Treatment of pairing correlations based on the equations of motion for zero-coupled pair operators

F. Andreozzi, A. Covello, A. Gargano, Liu Jian Ye,* and A. Porrino
Dipartimento di Fisica Nucleare, Struttura della Materia e Fisica Applicata, Università di Napoli, 80125 Napoli, Italy and Istituto Nazionale di Fisica Nucleare, 80125 Napoli, Italy
(Received 6 August 1984; revised manuscript received 18 March 1985)

The pairing problem is treated by means of the equations of motion for zero-coupled pair operators. Exact equations for the seniority-v states of N particles are derived. These equations can be solved by a step-by-step procedure which consists of progressively adding pairs of particles to a core. The theory can be applied at several levels of approximation depending on the number of core states which are taken into account. Some numerical applications to the treatment of $v=0$, $v=1$, and $v=2$ states in the Ni isotopes are performed. The accuracy of various approximations is tested by comparison with exact results. For the seniority-one and seniority-two problems it turns out that the results obtained from the first-order theory are very accurate, while those of higher order calculations are practically exact. Concerning the seniority-zero problem, a fifth-order calculation reproduces quite well the three lowest states.

I. INTRODUCTION

The problem of treating pairing correlations in nuclei without violating the particle-number conservation has been the subject of a considerable amount of theoretical work. In this context, numerous efforts have been made to improve the Bardeen-Cooper-Schrieffer (BCS) method. Most of the early work in this direction was based on number projection techniques resulting essentially in two different methods, usually denoted as PBCS and FBCS. While the BCS method with subsequent projection (PBCS) (Ref. 1) has the advantage of being straightforward in application, it is of no aid in the weak-pairing limit where BCS solutions go trivial. The method of projection before variation (FBCS) (Refs. 2 and 3) does not suffer from this drawback, but the simplicity of the BCS approach is completely lost.

Renewed interest has recently been focused on particle-number-conserving theories in connection with the important problem of the microscopic foundation of the interacting boson model.[4,5] In this context, an approximate projection method has been developed by Li[6] for constructing the physical states in the Dyson boson description of nuclear collective motion. We shall comment on the application of this method to the pairing model in Sec. III.

The problem of restoring broken symmetries arises in various approaches to the nuclear many-body problem, as for instance the Hartree-Fock-Bogoliubov theory, and has received a great deal of attention over the years. Consequently the projection techniques have reached a high degree of sophistication through the work of many authors. To touch on this topic is clearly beyond the scope of our presentation. For a comprehensive discussion including references we refer the reader to a few representative papers[7–9] and to the recent monograph by Ring and Schuck.[10]

A method of suppressing the particle-number fluctuation without actually having to perform the projection calculation is the well-known Lipkin-Nogami version of the BCS approximation,[11–14] in which a term proportional to $\hat{N}^2$ ($\hat{N}$ is the number operator) is subtracted from the pairing Hamiltonian. In this connection it may be mentioned that there have been attempts[15] in recent years to incorporate the Lipkin-Nogami procedure in more general quasiparticle calculations (modified Tamm-Dancoff approximation) involving mixing of different quasiparticle subspaces.

Another way of restoring the number conservation is to handle the residual interactions neglected in the BCS approximation by means of the quasiboson approximation.[16] This approach has also been applied[17] to the problem of improving on the isospin invariance in the treatment of the charge-independent pairing Hamiltonian through the use of a generalized BCS formalism.[18–20] While this method is very simple in philosophy, it is rather involved in application. Indeed, no explicit calculations were performed for the neutron-proton pairing problem in the general case of nondegenerate levels.

A quite different approach to the treatment of pairing correlations is that of devising approximation schemes without recourse to quasiparticle transformations. One attempt in this direction started from a degenerate model to treat cases of closely spaced single-particle levels by perturbation theory.[21] Subsequently, this study was extended to situations where the single-particle levels can be divided into two groups, each of them being almost degenerate.[22] Whereas this simple method may give accurate results in special cases, it is clearly very limited in scope.

A study of the exact eigenstates of the pairing Hamiltonian was made by Richardson, who gave analytical expressions for a restricted class of n-pair eigenstates[23] (subsequently the same approach was applied to the charge-independent pairing Hamiltonian[24]). However, the explicit calculation of the energies and wave functions of these states is very complicated in general, since it requires solv-

ing a system of n coupled, nonlinear, algebraic equations.

The usefulness of the equations-of-motion method as a means of dealing with the complexities of the nuclear many-body problem has been recognized for more than two decades,[25–27] and several treatments within its framework have been developed for the pairing-force problem. It should be stressed that in most of this work the study of the pairing model has been undertaken as a first step towards the treatment of more realistic interactions. In this context, the full treatment of the equations of motion for single-fermion operators (a detailed account of this approach and related developments is given in the recent review paper by Klein[28]) has led to methods involving iteration across even and odd nuclei simultaneously.[29–33] This is an undesirable feature which makes this kind of approach of limited practical value. In particular, it would be extremely difficult to deal with more complicated interactions. Actually, in the extension of the method to the treatment of neutron-proton correlations given by Campi, Vucetich, and Jean,[34] the numerical applications did not go beyond a simple model of four doubly degenerate levels.

