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Boson expansion theory in the seniority scheme

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A boson expansion formalism in the seniority scheme is presented and its relation with number-conserving quasiparticle calculations is elucidated. Accuracy and convergence are demonstrated numerically. A comparative discussion with other related approaches is given.

I. INTRODUCTION

Some two years ago, Kishimoto and Tamura[1] undertook a major reformulation of the boson expansion theory (BET). In it a unified derivation was given of various results of this theory that had been worked out earlier by a number of authors.[2,3] (Henceforth, we shall refer to Ref. 1 as KT3, as it has been done in a few of our papers that followed it.)

More recently, Tamura[4] showed, by using a major part of the results obtained in KT3, that the bosonization procedure can be simplified significantly. The new method, called a term-by-term bosonization (TTB) method, was subsequently used to derive, e.g., a boson expansion in the random phase approximation (RPA) representation,[5] and it will be used also in the present paper.

Although the eventual goal of any BET is its application to realistic (and complicated) nuclear systems, it is instructive to see how a theory works for simple models. It is particularly helpful in comparing different approaches, because an exact fermion calculation, against which every boson method is to be compared, can also be easily performed. The main purpose of the present paper is to take up a single-j shell model ($1j$-SM), and show how to bosonize it by using the TTB model.

It is well known that the $1j$-SM is characterized by the possibility of introducing the concept of *seniority* and of constructing basis states in the seniority scheme. A powerful method to calculate the matrix elements exists for this scheme, known as seniority reduction (SR). The SR will be used to full advantage in the present paper. More precisely, we will carry out the SR completely first, and then apply the TTB method to bosonize the reduced matrix element. The formalism is given in Sec. II and it will be called SR + BET.

The SR is a good example of what is usually done in the fermion stage of the formulation, before the bosonization itself is undertaken. Another well-known example of this is the use of the BCS (Bardeen-Cooper-Schrieffer) approximation. This theory is particularly powerful, thanks to the use of the quasiparticle (QP) description, and it has been extensively used in practical applications of our boson formalism, which has been sometimes called BCS + BET. Let us remark that BCS remains a reasonably accurate theory, so long as it is applied to relatively low-lying states in systems with a large effective degeneracy and a large number of valence nucleons. However, these restrictions naturally determine the bounds of applicability of the BCS + BET.

Recently, Li[6] set forth a method, called the number-conserving quasiparticle (NCQP) method, that removes the major part of the error in BCS, preserving, nevertheless, the merits of the quasiparticle description. In Sec. III we thus discuss the NCQP + BET and its relation to the SR + BET, as well as to the BCS + BET. An important aspect of the NCQP + BET is that it becomes an *exact theory* in the $1j$-SM, and is thus equivalent to the SR + BET, which is also exact. (When we say that a BET is exact, we implicitly assume that the boson expansion is carried out to a sufficiently higher order.) The BCS + BET is a common approximate of these two theories. The presentation of SR + BET in Sec. II and of NCQP + BET in Sec. III accomplishes the major goal of the present paper.

The bosonization of the $1j$-SM in the seniority scheme was also done earlier by Otsuka, Arima, and Iachello (OAI),[7] and by Otsuka, Arima, Iachello, and Talmi (QAIT).[8] In Sec. IV we thus recapitulate their formalisms. As we shall see, both OAI and OAIT make use of the SR in their bosonization procedure, but in two different ways, both of which differ from ours. These differences came about because different guiding principles were adopted for the bosonization.

In Sec. V we present some numerical results obtained by using SR + BET, OAI, and OAIT. These calculations are taken as a convenient test bench for checking and comparing the accuracy and the convergence rate that these three methods provide. We then discuss a few aspects and implications of the OAI and OAIT work and its relation to the IBA (interacting boson approximation).[9]

One important point we raise in connection with these boson theories is the significance of the s bosons. Based on the results of Secs. II and III, we shall argue that, although the s boson does carry some physical information about the original fermion system, its explicit presence in the boson formalism is not mandatory. What we show in the present paper is that the elimination of the s boson, which is a well-known fact in the phenomenological IBA, can be demonstrated and justified from a number conserving approach to boson mapping.

II. BOSONIZATION OF THE 1j-SM CALCULATION

A. Seniority reduction

As mentioned in the Introduction, we take up in the present paper a 1j-SM, a model also considered by OAI. Let $a_m^\dagger = a_{jm}^\dagger$ and $a_{\tilde{m}} = (-)^{j-m} a_{j,-m}$ be the creation and annihilation operators defined within this shell. The pair creation and the scattering operators are then defined by

$$B_{\lambda\mu}^\dagger = \frac{1}{\sqrt{2}} \sum_{mm'} (jmjm' \mid \lambda\mu) a_m^\dagger a_{m'}^\dagger \ , \qquad (2.1a)$$

$$C_{\lambda\mu}^\dagger = \sum_{mm'} (-)^{j-m'} (jmj-m' \mid \lambda\mu) a_m^\dagger a_{m'} \ . \qquad (2.1b)$$

Note that we are using the notation of KT3. Our $B_{\lambda\mu}^\dagger$ and $C_{\lambda\mu}^\dagger$ are, respectively, equal to $A_{\lambda\mu}^\dagger$ and $-U_{\lambda\mu}$ of OAI. In (2.1), $(jmjm' \mid \lambda\mu)$ is a Clebsch-Gordan coefficient. Later we also use $B_{\lambda\mu} = (B_{\lambda\mu}^\dagger)^\dagger$ and $B_{\lambda\tilde{\mu}} = (-)^{\lambda-\mu} B_{\lambda,-\mu}$, and so forth.

