)q_l(r)dr \quad ,$$

and since $G_l(kr)$ has a logarithmic singularity at the origin, this integral does, indeed, diverge.

Recently, we gave considerable attention to this problem and we have found that the difficulty encounterd above can be overcome if we proceed in an alternative manner from the very beginning. To make our formulation clear, we proceed as follows.

As before, we write down the Schrödinger equation for two particles interacting via a short-range nuclear potential of separable type (5) and a Coulomb potential

$$\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}-\frac{2\eta k}{r}\right]\psi_l(r)=\lambda_l q_l(r)J_l(k) \;, \tag{6}$$

where

$$J_l(k)=\int_0^\infty q_l(r')\psi_l(k,r')dr' \;.$$

It was shown by Ali *et al.*[9] that the function ψ_l in Eq. (6) is explicitly calculable in terms of the form factor q_l for all r. In order to derive the ELP, we now make use of equations (2) [but with the Coulomb term included in the left-hand side (lhs)], (3) (where we now replace u by ψ) and (6), obtaining (the primes denote differentiation with respect to r)

$$U_l(k,r)=-\frac{f_l''}{f_l}+\frac{2\psi_l h_l^2-2\psi_l' h_l+\lambda_l q_l J_l}{\psi_l} \;, \tag{7}$$

where $h_l=f_l'/f_l$ and

$$f_l(r)=\exp\left[-\int_r^\infty h_l(r')dr'\right] \;.$$

It is at this point that we make a departure from the previous method and note that the singularities in U could be avoided if in Eq. (7) we set

$$2\psi_l h_l^2-2\psi_l' h_l+\lambda_l q_l J_l=V_l(k,r)\psi_l \;, \tag{8}$$

where V is some function of r, depending also on l and k, whose properties we shall determine. The ELP U is then given by

$$U=V-h^2-h' \;, \tag{9}$$

where h is obtained in terms of V from (8) as

$$h=\frac{1}{2}\frac{\psi'}{\psi}\left\{1-\left[1+2V\left(\frac{\psi}{\psi'}\right)^2-2\lambda Jq\frac{\psi}{\psi'^2}\right]^{1/2}\right\} \tag{10}$$

and the minus sign has been chosen in front of the square root in order to obtain a regular function.

We are now going to consider a possible choice for V, assuming, for the sake of definiteness, that the form factors of the separable potentials have the form

$$q_l(r)=r^\alpha e^{-\beta r} \;, \tag{11}$$

where α and $\beta(\beta>0)$ are constants, possibly dependent on l. We can set

$$V_l(k,r)=W_l(k)r^{-\delta}e^{-\gamma r}, \quad (\gamma>0) \;. \tag{12}$$

The constant W,γ,δ will be chosen in such a way that the potential U is a regular, real function of r, less singular than r^{-2} at the origin and tending to zero at infinity. To this aim, the argument of the square root in Eq. (10) must be positive. This condition will be satisfied if

$$W_l(k)>\sup_{r\geq 0}\phi_l(k,r) \;, \tag{13}$$

where

$$\phi_l(k,r)=\frac{\lambda_l J_l(k)r^{\alpha+\delta}e^{(\gamma-\beta)r}}{\psi_l(k,r)}-\frac{1}{2}\left[\frac{\psi_l'(k,r)}{\psi_l(k,r)}\right]^2 r^\delta e^{\gamma r} \;.$$

Remembering that $\psi_l(r)\sim A_l r^{l+1}$ as $r\to 0$, the upper bound in inequality (13) will be finite for all $r\geq 0$ if $\gamma<\beta$ and $\alpha>l-1$. Then, the function h, defined in Eq. (10), will behave as follows:

$$h=\lambda Jq/\psi', \quad (\psi=0, r>0) \;,$$

$$h=-\tfrac{1}{2}(2V-2\lambda Jq/\psi)^{1/2}, \quad (\psi'=0, r>0) \;,$$

$$h\sim[2(l+1)]^{-1}(\lambda J r^{\alpha-l}/A-Wr^{1-\delta}), \quad (r\to 0) \;,$$

$$h\sim(\lambda Jq-V\psi/2)/\psi', \quad (\psi'\neq 0, r\to\infty) \;.$$

Considering the above behavior of h at the origin and Eq. (9), we find the further limitation $\delta<2$.

In conclusion, it is found that the given separable potential (5), (11) with $\beta>0$, $\alpha>l-1$, admits a class of equivalent local potentials defined by Eqs. (9), (10), (12), with the restrictions (13), $0<\gamma<\beta$ and $\delta<2$. As our example has shown, this class contains infinitely many potentials, corresponding to the permissible choices of W,γ,δ; all of these potentials depend both on energy and on angular momentum, are less singular at the origin than r^{-2}, and decrease exponentially at infinity. Of course, the considered class does not exhaust all equivalent local potentials, for the reason, at least, that other forms of V in Eq. (8) are compatible with our requirements.

The existence of infinite families of equivalent local potentials is already known in the literature. For instance, Capuzzi[10] introduced S-matrix conserving transformations, which produce generally different Perey damping factors for the two independent solutions of the nonlocal and equivalent local Schrödinger equations. Our family of equivalent local potentials produces different damping factors for the regular and irregular solutions and is a subset of the potentials in principle obtainable by S-matrix conserving transformations. On the other hand, the potential (4) is the only one for which the damping factors are identical.

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[1]H. Fiedeldey, Nucl. Phys. **A96**, 463 (1967).
[2]S. Ali, M. Rahman, and D. Husain, Phys. Rev. C **9**, 1657 (1974).
[3]G. Pantis and H. Fiedeldey, Z. Phys. A **277**, 343 (1976).
[4]A. A. Z. Ahmad, S. Ali, N. Ferdous, and M. Ahmad, Nuovo Cimento **30A**, 385 (1975).
[5]G. Cattapan, G. Pisent, and V. Vanzani, Nucl. Phys. **A241**, 204 (1975).
[6]M. Rahman, D. Husain, and S. Ali, Phys. Rev. C **10**, 1 (1974).
[7]D. Husain and S. Ali, Phys. Rev. C **2**, 1587 (1970).
[8]M. Shah Jahan, Phys. Rev. D **8**, 2744 (1973).
[9]S. Ali, M. Rahman, and D. Husain, Phys. Rev. D **6**, 1178 (1972).
[10]F. Capuzzi, Nuovo Cimento **11A**, 801 (1972).