The equations of motion for pair operators have also been considered by various authors. Some early studies,[35] however, made use of linearization procedures which could not be justified much beyond plausibility arguments. A most important turn in the development of this approach resulted from the work of Ref. 36. In that work it was shown that it is possible to find solutions to the pairing problem by a chain calculation which involves only even nuclei. At this point it is worth mentioning that there have been attempts to avoid chain calculations based on the algebra of fermion pairs.[37] The price one has to pay, however, is the difficulty of dealing with nonlinear equations.

In working with the equations-of-motion method one is in general faced with the problem of an overcomplete space. This gives rise to spurious components which may strongly contaminate the solutions when approximations are made. Although this problem is of crucial importance for the theory to give reliable answers, it has received little attention (if at all) in most of the works cited above. One notable exception is the work of Mauger and Evans[38] who have tackled this difficulty within the framework of the method of Ref. 36. However, as pointed out in a previous paper[39] (hereafter referred to as I), the use of the uncoupled representation makes their treatment very impractical if one wants to go beyond the lowest order of approximation.

In I, a new and improved version of the method of Ref. 36 was developed which provides a simple way of treating states of seniority $v=\sum_j v_j=0$ (v_j is the seniority of level j) in pairing-force theory. The major new elements in I, absent from the initial work,[36] are the following: (i) The explicit use of zero-coupled pair operators. (ii) The extension of the formalism to include excited intermediate states. (iii) The introduction of a reliable procedure for removing the redundant states at each step of the chain calculation. Numerical applications of the method to single-closed-shell nuclei[39,40] gave very accurate results even at the lowest order of approximation (which we call first-order theory).

It is highly important to point out that the formalism given in I has recently been generalized to properly include the effects of correlations between unlike nucleons.[41] It turned out that this formalism provides a most satisfactory framework for finding approximate solutions to a charge-independent pairing Hamiltonian conserving both number and isospin invariance.

We have given above a brief review of various approaches to the study of the pairing model. However, the literature on this subject is very extensive and we are well aware that our account is by no means complete. Other works related to the present discussion may be traced from Refs. 42—46.

Our experiences with the equations-of-motion method summarized in the foregoing discussion made it apparent that our approach to the pairing model was worth pursuing beyond the seniority-zero problem. Within the framework of our approach, the treatment of higher seniority states may be performed in two different ways. This is because the wave function for states of N particles and seniority v ($v>0$) can be related either to the seniority-zero states of the $(N-v)$-particle system or to the seniority-v states of the $(N-2)$-particle system.

The equations-of-motion formalism corresponding to the former approach and several applications to the treatment of $v=1$ and $v=2$ states have been presented in Refs. 47 and 48. A characteristic of this formalism is that the solutions for $v>0$ are obtained using as input the results for the seniority-zero problem.

The purpose of this paper is to discuss the treatment of pairing correlations corresponding to the second alternative which involves the use of the equations of motion for zero-coupled pair operators for any seniority v. Within this framework, the formalism has the same structure for all values of v.

The seniority-v states of N particles can be treated at several levels of approximation depending on the number of core states [seniority-v states of the $(N-2)$-particle system] which are taken into account. In this paper, we apply the theory at various orders of approximation to the seniority-one and seniority-two problems in the case of the Ni isotopes.

The paper is organized as follows. In Sec. II A we describe the general formalism of the theory. In Sec. II B the salient features of the first-order theory are outlined. In Sec. III we present the results of our calculations and test the accuracy of the various approximations by comparison with the exact results. In Sec. IV we summarize the main advantages of our method and discuss how the results of the present work compare with those of other approaches.

II. FORMALISM

A. General

The pairing Hamiltonian is written as

$$H=\sum_j \epsilon_j \hat{N}_j-\sum_{jj'} G_{jj'} A_j^\dagger A_{j'} , \tag{1}$$

where

$$\hat{N}_j=\sum_m a_{jm}^\dagger a_{jm} \ , \tag{2}$$

$$A_j^\dagger=\sum_{m>0} a_{jm}^\dagger a_{j\bar{m}}^\dagger \ , \tag{3}$$

and

$$a_{j\bar{m}}^\dagger=(-)^{j+m}a_{j-m}^\dagger \ .$$

The wave function of the seniority-v states of a system with N identical particles (N even or odd) is given the form

$$|\, N,\beta,L\,\rangle=\sum_{j\gamma} c_{j\beta\gamma}^L(N)A_j^\dagger\,|\, N-2,\gamma,L\,\rangle \ . \tag{4}$$

Here, the symbol L stands for all the seniority quantum numbers v_j, and β and γ distinguish states with the same value of L in the N- and $(N-2)$-particle systems, respectively.