We define the S_+, S_-, and S_0 operators as

$$S_+ = \sum_{m>0} a_m^\dagger a_{\tilde{m}}^\dagger \ ,$$
$$S_- = (S_+)^\dagger \text{ and } S_0 = (\hat{n} - \Omega)/2 \ , \qquad (2.2)$$

where $\Omega = j + \frac{1}{2}$, and $\hat{n} = \sum_m a_m^\dagger a_m$ is the number operator. These S_+, S_-, and S_0 operators are exactly the same as those in OAI.

We also define the $D_\mu^\dagger$ operator by

$$D_\mu^\dagger = P B_{2\mu}^\dagger \ ;$$
$$P = [1 + (4S_0 - 6)^{-1} S_+ S_-][1 + (2S_0 - 2)^{-1} S_+ S_-] \ . \qquad (2.3)$$

Thus our $D_\mu^\dagger$ is again exactly the same as that in OAI. As shown in OAI, the presence of the projection operator $\hat{P}$ guarantees that a state of the form

$$|v,v;\alpha\rangle = \frac{1}{\sqrt{(v/2)!}} \frac{1}{N_\alpha} D_{\mu_1}^\dagger D_{\mu_2}^\dagger \cdots D_{\mu_{v/2}}^\dagger |0\rangle \qquad (2.4)$$

is a highest seniority state with seniority v. In (2.4), $|0\rangle$ denotes the vacuum, and

$$\frac{1}{\sqrt{(v/2)!}} \frac{1}{N_\alpha}$$

is a normalization factor.

As seen, the "product state" of (2.4) is defined in the M representation (magnetic quantum number representation). Therefore, the parameter α stands for the set of magnetic quantum numbers $\mu_1, \mu_2, \ldots, \mu_{v/2}$. When we switch to the I representation, with good angular momenta, the resultant states are nothing but linear combinations of the states of (2.4). Therefore, such new states also maintain their highest seniority nature. In the I representation, we shall continue to use the same index α as in the M representation. It must be understood that now the parameter α denotes a particular scheme of angular momentum coupling.

States which are not of highest seniority may be constructed as

$$|n,v,\alpha\rangle = N_{n,v} (S_+)^{(n-v)/2} |v,v;\alpha\rangle \ , \qquad (2.5a)$$

$$N_{n,v} = \sqrt{[\Omega - (1/2)(n+v)]!} / \sqrt{(\Omega - v)! [(1/2)(n-v)]!} \ . \qquad (2.5b)$$

Equation (2.5a) is of course valid for $n \geq v$, reducing to an identity for $n = v$.

The seniority reduction allows one to express the matrix element of an operator between a bra state $\langle n',v' |$ and a ket state $| n,v \rangle$ in terms of the matrix elements of related operators between the states $\langle n_1',v' |$ and $| n_1,v \rangle$, with $n' \geq n_1' \geq v'$ and $n \geq n_1 \geq v$. If $n_1' = v'$ and $n_1 = v$, we say that a complete SR has been carried out; otherwise, only a partial (or incomplete) SR has been performed. In the following, when referring to the SR we always refer to the complete SR, unless otherwise specified.

Let us take here first $B_{\lambda\mu}^\dagger$ as such an operator. The seniority reduction formula in this case reads[10,11]

$$\begin{aligned} \langle n+2,v';\alpha' | B_{\lambda\mu}^\dagger | n,v;\alpha\rangle = {} & U(n+1,v+1) U(n,v) \langle v',v';\alpha' | \hat{B}_{\lambda\mu}^\dagger | v,v;\alpha\rangle \delta_{v';v+2} \\ & - U(n,v) V(n+2,v) \sqrt{2} \langle v',v';\alpha' | (\hat{C}_{\lambda\mu}^\dagger - \sqrt{\Omega}\delta_{\lambda 0}) | v,v;\alpha\rangle \delta_{v'v} [1 + (-)^\lambda]/2 \\ & - V(n+2,v-2) V(n+1,v-1) \langle v',v';\alpha' | \hat{B}_{\lambda\tilde{\mu}} | v,v;\alpha\rangle \delta_{v',v-2} \ . \end{aligned} \qquad (2.6)$$

Note that the right-hand side (rhs) of (2.6) [as well as (2.8) below] consists of three terms, corresponding to $v' = v+2$, v, and $v-2$. These terms will henceforth be called the $\Delta v = 2$, 0, and -2 terms (as also done in OAI). Note also that we have put a caret on the operators on the rhs, although the careted operators are here the same as those without the caret. We did this to emphasize the fact that they now appear only in the reduced matrix elements, being associated with a particular seniority selection rule, namely $\hat{B}_{2\mu}^\dagger$ is associated with $\Delta v = 2$, $\hat{B}_{2\mu}$ with $\Delta v = -2$, and $\hat{C}_{2\mu}^\dagger$ with $\Delta v = 0$. Furthermore, this new notation will be very conveniently used later in subsection C in the bosonization process.

