Collective $M1$ states in the classical limit of the neutron-proton interacting boson model

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Using the classical picture of the collective $M1$ mode in nuclei as an oscillation of the angle between the deformed valence protons and valence neutrons, we obtain an expression for the classical limit of the neutron-proton interacting boson model Hamiltonian. We use this result to contrast the excitation energy in the classical limit to the experimentally observed value for the 1^+ state in ^{156}Gd.

Due to the recent discovery of collective magnetic-dipole excitation modes in $^{156}_{64}\mathrm{Gd}_{92}$ (Ref. 1) and other nuclei,[2] there has been considerable theoretical[3–5] and experimental[6] interest in such modes. Although a collective $M1$ mode had been predicted and discussed in earlier works,[5] the motivation for the ^{156}Gd experiment was provided by theoretical studies[4] within the framework of the neutron-proton interacting boson model (or IBM-2).[7] In the IBM-2 picture, 1^+ states arise from the proton-neutron (π-ν) mixed symmetry representations of the group U(6) connected with the s and d bosons. Classically, within the SU(3) limit of the IBM-2, these 1^+ states can be regarded as small amplitude oscillations of the angle between the two symmetry axes of the deformed valence neutrons and valence protons. This classical picture of the collective 1^+ state has been referred to as the scissors mode[4,8] (see Fig. 1).

In an earlier paper[9] we looked at potential energy surfaces in the classical limit of the IBM-2. In the present investigation we expand our previous work so as to include the appropriate kinetic energy terms in the IBM-2 Hamiltonian, so that we can study the properties of the collective 1^+ states, mentioned above, in the classical limit. Specifically, we apply our results to $^{156}_{64}\mathrm{Gd}_{92}$, so that we can make a comparison with the recently obtained experimental results.

The general form of the IBM-2 Hamiltonian for s ($J=0$) and d ($J=2$) proton and neutron bosons is[7,10]

$$H^{\text{IBM-2}}=\epsilon(n_{d_\pi}+n_{d_\nu})+V_{\pi\nu}+H_{\pi\pi}+H_{\nu\nu}\ , \qquad (1)$$

where $H_{\rho\rho}$ ($\rho=\pi,\nu$) is the two-boson interaction among like bosons, ϵ is the single boson excitation energy, and $n_{d_\rho}=(\sum_m d_m^\dagger d_m)_\rho$ is the number operator for d_ρ bosons. The standard form taken for $V_{\pi\nu}$ is

$$V_{\pi\nu}=\kappa Q_\pi\cdot Q_\nu+M_{\pi\nu}\ , \qquad (2)$$

where

$$Q_\rho=[(s^\dagger\tilde{d}+d^\dagger s)^{(2)}+\chi_\rho(d^\dagger\tilde{d})^{(2)}]_\rho \qquad (3)$$

is the quadrupole operator and

$$M_{\pi\nu}=\xi_2(s_\nu^\dagger d_\pi^\dagger-d_\nu^\dagger s_\pi^\dagger)^{(2)}\cdot(s_\nu\tilde{d}_\pi-\tilde{d}_\nu s_\pi)^{(2)}+\sum_{k=1,3}\xi_k(d_\nu^\dagger d_\pi^\dagger)^{(k)}\cdot(\tilde{d}_\nu\tilde{d}_\pi)^{(k)} \qquad (4)$$

is the Majorana operator, which separates the configurations which are totally symmetric under the interchange of the protons and neutrons from those configurations which have mixed symmetry in the protons and neutrons.[10] In the above equations $\tilde{d}_m=(-1)^m d_{-m}$ = spherical tensor, $(\ )^{(i)}$ represents a tensor coupling of rank i, and $(\ )\cdot(\ )$ denotes a scalar product of two equal rank tensors. The quantities ϵ, κ, χ_π, χ_ν, ξ_1, ξ_2, and ξ_3 are variable parameters which are adjusted to obtain the best possible fit with the experimental energy levels for a given nucleus.[10]

We obtain the classical limit of the IBM-2 Hamiltonian (1) using the procedure of Hatch and Levit.[11] This procedure consists of replacing the boson operators s, $s^\dagger$, d_μ,