The equations of motion for the pair creation operators $A_j^\dagger$ are

$$[H,A_j^\dagger]=2\epsilon_j A_j^\dagger-\sum_{j'} G_{jj'}A_{j'}^\dagger(\Omega_j-\hat{N}_j) \ , \tag{5}$$

where $\Omega_j=j+\frac{1}{2}$. Taking matrix elements of (5) and introducing a set of intermediate states between $A_{j'}^\dagger$ and $\hat{N}_j$, one obtains, for each value of L, the eigenvalue equation

$$\sum_{j'\gamma'} M_{j\gamma j'\gamma'}^L X_{j'\beta\gamma'}^L(N)=E_{L\beta}(N)X_{j\beta\gamma}^L(N) \ , \tag{6}$$

with

$$\begin{aligned} M_{j\gamma j'\gamma'}^L=&\,[2\epsilon_j+E_{L\gamma}(N-2)]\delta_{jj'}\delta_{\gamma\gamma'} \\ &-G_{jj'}\Omega_j[\delta_{\gamma\gamma'}-2\rho_{j\gamma'\gamma}^L(N-2)] \ , \end{aligned} \tag{7}$$

where

$$X_{j\beta\gamma}^L(N)=\langle N,\beta,L\,|\,A_j^\dagger\,|\,N-2,\gamma,L\,\rangle \tag{8}$$

and

$$\rho_{j\gamma'\gamma}^L(N-2)=\langle N-2,\gamma',L\,|\,a_{jm}^\dagger a_{jm}\,|\,N-2,\gamma,L\,\rangle \ . \tag{9}$$

The two-particle transfer amplitudes $X_{j\beta\gamma}^L(N)$ are related to the coefficients $c_{j\beta\gamma}^L(N)$ of (4) through

$$X_{j\beta\gamma}^L(N)=\sum_{j'\gamma'} c_{j'\beta\gamma'}^L(N)d_{j'\gamma'j\gamma}^L(N-2) \ , \tag{10}$$

where

$$d_{j'\gamma'j\gamma}^L(N-2)=\langle N-2,\gamma',L\,|\,A_{j'}A_j^\dagger\,|\,N-2,\gamma,L\,\rangle \tag{11}$$

are the elements of the metric matrix of the N-particle states $A_j^\dagger\,|\,N-2,\gamma,L\,\rangle$. From the normalization condition for the states $|\,N,\beta,L\,\rangle$ it follows that

$$\sum_{j\gamma} c_{j\beta\gamma}^L(N)X_{j\beta\gamma}^L(N)=1 \ . \tag{12}$$

Making use of the commutators $[\hat{N}_j,A_{j'}^\dagger]$ and $[A_j,A_{j'}^\dagger]$, respectively, the density matrix (9) and the metric matrix (11) can be expressed as

$$\begin{aligned} \rho_{j\beta\beta'}^L(N)=&\,\frac{1}{\Omega_j}\sum_\gamma c_{j\beta'\gamma}^L(N)X_{j\beta\gamma}^L(N) \\ &+\sum_{j'\gamma\gamma'} c_{j'\beta'\gamma}^L(N)X_{j'\beta\gamma'}^L(N)\rho_{j\gamma'\gamma}^L(N-2) \end{aligned} \tag{13}$$

and

$$\begin{aligned} d_{j\beta j'\beta'}^L(N)=&\sum_\gamma X_{j'\beta\gamma}^L(N)X_{j\beta'\gamma}^L(N) \\ &+\Omega_j[\delta_{\beta\beta'}-2\rho_{j\beta\beta'}^L(N)]\delta_{jj'} \ . \end{aligned} \tag{14}$$

It is then clear that the solution of the seniority-v problem for N particles can be obtained by a step-by-step procedure starting from $N=v$. In other words, one can solve Eq. (6) for $N=v+2$, thus obtaining $E_{L\beta}(v+2)$, $c_{j\beta\gamma}^L(v+2)$, and $X_{j\beta\gamma}^L(v+2)$, then calculate the $\rho(v+2)$'s and the $d(v+2)$'s and proceed by successive steps up to N. The initial values needed to start the chain calculation are (the index β is dropped since for $N=v$ there is only one state for a given L) the following:

$$\begin{aligned} &E_L(N=v)=\sum_j \epsilon_j v_j \ , \\ &\rho_j^L(N=v)=\frac{v_j}{2\Omega_j} \ , \\ &d_{jj'}^L(N=v)=(\Omega_j-v_j)\delta_{jj'} \ . \end{aligned} \tag{15}$$

The formalism outlined above is exact in the sense that all the existing seniority-v states of the $(N-2)$-particle system are taken into account. However, the energy matrix M, for $N>v+2$, has a number of solutions which is larger than the correct one. These spurious solutions appear because the set of basis states $A_j^\dagger\,|\,N-2,\gamma,L\,\rangle$ is overcomplete. Special care is necessary in handling this problem when approximations are made (i.e., when one reduces the number of core states $|\,N-2,\gamma,L\,\rangle$), since the unphysical solutions may mix strongly with the good ones. In I, a procedure for removing the redundant states at any order of approximation was given, which involves the diagonalization of the metric matrix. Subsequently, we have developed an alternative technique which uses the Cholesky decomposition of symmetric positive definite matrices.[49] For lower-order pairing calculations both procedures are very simple and effective. The higher efficiency of the Cholesky decomposition, however, is absolutely crucial to treatment of more complex problems,[49] where many diagonalizations of large matrices may be totally impractical.