The coefficients $U(n,v)$ and $V(n,v)$ introduced in (2.6) are defined as

$$U(n,v) = \left[\frac{2\Omega - n - v}{2(\Omega - v)} \right]^{1/2} \ ,$$
$$V(n,v) = \left[\frac{n - v}{2(\Omega - v)} \right]^{1/2} \ . \qquad (2.7)$$

Notice that Eq. (2.6), together with (2.7), is exactly the same as the formula given just below Eq. (A3.23) by Lawson.[10] Here we have used the U and V notation, however, firstly for convenience and, secondly because it helps to directly relate the results of this section with those of Sec. III.

The seniority reduction of the $B_{\lambda\mu}$ operator can be obtained as the Hermitian conjugate of (2.6). The seniority reduction formula for the scattering operator $C^\dagger_{\lambda\mu}$ is given as

$$\langle n,v';\alpha' | C^\dagger_{\lambda\mu} | n,v;\alpha\rangle = \sqrt{2}U(n-1,v+1)V(n,v)\langle v',v';\alpha' | \hat{B}^\dagger_{\lambda\mu} | v,v;\alpha\rangle \delta_{v',v+2} + [U^2(n,v)-(-)^\lambda V^2(n,v)]\langle v',v';\alpha' | \hat{C}^\dagger_{\lambda\mu} | v,v;\alpha\rangle \delta_{vv'} + V^2(n,v)\sqrt{2\Omega}\delta_{\lambda 0}\delta_{vv'} + \sqrt{2}U(n-1,v-1)V(n,v-2)\langle v',v';\alpha' | \hat{B}_{\lambda\mu} | v,v;\alpha\rangle \delta_{v',v-2} \; . \tag{2.8}$$

We shall henceforth call the matrix elements that appear on the rhs of (2.6) and (2.8) the *reduced matrix elements*. By construction, these reduced matrix elements contain only the highest seniority states of the form of (2.4) as the basis states. In other words the S_+ operators have been completely eliminated from the basis states, a fact which now permits one to directly apply the bosonization method of KT3, combined with the TTB method. (This is because $[D^\dagger_\mu, D^\dagger_{\mu'}]=0$. Note that the BET of KT3 was formulated by assuming that all creation operators commute. However, we see that $[S_+, D^\dagger_\mu]\neq 0$.)

The S_+, S_-, and S_0 operators can still appear as operators in the matrix elements in (2.6) and (2.8), when we set $\lambda=\mu=0$, because

$$B^\dagger_{00}=\frac{1}{\sqrt{\Omega}}S_+ , \quad B_{00}=\frac{1}{\sqrt{\Omega}}S_- , \quad \text{and } C^\dagger_{00}=\frac{1}{\sqrt{2\Omega}}\hat{n} \; . \tag{2.9a}$$

With these operators, however, the relations in (2.6) and (2.8) are drastically simplified, in particular, because all the reduced matrix elements of the S_+ and S_- operators vanish identically. We in fact have

$$\langle n+2,v';\alpha' | B^\dagger_{00} | n,v;\alpha\rangle = \frac{1}{\sqrt{\Omega}}[(\Omega-v)U(n,v)V(n+2,v)]\delta_{\alpha\alpha'}\delta_{vv'} \; , \tag{2.9b}$$

$$\langle n,v';\alpha | C^\dagger_{00} | n,v;\alpha\rangle = \frac{n}{\sqrt{2\Omega}}\delta_{\alpha\alpha'}\delta_{vv'} \; . \tag{2.9c}$$

This means that the calculations of the matrix elements of the $B^\dagger_{00}$ and $C^\dagger_{00}$ operators can be completed analytically. Therefore, we may now concentrate on the calculation of the reduced matrix elements for the cases in which $\lambda=2$.

B. Calculation of the reduced matrix elements

In this subsection we shall present the calculation of the reduced matrix elements. We do this mainly because the expressions obtained in this subsection will be needed in what follows, but also because the algebra, although straightforward, becomes somewhat involved due to the presence of the projection operator $\hat{P}$.

The general form of the basis states we take to start with is that of (2.5), where $n \geq v$. Since we can now restrict ourselves to states with $n=v$ that involve $D^\dagger_\mu$ only, we may denote these as $| D^{v/2};\alpha\rangle$ instead of $| v,v;\alpha\rangle$.

The state with only one $D^\dagger_{2\mu}$ may be written as

$$| D;2\mu\rangle = D^\dagger_\mu | 0\rangle = \hat{P}B^\dagger_{2\mu} | 0\rangle = B^\dagger_{2\mu} | 0\rangle \; . \tag{2.10}$$

Namely, in this case the effect of the operator P is *nil*, because $B^\dagger_{2\mu} | 0\rangle$ is a pure highest seniority state already.

An explicit construction of states with two pairs requires somewhat lengthy algebra, and after carrying this out, we obtain

$$| D^2;IM\rangle\rangle = \frac{1}{\sqrt{2}}[D^\dagger_2 D^\dagger_2]_{IM} | 0\rangle = \frac{1}{\sqrt{2}}\{[B^\dagger_2 B^\dagger_2]_{IM} | 0\rangle + \sqrt{2\Omega}[(1+\delta_{I0})\Omega-2]^{-1} \times e_{22I}B^\dagger_{00}B^\dagger_{IM} | 0\rangle\} \; . \tag{2.11}$$

As seen, we have switched to the I representation, the notation $[B^\dagger_2 B^\dagger_2]_{IM}$ standing for the usual angular momentum coupling. The coefficient e_{22I} that appears in (2.11) is a special case of a more general coefficient

$$e_{\lambda\lambda'\lambda''}=2\hat{\lambda}\hat{\lambda}'W(jj\lambda\lambda';\lambda''j) \; , \tag{2.12}$$

where $\hat{\lambda}=\sqrt{2\lambda+1}$, while $W(jj\lambda\lambda';\lambda''j)$ is a Racah coefficient.