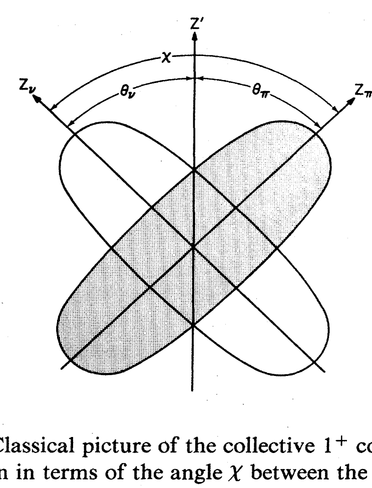

FIG. 1. Classical picture of the collective 1^+ configuration as an oscillation in terms of the angle χ between the two symmetry axes z_ν and z_π of the deformed valence neutrons and valence protons, respectively.

and $d_\mu^\dagger$ by the classical commuting variables, α_s, α_s^*, α_μ, and α_μ^*, respectively. These classical variables are then related to the nuclear deformation parameters q_μ and their canonically conjugate momenta p_μ in the following manner:

$$\alpha_s = e^{-iQ}\left[N - \tfrac{1}{2}\sum_\mu (p_\mu^* p_\mu + q_\mu^* q_\mu)\right]^{1/2} \tag{5}$$

and

$$\alpha_\mu = \frac{1}{\sqrt{2}}(q_\mu^* + ip_\mu)e^{-iQ} , \tag{6}$$

where N is the total number of bosons and

$$Q = i\ln[\alpha_s/\alpha_s^*]^{1/2} .$$

The parameters q_μ and p_μ can then be transformed from the space fixed coordinate system to the intrinsic coordinate system in the usual way,[12] leading to expressions in terms of the Euler angles and their conjugate momenta and in terms of the standard deformation coordinates β and γ and their conjugate momenta p_β and p_γ.[12,13]

In the classical limit of the IBM-2 this procedure must be carried out for both the proton operators and the neutron operators. The completely general expressions, which would be obtained by doing this, would contain ten coordinates, five for the protons and five for the neutrons, with their conjugate momenta. These ten coordinates can be expressed by β_π and γ_π for the protons, β_ν and γ_ν for the neutrons, three Euler angles for the proton distribution, and three Euler angles for the neutron distribution. The completely general result is quite complicated and difficult to obtain.

Since we are only interested in the collective $M1$ configurations, we are able to make a number of approximations, which greatly simplify the general classical IBM-2 results for the Hamiltonian (1). These approximations are the following: (1) the momenta p_{β_ρ} and p_{γ_ρ} ($\rho=\pi,\nu$) are zero, since the shapes of the proton and the neutron distributions are not changing; (2) the momenta related to the first and third proton and neutron Euler angles (which we denote by ϕ_ρ and ψ_ρ, $\rho=\pi,\nu$) are zero (i.e., $p_{\phi_\rho}=p_{\psi_\rho}=0$), since the equilibrium values of ϕ_ρ and ψ_ρ are zero and we consider no motion in terms of these variables; (3) the only motion is in terms of the second proton and neutron Euler angles θ_π and θ_ν, which we take to be $\theta_\pi = \chi/2$ and $\theta_\nu = -\chi/2$ (see Fig. 1), so that p_{θ_π} and p_{θ_ν} are nonzero; and (4) $\gamma_\pi=\gamma_\nu=0$ for prolate deformations and $\beta_\pi=\beta_{\pi_0}$ and $\beta_\nu=\beta_{\nu_0}$, their equilibrium values, which may not be equal. We will consider both the SU(3) values of β_{π_0} and β_{ν_0} (Ref. 14) and the actual equilibrium values for our classical form for the potential. We neglect the like-boson interactions $H_{\rho\rho}$, since they do not play an important role in the $M1$ configurations.

In terms of these approximations the angular momenta L_{1_ρ} and L_{3_ρ} are zero, so that the only nonzero momentum variables b_μ are

$$b_{\pm 1_\rho} = \mp \frac{L_{2_\rho}}{\sin\left[\gamma_\rho - \frac{4\pi}{3}\right]}\left[\frac{1}{2\sqrt{2}\beta_\rho}\right] = \frac{\pm L_{2_\rho}}{\sqrt{6}\beta_{\rho_0}} , \tag{7}$$

where $L_{2_\rho} = p_{\theta_\rho}$ in units of $\hbar = 1$. We then obtain the following expression for the potential energy surface in the classical limit at the minimum conditions described above:

$$\begin{aligned}
V(\beta_{\pi_0},\beta_{\nu_0},\gamma_\pi=\gamma_\nu=0,\chi)
&= \frac{\epsilon}{2}(\beta_{\pi_0}^2+\beta_{\nu_0}^2) + \kappa\Bigg\{2[(N_\pi-\tfrac{1}{2}\beta_\pi^2)(N_\nu-\tfrac{1}{2}\beta_\nu^2)]^{1/2}\beta_{\pi_0}\beta_{\nu_0}(1-\tfrac{3}{2}\sin^2\chi)\\
&\quad -(1-\tfrac{3}{2}\sin^2\chi)\left[\frac{\chi_\nu}{\sqrt{7}}\beta_{\nu_0}^2\beta_{\pi_0}(N_\pi-\tfrac{1}{2}\beta_{\pi_0}^2)^{1/2} + \frac{\chi_\pi}{\sqrt{7}}\beta_{\pi_0}^2\beta_{\nu_0}(N_\nu-\tfrac{1}{2}\beta_{\nu_0}^2)^{1/2}\right]\\
&\quad + \tfrac{1}{14}\chi_\pi\chi_\nu\beta_{\pi_0}^2\beta_{\nu_0}^2(1-3\sin^2\chi)\Bigg\} - \frac{9\xi_1}{160}\beta_{\pi_0}^2\beta_{\nu_0}^2\sin^2\chi\\
&\quad + \tfrac{1}{2}\xi_2\left\{\beta_{\pi_0}^2\left[N_\nu - \frac{\beta_{\nu_0}^2}{2}\right] + \beta_{\nu_0}^2\left[N_\pi - \frac{\beta_{\pi_0}^2}{2}\right] - \beta_{\pi_0}\beta_{\nu_0}(3\cos^2\chi - 1)\left[\left[N_\nu - \frac{\beta_{\nu_0}^2}{2}\right]\left[N_\pi - \frac{\beta_{\pi_0}^2}{2}\right]\right]^{1/2}\right\}\\
&\quad - \frac{3\xi_3}{16}\beta_{\pi_0}^2\beta_{\nu_0}^2\left[\tfrac{1}{5}\sin^2\chi + \tfrac{1}{2}\sin^4\frac{\chi}{2}\right] .
\end{aligned} \tag{8}$$

Note that in the limit $\chi=0$, the above equation reduces to Eq. (7) in Ref. 9 provided that $\beta_\pi=\beta_{\pi_0}$, $\beta_\nu=\beta_{\nu_0}$, and $\gamma_\pi=\gamma_\nu=0$.

We wish to treat the Hamiltonian (1) in the oscillator approximation, assuming that the collective $M1$ mode in Fig. 1 is the *lowest* excited state of a harmonic oscillator in terms of the variables χ and $\dot{\chi}$. That is, we wish to write Eq. (1) in the classical limit in the form

$$H_{\rm cl} = T + V \cong \tfrac{1}{2}\mu\dot{\chi}^2 + C + D\chi^2 , \tag{9}$$

where C and D are quantities independent of $\mathcal{X}$ and $\dot{\mathcal{X}}$,

$$\mu = \text{reduced "mass"} = \frac{m_\pi m_\nu}{m_\pi + m_\nu} , \tag{10}$$

$$\frac{1}{m_\rho} \equiv \left[\frac{\partial^2 H_{\rm cl}}{\partial p_{\theta_\rho}^2} \right]_{\text{all } p=0} , \tag{11}$$

so that the "oscillator energy" is given by

$$\hbar\omega = \sqrt{2D/\mu} . \tag{12}$$

As can be easily seen, the term referred to as C in Eq. (9) is simply the potential energy term we obtained in Ref. 9, Eq. (7), for $\beta_\pi = \beta_{\pi_0}$, $\beta_\nu = \beta_{\nu_0}$, and $\gamma_\pi = \gamma_\nu = 0$.