In the present work, we have found it convenient to proceed through the diagonalization of the metric matrix in the first-order theory, and to resort to the Cholesky decomposition for the higher-order calculations.

In the above, we have given the formulation of the theory in terms of particles. The hole formalism can be readily derived by proceeding precisely as before, and will not be given explicitly here. In this case, the wave function for the seniority-v states of N particles is written as

$$|\, N,\beta,L\,\rangle=\sum_{j\alpha} c_{j\beta\alpha}^L(N)A_j\,|\,N+2,\alpha,L\,\rangle \ , \tag{16}$$

and one has to use the equations of motion for pair an-

nihilation operators A_j,

$$[H,A_j]=2(G_j-\epsilon_j)A_j-\sum_{j'}G_{jj'}A_{j'}(\hat{N}_j-\Omega_j)\ , \tag{17}$$

where $G_j\equiv G_{jj}$. Of course, when the sums on γ and α run over all existing core states, either formalism permits one to solve the seniority-v problem exactly. In the approximate versions of the theory, however, systems beyond midshell are better described in terms of holes.

B. First-order theory

We now focus attention on the simplest stage of approximation of the previously described theory, wherein the core states are restricted to a single state. A complete presentation of this approximation scheme and a detailed account of how to remove the unphysical states within its framework was given in Ref. 40 for the seniority-zero problem. Since, as already emphasized in the Introduction, the formalism is essentially the same for all values of v, we only give here a brief survey of the salient features of the first-order theory. In this case the wave function (4) becomes

$$|\,N,\beta,L\,\rangle=\sum_j c^L_{j\beta\gamma_1}(N)A^\dagger_j\,|\,N-2,\gamma_1,L\,\rangle\ , \tag{18}$$

where $|\,N-2,\gamma_1,L\,\rangle$ denotes the lowest state of a given L.

For the usual constant pairing force, $G_{jj'}=G$ (which is used in our calculations), the eigenvalue problem reduces to the simple dispersion relation

$$\frac{1}{G}=\sum_{jj'}\frac{\Omega_j[1-2\rho^L_{j\gamma_1}(N-2)]}{2\epsilon_j-e_{L\beta\gamma_1}(N)}\xi^L_{j'j}\ , \tag{19}$$

where

$$e_{L\beta\gamma_1}(N)=E_{L\beta}(N)-E_{L\gamma_1}(N-2)\ .$$

In Eq. (19) the matrix ξ^L eliminates the unphysical states and is easily constructed from the eigenvectors of the metric matrix. The reader is referred to Ref. 40 for a detailed discussion of this important point.

As has already been pointed out in the seniority-zero case,[39] in the first-order theory the occupation numbers are best obtained if no explicit use is made of the coefficients $c^L_{j\beta\gamma_1}(N)$. This can be done by using the following relation:

$$\rho^L_{j\beta_1}(N)=\rho^L_{j\gamma_1}(N-2)+\frac{(X^L_{j\beta_1\gamma_1}(N))^2}{\Omega_j(\Omega_j-v_j)\sum_{j'}\dfrac{(X^L_{j'\beta_1\gamma_1}(N))^2}{\Omega_{j'}-v_{j'}}}\ , \tag{20}$$

which is the straightforward generalization of that given in Ref. 40 for $v=0$.

It should be noted that, for a given L, Eq. (19) yields (n_j-n_s) solutions (n_j is the number of single-particle levels and n_s the number of spurious states). The first-order theory, however, is expected to describe with good accuracy only the lowest state for each value of L. In particular, as we shall see in Sec. III, to obtain accurate results for the first two seniority-zero excited states a fifth-order calculation is required.

Clearly the first-order theory is exact for $N=v+2$. In this case Eq. (19) becomes

$$\frac{1}{G}=\sum_j\frac{\Omega_j-v_j}{2\epsilon_j+\sum_{j'}\epsilon_{j'}v_{j'}-E_{L\beta}(v+2)}\ . \tag{21}$$

It is worth noting that Eq. (21) comprises the well-known eigenvalue equations[50] for the two- and three-particle problems ($v=0$ and $v=1$, respectively).