To be remarked about the result of (2.11) is the appearance of the $B^\dagger_{00}$ and the $B^\dagger_{IM}$ (with $I=0$, 2, and 4) operators. In other words, in spite of our restriction to the $D^\dagger_{2\mu}$ operators in constructing the highest seniority states, the $B^\dagger_{IM}$ with $I\neq 2$ appears in an explicit construction of these states. Since this fact does not play any significant role within the context of the present paper, however, we shall not discuss this point any further. (Note that a problem related to this has been discussed in detail in KT3. It normally makes it necessary to introduce a norm matrix, rather than simple norm quantities.)

The states in (2.11) are orthogonal with respect to I, but they are not normalized. The norm of these states can be easily calculated, and it is given as

$$N_I=\sqrt{1-y_I} \; , \tag{2.13}$$

with

$$y_I = 50 \begin{Bmatrix} j & j & 2 \\ j & j & 2 \\ 2 & 2 & I \end{Bmatrix} + 100 \frac{W^2(jj22;Ij)}{(1+\delta_{I0})\Omega - 2} . \quad (2.14)$$

Here the curly bracket stands for a nine-j symbol, and W denotes a Racah coefficient.

Note that the leading term of N_I is 1 (rather than, e.g., 2), when N_I is expressed as a power series of y_I. This convenient normalization of N_I resulted because we had the $(1/\sqrt{2})$ factor in (2.11), which is a particular case of the $[(v/2)!]^{-1/2}$ factor originally introduced in (2.4).

The normalized two-pair state, which we denote by $|D^2;IM\rangle$, is now given as

$$|D^2;IM\rangle = \frac{1}{\sqrt{2}N_I}[D_2^\dagger D_2^\dagger]_{IM}|0\rangle . \quad (2.15)$$

[Compare this with (2.4).] The one-pair state $|D;2\mu\rangle$ in (2.10) is already normalized, and we shall now calculate the matrix element in $B_{2\mu}^\dagger$, between this state and the two-pair state of (2.15). By more or less repeating the algebra, through which we obtained (2.13) and (2.14), we find that

$$\langle D^2;IM | B_{2\mu}^\dagger | D;2\mu'\rangle = \sqrt{2}N_I(2\mu 2\mu' | IM) . \quad (2.16)$$

This result will be found very convenient in obtaining the boson image of $B_{2\mu'}^\dagger$ in the next subsection.

C. Bosonization

As remarked in the Introduction, we carry out here the bosonization in the seniority scheme by using the TTB method, the basic idea of which is as follows. The TTB method begins by recognizing that a matrix element of a fermion operator is, in general, a reducible tensor, and thus can be expanded as a sum of irreducible tensors. (In KT3, this was called a linked-cluster expansion of a fermion matrix element.)

This expansion has, in general, a rather fixed pattern, so that each term can be easily replaced by a matrix element of a boson operator, the pair of boson states that are used to construct these matrix elements being common to all the elements. The sum of the above boson operators can then be regarded as the boson image of the original fermion operator.

We shall illustrate the use of the TTB method by first taking the operator $B_{2\mu'}^\dagger$. We find it easiest to present an algebraic step first, and then explain what has been done. The algebraic step is that

$$\begin{aligned}(1/\sqrt{2})\sum_{IM}(2\rho 2\rho' | IM)\langle D^2;IM | B_{2\mu'}^\dagger | D;2\mu\rangle &= \sum_{IM}(2\rho 2\rho' | IM)N_I(2\mu 2\mu' | IM) \\ &= \sum_{IM}(2\rho 2\rho' | IM)(2\mu 2\mu' | IM)[1+(N_I-1)] \\ &= \Delta_{\rho\rho';\mu\mu'} + \sum_{IM}(N_I-1)(2\rho 2\rho' | IM)(2\mu 2\mu' | IM) . \end{aligned} \quad (2.17)$$

The first line of (2.17) was obtained by multiplying $(1/\sqrt{2})(2\rho 2\rho' | IM)$ onto the left-hand side (lhs) of (2.16) and then summing over I and M. The purpose of doing this is to create a quantity which is a tensor of rank 2. The first equality of (2.17) is a result of using (2.16). We then replaced N_I by $1+(N_I-1)$, explaining the second equality. The last equality then shows that the summations over I and M are carried out analytically for the "1" term, resulting in the first term of the last line of (2.17). There

$$\Delta_{\rho\rho';\mu\mu'} = (\delta_{\rho\mu}\delta_{\rho'\mu'} + \rho_{\rho\mu'}\delta_{\rho'\mu})/2 .$$

This term is to be interpreted as consisting of a tensor of rank 0, being multiplied by a Kronecker delta $\Delta_{\rho\rho';\mu\mu'}$ of rank 2. On the other hand the second term of the last line of (2.17) is already an irreducible tensor of rank 2. In this way the reduction of the starting tensor into its irreducible components has been completed.