Proceeding in the manner described above, we obtain D and $1/\mu$ to be

$$D = \tfrac{3}{2}\kappa\beta_{\pi_0}\beta_{\nu_0}\left\{ \frac{1}{\sqrt{14}}[\mathcal{X}_\nu\beta_{\nu_0}(2N_\pi-\beta_{\pi_0}^2)^{1/2}+\mathcal{X}_\pi\beta_{\pi_0}(2N_\nu-\beta_{\nu_0}^2)^{1/2}] - [(2N_\pi-\beta_{\pi_0}^2)(2N_\nu-\beta_{\nu_0}^2)]^{1/2} - \frac{\mathcal{X}_\pi\mathcal{X}_\nu}{7}\beta_{\pi_0}\beta_{\nu_0} \right\}$$
$$- \tfrac{3}{80}\beta_{\pi_0}^2\beta_{\nu_0}^2(\tfrac{3}{2}\xi_1+\xi_3) + \tfrac{3}{2}\xi_2\beta_{\pi_0}\beta_{\nu_0}\left[\left[N_\nu - \frac{\beta_{\nu_0}^2}{2}\right]\left[N_\pi - \frac{\beta_{\pi_0}^2}{2}\right]\right]^{1/2} , \tag{13}$$

$$\frac{1}{m_\rho} = \frac{\epsilon}{3\beta_{\rho_0}^2} + \frac{\kappa}{3(2N_\rho - \beta_{\rho_0}^2)^{1/2}}\left[\frac{\beta_{\rho'_0}}{\beta_{\rho_0}}\right]\left[(2N_{\rho'}-\beta_{\rho'_0}^2)^{1/2}\left[1+\frac{\mathcal{X}_\rho}{\sqrt{14}\beta_{\rho_0}}\right] - \frac{\mathcal{X}_{\rho'}}{\sqrt{14}}\left[\frac{\beta_{\rho'_0}}{\beta_{\rho_0}}\right]\left[1+\frac{\mathcal{X}_\rho}{\sqrt{14}}\right]\right]$$
$$-\left[\frac{\beta_{\rho'_0}}{\beta_{\rho_0}}\right]\left[\frac{\xi_1}{20}+\frac{\xi_3}{30}\right] + \frac{\xi_2}{3}\left\{\frac{1}{2}\left[\frac{\beta_{\rho'_0}}{\beta_{\rho_0}}\right]^2 - \frac{1}{2}\left[\frac{\beta_{\rho'_0}}{\beta_{\rho_0}}\right]\frac{[N_{\rho'}-(\beta_{\rho'_0}^2/2)]^{1/2}}{[N_\rho-(\beta_{\rho_0}^2/2)]^{1/2}} + \frac{4}{\beta_{\rho_0}^2}\left[N_{\rho'}-\frac{\beta_{\rho'_0}^2}{2}\right]\right\} , \tag{14}$$

where $\rho = \pi$, $\rho' = \nu$ or $\rho = \nu$, $\rho' = \pi$, and

$$\frac{1}{\mu} = \frac{1}{m_\pi} + \frac{1}{m_\nu} . \tag{15}$$

In the above results we have kept only the lowest order terms in $\mathcal{X}$ for small $\mathcal{X}$, namely terms of the order $\sin^2\mathcal{X} \simeq \mathcal{X}^2$ and $\cos^2\mathcal{X} \simeq 1 - \mathcal{X}^2$, where $\theta_\pi = \mathcal{X}/2$ and $\theta_\nu = -\mathcal{X}/2$ (cf. Fig. 2).

Since the first collective $M1$ state was observed in ^{156}Gd, we would like to see what Eq. (12) predicts for the excitation energy of the 1^+ state in this nucleus. To determine the values of D and μ appropriate for ^{156}Gd, we use the empirical IBM-2 parameter values obtained by Scholten, as reported in Ref. 15, except for the values of ξ_1, ξ_2, and ξ_3, which, at that time, were chosen at random and *not* to fit any particular level. For ξ_1, ξ_2, and ξ_3, we use Scholten's present values[16] of

$$\xi_1 = \xi_3 = -2\xi_2 = -0.30 \text{ MeV} ,$$

which were obtained by an IBM-2 fit of the 1^+ energy in ^{156}Gd.