III. APPLICATIONS

In order to assess the practical value of the present approach, we have performed several calculations for the even and odd Ni isotopes. The single-particle energies and the coupling strength G are the same as those used in I. Some results are presented in Tables I–VII. For notational simplicity the quantum numbers L are omitted in Tables I–III, while in Tables IV–VII the physical quantities are specified by the angular momenta of the unpaired particles.

In Table I we compare the energies of the three lowest seniority-zero states obtained from the fifth-order application of the theory [the $v=0$ core states in (4) are restricted to the lowest five states] with the exact ones; for completeness we also give the first-order ground-state energies (they have already been published in I). Tables II and III show the improvement in the ground-state occupation probabilities and in the g.s.→g.s. two-particle transfer amplitudes as we move from the first- to the fifth-order

TABLE I. Values of the energies $E_\beta(N)$ of the three lowest seniority-zero states for the Ni isotopes (MeV). N is the number of valence neutrons. The columns labeled I and V list the results obtained from the first- and fifth-order calculations, respectively.

N / State	6			8			10		
	Exact	I	V	Exact	I	V	Exact	I	V
0_1	−1.75	−1.73	−1.75	−0.50	−0.53	−0.50	1.70	1.60	1.72
0_2	0.76		0.76	2.06		2.06	3.60		3.60
0_3	1.38		1.38	2.57		2.57	5.65		5.61

TABLE II. Values of the occupation numbers $\rho_{j\beta_1}(N)$ of the seniority-zero ground state for the Ni isotopes.

j \ N	6			8			10		
	Exact	I	V	Exact	I	V	Exact	I	V
$\frac{3}{2}$	0.764	0.729	0.766	0.859	0.812	0.862	0.934	0.876	0.944
$\frac{5}{2}$	0.404	0.413	0.405	0.631	0.632	0.632	0.856	0.833	0.858
$\frac{1}{2}$	0.153	0.169	0.154	0.252	0.278	0.255	0.408	0.466	0.421
$\frac{9}{2}$	0.021	0.027	0.020	0.027	0.040	0.025	0.031	0.057	0.023

TABLE III. Values of the g.s.→g.s. two-particle transfer amplitudes $X_{j\beta_1\gamma_1}(N)$ for the even Ni isotopes.

j \ N	6			8			10		
	Exact	I	V	Exact	I	V	Exact	I	V
$\frac{3}{2}$	1.061	0.995	1.061	0.891	0.800	0.890	0.709	0.649	0.708
$\frac{5}{2}$	1.706	1.827	1.705	1.835	1.946	1.834	1.673	1.730	1.667
$\frac{1}{2}$	0.373	0.383	0.373	0.460	0.458	0.460	0.551	0.558	0.553
$\frac{9}{2}$	0.711	0.723	0.712	0.810	0.799	0.815	0.862	0.829	0.874

TABLE IV. Values of the energies $E_{j\beta_1}(N)$ of the various seniority-one states for the Ni isotopes. The columns labeled I and V list the results obtained from the first- and fifth-order calculations, respectively.

j \ N	7			9			11		
	Exact	I	V	Exact	I	V	Exact	I	V
$\frac{3}{2}$	0.34	0.19	0.34	2.34	2.09	2.34	5.55	5.26	5.57
$\frac{5}{2}$	−0.07	−0.14	−0.07	1.69	1.45	1.70	4.80	4.47	4.81
$\frac{1}{2}$	0.30	0.28	0.30	1.73	1.63	1.73	4.07	3.86	4.07
$\frac{9}{2}$	2.95	2.96	2.95	4.23	4.19	4.24	6.43	6.32	6.45

TABLE V. Values of the occupation numbers $\rho^{j'}_{j\beta_1}(N)$ of the seniority-one ground state for the Ni isotopes. The angular momentum j' is $\frac{5}{2}$ for $N=7,9$ and $\frac{1}{2}$ for $N=11$.

j \ N	7			9			11		
	Exact	I	V	Exact	I	V	Exact	I	V
$\frac{3}{2}$	0.862	0.845	0.863	0.945	0.914	0.948	0.985	0.944	0.985
$\frac{5}{2}$	0.514	0.507	0.514	0.747	0.728	0.748	0.977	0.953	0.977
$\frac{1}{2}$	0.151	0.170	0.152	0.270	0.308	0.274	0.500	0.500	0.500
$\frac{9}{2}$	0.017	0.024	0.016	0.020	0.036	0.017	0.019	0.051	0.019

TABLE VI. Values of the energies $E_{j_1 j_2 \beta_1}(N)$ of the various seniority-two states for the Ni isotopes (MeV). The columns labeled I and IV list the results obtained from the first- and fourth-order calculations, respectively.