Our goal here is to bosonize the reduced matrix elements that appear on the rhs of (2.6) and (2.8), or, in operator language, to obtain the boson images of the careted operators introduced there. Note that the "caret" notation, which was redundant in (2.6) and (2.8), is not so any more because the careted operators are precisely the objects to be bosonized.

We start the bosonization procedure by constructing one and two quadrupole boson states as

$$|d;2\mu) = d_\mu^\dagger |0) \text{ and } |d^2;IM) = \frac{1}{\sqrt{2}}[d^\dagger d^\dagger]_{IM}|0) , \quad (2.18)$$

$|0)$ being the boson vacuum. Our objective is to find a boson image $(\hat{B}_{2\mu}^\dagger)_B$ of $\hat{B}_{2\mu}^\dagger$ such that

$$\frac{1}{\sqrt{2}}\sum_{IM}(2\rho 2\rho' | IM)(d^2;IM | (\hat{B}_{2\mu'}^\dagger)_B | d;2\mu) \quad (2.19)$$

again gives rise to the expression in the last line of (2.17). This goal is achieved essentially by inspection, and the result is that

$$(\hat{B}_{2\mu}^\dagger)_B = d_\mu^\dagger + \sum_I (N_I-1)(\hat{I}/\sqrt{5})[[d^\dagger d^\dagger]_I \tilde{d}]_{2\mu} . \quad (2.20)$$

When inserted in (2.19), the first and second terms of (2.20) give rise, respectively, to the first and second terms in the last line of (2.17). This fact is the reason why we call the method used above a TTB method.

By obtaining (2.20), we have accomplished the bosonization of $\hat{B}^{\dagger}_{2\mu}$ to the third-order term (a term containing three $d^{\dagger}$ and/or d factors). If we repeat the above procedure, by first calculating the matrix element of $B^{\dagger}_{2\mu}$ between two and three pairs states, we can obtain $(\hat{B}^{\dagger}_{2\mu})_B$, including the fifth-order term; and so forth.

The bosonization of the $\hat{C}^{\dagger}_{2\mu}$ operator can be done similarly. To second order, it is given as

$$(\hat{C}^{\dagger}_{2\mu})_B = 10W(jj22;2j)[d^{\dagger}\tilde{d}]_{2\mu} . \tag{2.21}$$

After thus completing the bosonization of the two basic pair operators, we may summarize what has been done in the present section. Suppose we want to calculate the matrix element $\langle n,v+2;\alpha' | C^{\dagger}_{2\mu} | n;v;\alpha\rangle$. The calculation is done in the following two steps;

$$\begin{aligned}\langle n,v+2;\alpha' | C^{\dagger}_{2\mu} | n,v;\alpha\rangle &= \sqrt{2}U(n-1,v+1)V(n,v)\langle D^{(v/2)+1};\alpha' | B^{\dagger}_{2\mu} | D^{v/2};\alpha\rangle \\ &= \sqrt{2}U(n-1,v+1)V(n,v)(d^{(v/2)+1};\alpha' | (\hat{B}^{\dagger}_{2\mu})_B | d^{v/2};\alpha) .\end{aligned} \tag{2.22}$$

In other words, the seniority reduction is done first, and then the reduced matrix element is bosonized.

When bosonizing a fermion problem, one often speaks of the mapping of both the operators and the states. In the procedure we employed above only the reduced (highest seniority) states are mapped to boson states, the latter including only the $d^{\dagger}$ bosons. In other words, we do not map the full fermion states $| n,v;\alpha\rangle$ onto boson states that may include also the monopole $s^{\dagger}$ bosons. This, however, does not mean that the boson space we work in is narrower than the original fermion space spanned by $| n,v;\alpha\rangle$. Any calculation is done for a fixed n (which in reality corresponds to choosing a particular isotope). For a fixed n the number of states of the form $(S_+)^{(n-v)/2} | D^{v/2};\alpha\rangle$ is of course the same as the number of states of the form $| D^{v/2};\alpha\rangle$ [and of the form $| d^{v/2};\alpha)$]. There is a perfect one-to-one correspondence between these sets of states.

Throughout this section we have used the matrix element representation for the pair creation and scattering operators. For convenience we shall also give here the full boson images of these operators in their *operator forms.* This is to integrate the U and V factors into the operator part. This allows us to group together the three pieces of matrix elements, with $\Delta v = 2$, 0, and -2, into one, and thus to express the boson images in compact forms. The results are given as

$$\begin{aligned}(B^{\dagger}_{2\mu})_B &= (\hat{B}^{\dagger}_{2\mu})_B U(n+1,2\hat{n}_d+1)U(n,2\hat{n}_d) \\ &\quad -\sqrt{2}(\hat{C}^{\dagger}_{2\mu})_B U(n,2\hat{n}_d)V(n+2,2\hat{n}_d) \\ &\quad -V(n+2,2\hat{n}_d)V(n+1,2\hat{n}_d+1)(\hat{B}_{2\tilde{\mu}})_B\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}(C^{\dagger}_{2\mu})_B &= \sqrt{2}(\hat{B}^{\dagger}_{2\mu})_B U(n-1,2\hat{n}_d+1)V(n,2\hat{n}_d) \\ &\quad +(\hat{C}^{\dagger}_{2\mu})_B[U^2(n,2\hat{n}_d)-V^2(n,2\hat{n}_d)] \\ &\quad +\sqrt{2}U(n-1,2\hat{n}_d+1)V(n,2\hat{n}_d)(\hat{B}_{2\tilde{\mu}})_B .\end{aligned} \tag{2.24}$$

As seen in (2.23) and (2.24) the U and V factors are now functions of the (d boson number) operator $\hat{n}_d$, which replaces the seniority dependence we had in the matrix element form. The explicit forms of these U and V operators are obtained from (2.7), by simply replacing v there by $2\hat{n}_d$.