For $^{156}_{65}$Gd$_{92}$, $N_\pi = 7$, and $N_\nu = 5$, and Scholten's IBM-2 parameter values[15] are $\epsilon = 0.46$ MeV, $\kappa = -0.081$ MeV, $\mathcal{X}_\pi = -1.0$, and $\mathcal{X}_\nu = -1.1$. We consider two choices for β_{π_0} and β_{ν_0}: (1) the values obtained in the SU(3) limit,[14] i.e., $\beta_{\pi_0} = 2(N_\pi/3)^{1/2}$ and $\beta_{\nu_0} = 2(N_\nu/3)^{1/2}$ and (2) the values obtained by minimizing our potential energy surface Eq. (8), namely, $\beta_{\pi_0} = 0.929(N_\pi)^{1/2}$ and $\beta_{\nu_0} = 0.958(N_\nu)^{1/2}$. These latter values for β_{π_0} and β_{ν_0}, obtained from minimizing our potential energy surface, are the same as those obtained by Pittel[17] applying a Hartree-Bose approach to this problem. For the SU(3) limit values we obtain $D = 13.22$ MeV and $1/\mu = 0.139$ MeV, so that $\hbar\omega = 1.92$ MeV. This energy is too low, clearly indicating that ^{156}Gd does not satisfy the SU(3)

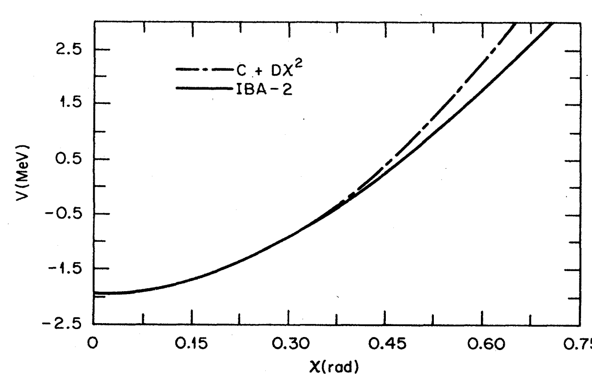

FIG. 2. Comparison of the approximate expression $C + D\mathcal{X}^2$, Eq. (9), with the actual potential energy surface, Eq. (8), for ^{156}Gd. The phenomenological IBA-2 parameters are taken from Refs. 15 and 16.

limit. Using the equilibrium values obtained from Eq. (8), we obtain $D=11.51$ MeV and $1/\mu=0.317$ MeV, so that $\hbar\omega=2.70$ MeV.

Since we have performed what is equivalent to a $1/N$ expansion, where $N_\nu=5$ and $N_\pi=7$ for ^{156}Gd, our result should be accurate to approximately $\pm20\%$ although our actual result agrees to within 12%. Hence, in our model, the proton-neutron antisymmetric collective 1^+ state in $^{156}_{64}Gd_{92}$ occurs at an excitation energy of

$$\hbar\omega \cong 2.70 \text{ MeV} \pm 0.56 \text{ MeV} . \quad (16)$$

This classical result is in good agreement with the observed excitation energy of the collective $M1$ state in ^{156}Gd, namely[1] $E_x(1^+,{}^{156}\text{Gd}) \simeq 3.075$ MeV. Since our potential energy surface produces the same equilibrium values of β_{π_0} and β_{ν_0} as different approaches,[17] the reason for our calculated value of the 1^+ excitation energy being lower than the experimental value is probably due to our simplified treatment of the momentum terms, and, in particular, the use of Eq. (11) to calculate the effective proton and neutron masses.

Having obtained an analytic expression for the energy of the $M1$ state in terms of the phenomenological IBM-2 parameters, we can study how the energy of the collective $M1$ state would change as a function of those parameters. A systematic study of the low-lying energy levels of the deformed Gd isotopes with $A=152$, 154, and 156 has been carried out by Scholten,[15] and the parameters ϵ, κ, χ_π, and χ_ν were determined. Of course, since the $M1$ state is not yet observed for ^{152}Gd and ^{154}Gd, the values of the parameters ξ_1, ξ_2, and ξ_3 cannot be determined. Adopting the convention $\xi_1=\xi_3=-2\xi_2=-\xi$, we plot the energy of the collective $M1$ state, Eq. (12), for these isotopes as a function of ξ in Fig. 3. As expected, for $\xi=0$, the energy of the antisymmetric 1^+ state would be much lower: It is the Majorana term in Eq. (2) which pushes the energy of this state up. Furthermore, the energy of this state varies smoothly as ξ increases, and for similar values of ξ it is the highest for lighter isotopes. This is a result of the dependence of β_{π_0} and β_{ν_0} on the $(N_\pi)^{1/2}$ and $(N_\nu)^{1/2}$, respectively, so a similar trend would have been found using the SU(3) values for β_{π_0} and β_{ν_0}. Hence, *if ξ does not change appreciably* for those Gd isotopes, then we expect the energy of the yet-to-be-observed neutron-proton antisymmetric 1^+ states in ^{154}Gd and ^{152}Gd to be higher than 3 MeV. Recent microscopic calculations[18,19] of ξ_1, ξ_2, and ξ_3 for *nondeformed* nuclei also suggest that the energy of the collective 1^+ state moves down in energy as the neutron boson number increases; however, these calculations also find that ξ_1, ξ_2, and ξ_3 are strongly dependent on N_ν and that $\xi_1 \neq \xi_3 \neq -2\xi_2$. Recent data[20] suggest that in *deformed* nuclei these levels go up in energy as the number of bosons increases. Further data on the properties of these states in both deformed and nondeformed regions are needed.