$j_1 j_2$ \ N	6			8			10		
	Exact	I	IV	Exact	I	IV	Exact	I	IV
$\frac{3}{2}\frac{3}{2}$	1.10	1.11	1.10	2.93	2.93	2.93	6.01	6.06	6.01
$\frac{3}{2}\frac{5}{2}$	0.68	0.57	0.68	2.26	2.02	2.27	5.25	4.63	5.26
$\frac{3}{2}\frac{1}{2}$	1.14	1.05	1.14	2.39	2.19	2.39	4.52	4.20	4.52
$\frac{3}{2}\frac{9}{2}$	3.90	3.82	3.90	5.01	4.85	5.01	7.02	6.76	7.03
$\frac{5}{2}\frac{5}{2}$	0.35	0.36	0.35	1.59	1.38	1.59	4.50	3.99	4.50
$\frac{5}{2}\frac{1}{2}$	0.97	1.00	0.97	1.89	1.77	1.89	3.76	3.45	3.76
$\frac{5}{2}\frac{9}{2}$	3.79	3.82	3.79	4.59	4.50	4.59	6.36	6.10	6.35
$\frac{1}{2}\frac{9}{2}$	4.39	4.42	4.39	4.96	4.94	4.96	6.41	6.29	6.41
$\frac{9}{2}\frac{9}{2}$	7.18	7.22	7.18	7.63	7.64	7.63	8.95	8.89	8.95

theory. In Table IV the energies derived from the first- and fifth-order theories for the seniority-one states (the lowest one for each value of L) in the odd Ni isotopes are compared with the exact ones. In Table V we report the occupation probabilities of the seniority-one ground state. The results for the seniority-two states are given in Tables VI and VII.

From the results obtained, it appears that the lowest order of approximation yields very accurate results for the seniority-zero ground state and the seniority-two states of even-even nuclei as well as for the seniority-one states in odd-A nuclei. It can be easily verified that our first-order energies are much better than those of the BCS approximation for all values of v and N. The fifth-order calculation reproduces extremely well the ground state and the first two excited states of seniority zero, while for the seniority-one and seniority-two problems the results turn out to be practically exact in the fifth- and fourth-order theories, respectively.

At this point it should be emphasized that the quality of the approximation need not systematically improve with the order of truncation, i.e., the number of core states in (4). This is essentially due to the step-by-step procedure required to solve the N-particle problem (6). In fact, if at a given step some of the higher-lying states are badly represented, then their use as core states may somewhat damage the low-lying states at the next step. Actually, in our numerical applications to the model of the Ni isotopes it turned out that the fourth- and fifth-order approximations (for $v=2$ and $v=0,1$, respectively) suffice to produce the various physical quantities with equal accuracy for all values of N.

It should be pointed out, however, that any state which is poorly described can be easily identified and removed through a check on the orthogonality of the solutions at each step of the chain calculation. More precisely, one has to exclude from the expansion (4) any core state which has unduly large scalar products with lower-lying states. Of course, this procedure causes the order of approximation to decrease with respect to the initial one, and is therefore of limited use in the lower-order applications of the theory. However, when considering systems with rather large numbers of valence particles (e.g., Sn isotopes) the above procedure may profitably be applied[51] to obtain accurate results with a very limited amount of computational work.

TABLE VII. Values of the occupation numbers $\rho^{j_1 j_2}_{j\beta_1}(N)$ of the lowest seniority-two excited state for the Ni isotopes. The angular momenta $(j_1 j_2)$ are $(\frac{5}{2}\frac{5}{2})$ for $N=6,8$ and $(\frac{5}{2}\frac{1}{2})$ for $N=10$.

j \ N	6			8			10		
	Exact	I	IV	Exact	I	IV	Exact	I	IV
$\frac{3}{2}$	0.871	0.830	0.871	0.957	0.953	0.958	0.987	0.965	0.987
$\frac{5}{2}$	0.389	0.403	0.389	0.631	0.611	0.631	0.820	0.807	0.821
$\frac{1}{2}$	0.055	0.074	0.055	0.137	0.164	0.138	0.500	0.500	0.500
$\frac{9}{2}$	0.008	0.011	0.008	0.011	0.019	0.011	0.013	0.030	0.012

IV. SUMMARY AND DISCUSSION

In this paper we have presented a method for the treatment of pairing interactions based on the development of the equations of motion for zero-coupled pair operators. The method can be applied at various levels of approximation which are by their nature exactly number conserving.

The advantages of this approach can be summarized as follows:

(i) The first-order theory provides a very accurate treatment of the seniority-zero ground state and of the various seniority-one and seniority-two states for any value of the interaction strength. A particularly attractive feature of this approximation scheme lies in the fact that it is comparable in simplicity to the usual BCS method. In fact, the numerical work consists only in diagonalizing at each step of the chain calculation matrices of order n_j (we remind the reader that n_j is the number of single-particle levels).

(ii) It is straightforward to improve on the first-order theory. In particular, accurate results for the seniority-zero excited states are easily obtained from higher-order calculations.

(iii) It is possible to treat nonconstant pairing forces at any order of approximation with no increase in complexity.

(iv) As remarked also in the Introduction, the method has been successfully used on the charge-independent pairing model.