Note that with the boson images given in the operator forms of (2.23) and (2.24), the calculations for a given system with n particles, can be entirely carried out within the boson description. Care must be taken, however, in using Eq. (2.23) since $(B^{\dagger}_{2\mu})_B$ connects states with different particle numbers. In Eqs. (2.23) n refers to the ket state.

III. BCS AND NCQP TREATMENTS OF THE 1j-SM PROBLEM

Let us first recapitulate the BCS formalism for the 1j-SM problem. As is well known, the use of the BCS theory begins by performing the Bogoliubov transformation written as

$$a^{\dagger}_m = U\chi^{\dagger}_m + V\chi_{\tilde{m}};\quad a_{\tilde{m}} = U\chi_{\tilde{m}} - V\chi^{\dagger}_m . \tag{3.1}$$

In (3.1) $\chi^{\dagger}_m$ and $\chi_{\tilde{m}}$ are the quasiparticle (QP) operators, while the U and V factors are given as

$$U=\sqrt{1-n/2\Omega};\quad V=\sqrt{n/2\Omega} . \tag{3.2}$$

It is clear that, if we set $v=0$ in the U and V factors in (2.7), they reduce to those in (3.2).

The results of the Bogoliubov transformation applied to the pair operators are given (limiting for simplicity to $\lambda=2$) as

$$\begin{aligned}B^{\dagger}_{2\mu} &= U^2\bar{B}^{\dagger}_{2\mu} - \sqrt{2}UV\bar{C}^{\dagger}_{2\mu} - V^2\bar{B}_{2\mu} . \\ C^{\dagger}_{2\mu} &= \sqrt{2}UV(\bar{B}^{\dagger}_{2\mu}+\bar{B}_{2\tilde{\mu}})+(U^2-V^2)\bar{C}^{\dagger}_{2\mu} .\end{aligned} \tag{3.3}$$

On the rhs of (3.3), $\bar{B}_{2\mu}$, etc., are the QP pair operators. The BCS states may be denoted by $| v;\alpha\rangle\rangle$ and are given as

$$| v;\alpha\rangle\rangle = \bar{B}^{\dagger}_{2\mu_1}\bar{B}^{\dagger}_{2\mu_2}\cdots\bar{B}^{\dagger}_{2\mu_{v/2}} | 0,\text{BCS}\rangle . \tag{3.4}$$

In (3.4), $| 0;\text{BCS}\rangle$ stands for the BCS vacuum.

In the framework of the NCQP method, we first replace the BCS state (3.4) by

$$| v;\alpha\rangle = \hat{P} | v;\alpha\rangle\rangle / \sqrt{\langle\langle \alpha;v | \hat{P} | v;\alpha\rangle\rangle} , \tag{3.5}$$

where $\hat{P}$ is an operator that projects the spurious components out of the BCS states. In the 1j-SM application, the new state $| v;\alpha\rangle$ then acquires a good seniority nature, i.e., v, which has so far been used as the QP number, can now be considered as the seniority as well. However, the number projection has not been done yet, and thus $|v;\alpha\rangle$ is a linear combination of the states of the form (2.5).

A projection operator called Λ_n is then introduced, which, when operating upon $|v;\alpha\rangle$, projects out and renormalizes the component of the correct particle number n and correct seniority v. It also has the property that $\Lambda_n\hat{P}=\Lambda_n$. Therefore, the state $\Lambda_n|v;\alpha\rangle$ is exactly the same as the state given in (2.5). Thus, if we define an operator

$$(a_m^\dagger)_n=\Lambda_{n+1}^\dagger a_m^\dagger\Lambda_n \ , \tag{3.6}$$

and construct the matrix element $\langle v';\alpha'|(a_m^\dagger)_n|v;\alpha\rangle$ $(v'=v\pm1)$, it is clear that the following equality holds;

$$\langle v';\alpha'|(a_m^\dagger)_n|v;\alpha\rangle=\langle n+1,v';\alpha'|a_m^\dagger|n,v;\alpha\rangle \ . \tag{3.7}$$

Namely, in spite of the use of the BCS states, we now have a matrix element calculated in the number conserving way.