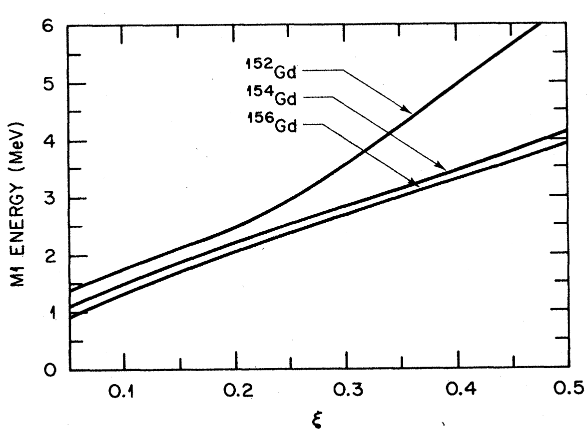

FIG. 3. Variation of the energy of the neutron-proton antisymmetric $M1$ state as a function of the strength ξ of the Majorana term with $\xi_1=\xi_3=-2\xi_2=-\xi$ for the Gd isotopes. The phenomenological IBM-2 parameters are taken from Ref. 15.

From our Eqs. (13)—(15), it is worth noting that the ξ_2 and ξ_3 terms, as well as the ξ_1 term, contribute to the magnitudes of D and $1/\mu$. In fact, the contribution of the ξ_2 term to $1/\mu$ is two or three times the size of the contribution of the ξ_1 and ξ_3 terms because of a larger numerical coefficient and a slightly different dependence on N_π and N_ν. At first, this may seem surprising, since one might expect the ξ_1 term to dominate in the contribution to the antisymmetric 1^+ state. This is true if only two d bosons (one d_π and one d_ν) can occur in the nuclear wave function. If three or more d bosons can occur, then the ξ_2 and ξ_3 terms in the Majorana interaction also contribute to the energy of the 1^+ state.

To study this effect, we first diagonalized the IBM-2 Hamiltonian (1) for ^{156}Gd using Scholten's parameter values[15] and

$$\xi_1=\xi_3=-2\xi_2=-0.30 \text{ MeV}$$

and obtained an excitation energy of 3.09 MeV for the first 1^+ state. The wave function for this state was highly collective with almost equal probabilities for components with four d bosons up to those with eight d bosons. We then repeated the calculation changing only the parameters ξ_1, ξ_2, and ξ_3 to the values $\xi_1=-0.30$ MeV and $\xi_2=\xi_3=0.0$ MeV. This yielded a 1^+ excited state at 2.38 MeV with the wave function now being 23% two d bosons and 36% four d bosons. Finally, we recovered the experimental value of the 1^+ excitation energy by setting $\xi_1=-0.60$ MeV and $\xi_2=\xi_3=0.0$ MeV, which produced $E_x(1^+,{}^{156}\text{Gd})=3.08$ MeV with the wave function being 51% two d bosons and 34% four d bosons. Hence, it is possible to obtain the experimental value for the excitation energy of the 1^+ state using only the ξ_1 term, but this yields a far less collective 1^+ state than one obtained at the same energy with $\xi_1=\xi_3=-2\xi_2$.

On the other hand, the microscopic calculations mentioned previously[18,19] indicate that $\xi_1 \neq \xi_3 \neq -2\xi_2$. Hence, further work is needed to understand the relationship among ξ_1, ξ_2, and ξ_3, if indeed any such relationship does exist.