Further, it should be mentioned that the method can be straightforwardly applied to models involving nondegenerate pair levels and may therefore provide a profitable way of treating strongly deformed nuclei.

That the present approach represents an advantageous alternative to the BCS theory is proved by the fact that even the lowest order of approximation is capable of yielding results which are vastly improved over those of the usual quasiparticle approximation for both even and odd nuclei (a comparison for the ground-state energy of the even Ni and Sn isotopes may be found in Ref. 52).

At this point, it is of interest to examine the merit of our approach as compared to other methods of dealing with the problem of number conservation in pairing-force theory. The Lipkin-Nogami version of the BCS method yields very good results (not better, however, than our first-order results) for the energies of the seniority-zero ground states,[11,12] but does not lead to any significant improvement over BCS for the states with nonzero seniority.[11] Unlike the BCS approximation, this approach can also describe seniority-zero excited states.[11] In this case, however, the accuracy is not very satisfactory. This situation may be improved by further refinements of the method,[13,14] but the simplicity of this kind of approach is then lost to a large extent. For instance, in the case of the seniority-zero excited states the method has to be combined with number projection and orthogonalization.[14]

Concerning the method of Hara,[22] it yields results which are superior to the BCS ones, but still not very accurate. In addition, the accuracy of the method is strongly dependent on the distribution of the single-particle levels.

The results obtained by Do Dang and Klein[30] and by Jean *et al.*[32] using their lowest order of approximation are very good (they are only slightly worse than our first-order results) for the ground-state energy of the even Ni isotopes, but compare rather badly with the exact results for the energy of the seniority-one states (the results reported in Ref. 32 are somewhat worse than the corresponding BCS ones). The improved (but more complicated) approximation scheme used in Ref. 31 leads to values of the ground-state energy for the even isotopes which are quite close to the exact ones, but is not capable of attaining a good accuracy for the seniority-zero excited states.

The results obtained by Li[6] for the ground-state energies and occupation probabilities for the even Ni isotopes are slightly better than our first-order results. A more complete comparison is not possible since no results for the excited states have been reported in Ref. 6. In any case, this approach is certainly more complicated than ours and therefore does not appear particularly suited for the treatment of the simple pairing-force Hamiltonian.

On the above grounds we may claim that our equations-of-motion approach provides a really simple and effective way of treating pairing correlations in a number conserving manner. However, we consider this achievement not an end in itself but a most useful step towards our main objective. This consists in extending our method to treat further problems of nuclear structure involving more complicated Hamiltonians. Indeed, it is our belief that the method has the power to deal with the relevant features of a many-body problem in a particularly economical and practical way. Efforts in this direction are under way.

---

*Permanent address: Institute of Modern Physics, Lanzhou, Academia Sinica, People's Republic of China.

[1]A. K. Kerman, R. D. Lawson, and M. H. MacFarlane, Phys. Rev. **124**, 162 (1961).

[2]B. F. Bayman, Nucl. Phys. **15**, 33 (1960).

[3]K. Dietrich, H. J. Mang, and J. H. Pradal, Phys. Rev. **135**, B22 (1964); H. J. Mang, J. O. Rasmussen, and M. Rho, *ibid.* **141**, 941 (1966).

[4]A. Arima and F. Iachello, Ann. Phys. (N.Y.) **99**, 253 (1976); **111**, 201 (1978); **123**, 468 (1979).

[5]*Interacting Bosons in Nuclei,* edited by J. S. Dehesa, J. M. G. Gomez, and J. Ros (Springer, Berlin, 1982).

[6]C. T. Li, Phys. Rev. C **29**, 2309 (1984); see also Nucl. Phys. **A417**, 37 (1984).

[7]H. J. Mang, Phys. Rep. **18**, 325 (1975).

[8]K. Hara and S. Iwasaki, Nucl. Phys. **A332**, 61 (1979); K. Hara, S. Iwasaki, and K. Tanabe, *ibid.* **A332**, 69 (1979); K. Hara and S. Iwasaki, *ibid.* **A348**, 200 (1980).

[9]K. W. Schmid, F. Grümmer, and A. Faessler, Phys. Rev. C **29**, 291 (1984); **29**, 308 (1984).