For practical purposes, the next important step is to derive a concrete form for the operator (3.6). In the NCQP method this is achieved by introducing a non-Hermitian number projection. Thus, let T_n and T_{n+1} denote the number projection operators for the n and $n+1$ particle systems, respectively. Then, it can be shown[6] that the use of just the first-order expansion (in the number operator) for T_n and T_{n+1} is sufficient to calculate the two matrix elements $\langle v';\alpha'|a_m^\dagger\vec{T}_n|v;\alpha\rangle$ and $\langle v';\alpha'|\overleftarrow{T}_{n+1}a_m^\dagger|v;\alpha\rangle$ exactly. It can be further shown that the square root of the product of these two matrix elements exactly equals the matrix element on the rhs of (3.7). Summarizing, this procedure permits one to express the operator $(a_m^\dagger)_n$ as

$$(a_m^\dagger)_n=\hat{P}[\chi_m^\dagger U(n,\hat{N})+V(n+1,\hat{N})\chi_{\tilde{m}}]\hat{P} \ , \tag{3.8}$$

where $\hat{N}$ is the QP number operator, and $U(n;\hat{N})$ and $V(n;\hat{N})$ are functions of n and $\hat{N}$. [Their functional forms are again given by (2.7), if v there is replaced by $\hat{N}$.] The significance of having this relation is that we can now operate $(a_m^\dagger)_n$ directly upon the original BCS states for number conserving calculations.

In (3.8), the factors $U(n,\hat{N})$ and $V(n,\hat{N})$ are functions of $\hat{N}$, and thus are not c numbers. Because of this fact, and because of the resemblance of (3.8) to (3.1), it is quite legitimate to call (3.8) a *quantized Bogoliubov transformation*. In other words, the NCQP method is simply understood as a method which permits one to replace the classical Bogoliubov transformation by its quantum version. This is why the QP nature of the original BCS theory is maintained, yet the number conservation which was lost in the BCS theory is recovered.

The transformation of (3.8) for single-particle operators can of course be extended to the transformations for the pair creation and scattering operators. For this, we note that, as shown in Ref. 6,

$$(a_\alpha^\dagger a_\beta^\dagger)_n=\Lambda_{n+2}^\dagger a_\alpha^\dagger a_\beta^\dagger\Lambda_n=(a_\alpha^\dagger)_{n+1}(a_\beta^\dagger)_n \tag{3.9a}$$

and

$$(a_\alpha^\dagger a_\beta)_n=\Lambda_n^\dagger a_\alpha^\dagger a_\beta\Lambda_n=(a_\alpha^\dagger)_{n-1}(a_\beta)_{n-1} \ . \tag{3.9b}$$

Thus, what we have to do is simply to form products of (3.8) with itself or its Hermitian conjugate, keeping track of the correct particle number. After these manipulations, the $B_{2\mu}^\dagger$ and $C_{2\mu}^\dagger$ operators can be written as

$$(B_{2\mu}^\dagger)_n=\hat{P}[\bar{B}_{2\mu}^\dagger U(n+1,\hat{N}+1)U(n,\hat{N})-\sqrt{2}\,\bar{C}_{2\mu}^\dagger U(n,\hat{N})V(n+2,\hat{N})-V(n+2,\hat{N})V(n+1,\hat{N}+1)\bar{B}_{2\tilde{\mu}}]\hat{P} \tag{3.10a}$$

and

$$(C_{2\mu}^\dagger)_n=\hat{P}\{\sqrt{2}\,\bar{B}_{2\mu}^\dagger U(n-1,\hat{N}+1)V(n,\hat{N})+\bar{C}_{2\mu}^\dagger[U^2(n,\hat{N})-V^2(n,\hat{N})]+\sqrt{2}U(n-1,\hat{N}+1)V(n,\hat{N})\bar{B}_{2\tilde{\mu}}]\}\hat{P} \ . \tag{3.10b}$$

If we now consider, e.g., the matrix element $\langle v';\alpha'|(B_{2\mu}^\dagger)_n|v;\alpha\rangle$, it is easy to see that it is expressed in exactly the same form as the rhs of (2.6), except that the matrix element $\langle v+2,v+2;\alpha'|\hat{B}_{2\mu}^\dagger|v,v;\alpha\rangle$ is replaced by $\langle v+2;\alpha'|\bar{B}_{2\mu}^\dagger|v;\alpha\rangle$ in the NCQP formalism. It is trivial to see, however, that these two matrix elements are exactly the same (numerically). This shows that the NCQP approach is totally equivalent to the SR method in the fermion problem of the $1j$-SM.

Using the same bosonization procedure of Sec. II, it is straightforward to bosonize the operators of (3.10a) and (3.10b). The resulting boson expansions are exactly the same as those obtained by the SR + BET method in Sec. II, and are given by Eqs. (2.23) and (2.24). This shows that the above equivalence of the SR and NCQP methods holds in the boson description as well. Since the SR + BET is exact, this equivalence confirmed that the NCQP approach is also exact. This is a quite gratifying and encouraging result, because the NCQP + BET method can be easily generalized to realistic, nondegenerate many-j problems, thus providing us with a powerful tool to study nuclear collective motions.

We shall conclude this section with a few remarks on the BCS approximation. From the above presentation, it is clear that to use the BCS approximation means to calculate the matrix elements using the operators of (3.3) and the space defined by (3.4), instead of the operators of (3.10) and the space defined by (3.5). As one can easily see, two kinds of approximations are involved. One is the replacement of the exact U and V factors by the corresponding BCS U and V. The other is related to the neglect of the projection operator $\hat{P}$ of (3.5) in the basis states, which leads to an error due to the presence of spurious components in the BCS states.