To summarize, we have obtained a classical form of the

IBM-2 Hamiltonian appropriate for the collective, magnetic-dipole 1^+ state. Using the empirical parameter values, obtained by fitting the quantum mechanical IBM-2 Hamiltonian to the data for ^{156}Gd, we calculated from our classical result a 1^+ excitation energy in reasonable agreement with the observed 1^+ energy in ^{156}Gd. We also found that the 1^+ excitation energy varied smoothly with changes in the Majorana strength term, i.e., $-\xi=\xi_1=\xi_3=-2\xi_2$, so that one can study how the location of the collective 1^+ state changes with ξ in neighboring nuclei. Our results would suggest that the 1^+ excitation energy decreases as the neutron boson number increases for constant values of ξ_1, ξ_2, and ξ_3. Microscopic studies for *nondeformed* nuclei indicate a similar trend for the 1^+ excitation energy with respect to N_ν, but for $\xi_1\neq\xi_3\neq-2\xi_2$ and for values of ξ_1, ξ_2, and ξ_3 changing with N_ν. Some recent experimental results for *deformed* nuclei show an increase in the 1^+ excitation energy as N_ν increases. Further experimental and theoretical studies are required to clarify our understanding of these 1^+ states.

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[1]D. Böhle *et al.*, Phys. Lett. **137B**, 27 (1984).

[2]A. E. L. Dieperink, invited talk, International Symposium on Nuclear Shell Models, Drexel University, 1984.

[3]D. R. Bes and R. A. Broglia, Phys. Lett. **137B**, 141 (1984); E. Lipparini and S. Stringari, *ibid.* **130B**, 139 (1983); I. Hamamoto and S. Åberg, *ibid.* **145B**, 163 (1984); P. Van Isacker *et al.*, *ibid.* **144B**, 1 (1984); S. Pittel, J. Dukelsky, R. P. J. Perazzo, and H. M. Sofia, *ibid.* **144B**, 145 (1984); B. R. Barrett and P. Halse, University of Arizona report, 1984.

[4]F. Iachello, Nucl. Phys. **A358**, 89c (1981); A. E. L. Dieperink, Prog. Part. Nucl. Phys. **9**, 121 (1983); M. Sambataro, O. Scholten, A. E. L. Dieperink, and G. Piccito, Nucl. Phys. **A423**, 333 (1984).

[5]N. Lo Iudice and F. Palumbo, Phys. Rev. Lett. **41**, 1532 (1978); Nucl. Phys. **A326**, 193 (1979); T. Suzuki and D. J. Rowe, *ibid.* **A289**, 461 (1977).

[6]U. E. P. Berg *et al.*, University of Giessen report, 1984.

[7]A. Arima, T. Otsuka, F. Iachello, and I. Talmi, Phys. Lett. **66B**, 205 (1977); T. Otsuka, A. Arima, F. Iachello, and I. Talmi, *ibid.* **76B**, 139 (1978).

[8]A. Richter, in *Proceedings of the International Conference on Nuclear Physics, Florence, 1983,* edited by P. Blasi and R. A. Ricci (Tipografia Compositori, Bologna, 1983), Vol. 2, pp. 189—217.

[9]A. B. Balantekin, B. R. Barrett, and S. Levit, Phys. Lett. **129B**, 153 (1983).

[10]B. R. Barrett, Rev. Mex. Fis. **27**, 533 (1981); in *Interacting Bosons in Nuclear Physics,* edited by F. Iachello (Plenum, New York, 1979).

[11]R. L. Hatch and S. Levit, Phys. Rev. C **25**, 614 (1982).

[12]J. M. Eisenberg and W. Greiner, *Nuclear Models* (North-Holland, Amsterdam, 1970).

[13]A. Bohr and B. R. Mottelson, *Nuclear Structure* (Benjamin, New York, 1975), Vol. II.

[14]A. E. L. Dieperink, Nucl. Phys. **A421**, 189 (1984).

[15]O. Scholten, Kernfysisch Versneller Institut Annual Report, 1979, p. 105; Ph.D. thesis, University of Groningen, 1980, p. 73.

[16]O. Scholten, private communication.

[17]S. Pittel, private communication.

[18]A. van Egmond and K. Allaart, Nucl. Phys. **A425**, 275 (1984).

[19]C. H. Druce, S. Pittel, B. R. Barrett, and P. D. Duval, University of Arizona report, 1984.

[20]D. Böhle, G. Küchler, A. Richter, and W. Steffen, Technische Hochschule, Darmstadt report, 1984.