[10]P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer, New York, 1980), Chap. 11.
[11]Y. Nogami, Phys. Rev. **134**, B313 (1964).
[12]Y. Nogami and I. J. Zucker, Nucl. Phys. **60**, 203 (1964).
[13]J. F. Goodfellow and Y. Nogami, Can. J. Phys. **44**, 1321 (1966).
[14]H. C. Pradhan, Y. Nogami, and J. Law, Nucl. Phys. **A201**, 357 (1973).
[15]R. Raj and M. L. Rustgi, Phys. Rev. C **26**, 243 (1982); **28**, 935 (1983).
[16]I. Unna and J. Weneser, Phys. Rev. **137**, B1455 (1965).
[17]J. N. Ginocchio and J. Weneser, Phys. Rev. **170**, 859 (1968).
[18]M. K. Pal and M. K. Banerjee, Phys. Lett. **13**, 155 (1964).
[19]A. Goswami, Nucl. Phys. **60**, 228 (1964).
[20]P. Camiz, A. Covello, and M. Jean, Nuovo Cimento **36**, 663 (1965); **42**, 199 (1966).
[21]K. Hara, Nucl. Phys. **A95**, 385 (1967).
[22]K. Hara, Z. Phys. **202**, 504 (1967).
[23]R. W. Richardson, Phys. Lett. **3**, 277 (1963); R. W. Richardson and N. Sherman, Nucl. Phys. **52**, 221 (1964). See also R. W. Richardson, J. Math. Phys. **6**, 1034 (1965), where it is indicated how to remove the restrictions mentioned in the text.
[24]R. W. Richardson, Phys. Rev. **144**, 874 (1966); **154**, 1007 (1967).
[25]M. Baranger, Phys. Rev. **120**, 957 (1960).
[26]A. K. Kerman and A. Klein, Phys. Rev. **132**, 1326 (1963); A. Klein and A. K. Kerman, *ibid.* **138**, B1323 (1965).
[27]D. J. Rowe, Rev. Mod. Phys. **40**, 153 (1968).
[28]A. Klein, *Progress in Particle and Nuclear Physics,* edited by D. Wilkinson (Pergamon, Oxford, 1983), Vol. 10, p. 39.
[29]M. Jean, Nuovo Cimento **40**, 1224 (1965).
[30]G. Do Dang and A. Klein, Phys. Rev. **143**, 735 (1966).
[31]G. Do Dang and A. Klein, Phys. Rev. **147**, 689 (1966).
[32]M. Jean, X. Campi, and H. Vucetich, in *Fundamentals in Nuclear Theory* (International Atomic Energy Agency, Vienna, 1967), p. 807.
[33]G. Do Dang, R. M. Dreizler, A. Klein, and Chi-Shiang Wu, Phys. Rev. **172**, 1022 (1968).
[34]X. Campi, H. Vucetich, and M. Jean, Nuovo Cimento **55B**, 185 (1968).
[35]R. R. Chasman, Phys. Rev. **156**, 1197 (1967); **169**, 782 (1968).
[36]A. Covello and E. Salusti, Phys. Rev. **162**, 859 (1967).
[37]S. C. Pang and A. Klein, Can. J. Phys. **50**, 655 (1972); C. Dasso, A. Klein, C. Y. Wang-Keiser, and G. J. Dreiss, Nucl. Phys. **A205**, 200 (1973).
[38]J. Mauger and J. A. Evans, Nucl. Phys. **A167**, 16 (1971).
[39]F. Andreozzi, A. Covello, and A. Porrino, Phys. Rev. C **21**, 1094 (1980).
[40]A. Covello, F. Andreozzi, and A. Porrino, in *Proceedings of the Nuclear Physics Workshop, International Center for Theoretical Physics, Trieste, 1981,* edited by C. H. Dasso, R. A. Broglia, and A. Winther (North-Holland, Amsterdam, 1982), p. 311.
[41]F. Andreozzi, A. Covello, A. Gargano, E. E. Maqueda, and R. P. J. Perazzo, Phys. Rev. C **27**, 370 (1983).
[42]M. Bishari, I. Unna, and A. Mann, Phys. Rev. C **3**, 1715 (1971).
[43]M. Hasegawa and N. Kanesaki, Prog. Theor. Phys. **63**, 469 (1980).
[44]D. Janssen and P. Schuck, Z. Phys. A **301**, 255 (1981).
[45]J. Y. Zeng and T. S. Cheng, Nucl. Phys. **A405**, 1 (1983).
[46]M. Hasegawa, Nucl. Phys. **A414**, 173 (1984).
[47]F. Andreozzi, A. Covello, and A. Porrino, Phys. Rev. C **22**, 2555 (1980).
[48]F. Andreozzi, A. Covello, A. Gargano, and Liu Jian Ye, to be submitted to Phys. Rev. C.
[49]F. Andreozzi, A. Covello, A. Gargano, and A. Porrino, in *Proceedings of the Nuclear Shell Models Symposium in Honor of the 60th Birthday of Igal Talmi, Philadelphia, 1984,* edited by D. H. Feng, M. Vallieres, and B. H. Wildenthal (World Scientific, Singapore, 1985).
[50]A. M. Lane, *Nuclear Theory* (Benjamin, New York, 1964), Chap. 1.
[51]F. Andreozzi, A. Covello, A. Gargano, Liu Jian Ye, and A. Porrino (unpublished).
[52]A. Covello, in *Proceedings of the International School of Physics, Enrico Fermi, Course XCI, Varenna, 1984,* edited by A. Molinari and R. A. Ricci (North-Holland, Amsterdam, 1985).