The resemblance of (3.3) to (3.10) is not accidental. The BCS theory is an approximation of the NCQP (and SR) theories. And the same holds for the boson formalisms based on them, namely for the BCS + BET and NCQP + (or SR +)BET. In spite of its approximate nature, the BCS theory is extremely convenient, because of its QP nature, and has been used extensively, e.g., in most of the practical BET calculations.[12] Also, the so-called BCS error is not very severe, so long as we use this theory under the restriction that $v \ll n$. However, the NCQP approach allows us now to construct a boson theory free from the major part of the BCS error, yet keeping the QP nature of BCS. The NCQP + BET will have a larger region of applicability than BCS + BET, so that we will be able to perform new (realistic) calculations. Furthermore, we will also be able to get a quantitative estimate of the BCS error in the various realistic cases, by comparing the results of the calculations obtained by using BCS and NCQP.

IV. BOSONIZATION TECHNIQUE OF OAI

In this section we shall recapitulate the bosonization technique of OAI, pointing out the aspects that differentiate the OAI approach from ours. In doing this we have a twofold purpose in mind. Firstly, we want to set the basis for a comparative discussion of the OAI method and our BET to be presented in Sec. V. Secondly, we want to remove a certain degree of confusion present in the OAI work, due to the existence of two different boson formalisms, one developed in OAI, and another in an earlier paper (OAIT). As we shall see below, these two formalisms have many similarities, but also one important difference. The confusion mentioned above arises from the fact that in the OAI paper a new formalism was presented, but the numerical results given there can actually be obtained only by using the boson expansion of OAIT.

In order to make the presentation as simple as possible, we shall first concentrate on the $\Delta v=2$ matrix element of the scattering operator $C^\dagger_{2\mu}$, i.e., on

$$\langle n,v;\alpha' \mid C^\dagger_{2\mu} \mid n,v-2;\alpha\rangle \,.$$

This matrix element was discussed in detail in Sec. II, but note that we have shifted v by two units, compared with the example taken up in Sec. II. This was done so that the results given below can be compared directly with those of OAI and OAIT.

Both OAI and OAIT make use of the SR formula written as

$$\langle n,v;IM \mid C^\dagger_{2\mu} \mid n,v-2;I'M'\rangle = U(n-1,v-1)\sqrt{(n-v+2)/2}\,\langle v,v;IM \mid C^\dagger_{2\mu} \mid v,v-2;I'M'\rangle \,. \tag{4.1}$$

One will easily recognize that Eq. (4.1) is the same as Eq. (3.1) of OAI. One will also notice that (4.1) is not in the same form as the first term on the rhs of (2.8). The reason is that, in obtaining (2.8) we carried out the SR completely, while the same was not done in deriving (4.1). It is straightforward, however, to see that these two expressions are, in fact, equivalent.

Both OAI and OAIT methods of bosonization start by writing down the equality

$$\langle n,v;IM \mid C^\dagger_{2\mu} \mid n,v-2;I'M'\rangle = (d^{v/2},s^{(n-v)/2};IM \mid (C^\dagger_{2\mu})_B \mid d^{(v/2)-1},s^{(n-v)/2+1};I'M') \,. \tag{4.2}$$

The following step is where the two methods begin to differ. Namely, OAIT write down the boson image $(C^\dagger_{2\mu})_B$ as

$$(C^\dagger_{2\mu})_{\rm OAIT} = U(n-1,2\hat{n}_d-1) \times s\left\{\alpha_0 d^\dagger_\mu + \sum_I \alpha_{1;I}[[d^\dagger d^\dagger]_I\tilde{d}]_{2\mu}\right\}, \tag{4.3a}$$

while OAI assume the form

$$(C^\dagger_{2\mu})_{\rm OAI} = s\left\{\alpha'_0 d^\dagger_\mu + \sum_I \alpha'_{1;I}[[d^\dagger d^\dagger]_I\tilde{d}]\right\}. \tag{4.3b}$$

Then, both OAI and OAIT determine the unknown coefficients α_0 and $\alpha_{1;I}$, and α'_0 and $\alpha'_{1;I}$, respectively, by requiring that (4.2) holds exactly, first for $v=2$ and then for $v=4$. One thus eventually obtains $(C^\dagger_{2\mu})_{\rm OAIT}$ and $(C^\dagger_{2\mu})_{\rm OAI}$ in the form

$$(C^\dagger_{2\mu})_{\rm OAIT} = U(n-1,2\hat{n}_d-1)s\left\{\sqrt{2/\Omega}d^\dagger_\mu + \sum_I \frac{\hat{I}}{\sqrt{5}}(\sqrt{2/(\Omega-2)}N_I - \sqrt{2/\Omega})[[d^\dagger d^\dagger]_I\tilde{d}]_{2\mu}\right\} \tag{4.4a}$$

and

$$(C^\dagger_{2\mu})_{\rm OAI} = s\left\{\left[\frac{2\Omega-n}{\Omega(\Omega-1)}\right]^{1/2} d^\dagger_\mu + \sum_I \frac{\hat{I}}{\sqrt{5}}\left\{\left[\frac{2\Omega-n-2}{(\Omega-2)(\Omega-3)}\right]^{1/2} N_I - \left[\frac{2\Omega-n}{\Omega(\Omega-1)}\right]^{1/2}\right\}[[d^\dagger d^\dagger]_I\tilde{d}]_{2\mu}\right\}. \tag{4.4b}$$

The $d^\dagger_\mu$ terms in (4.4a) and (4.4b) agree, respectively, with those in Eq. (7) of OAIT and Eq. (3.19) of OAI. However, neither OAIT nor OAI gave the explicit form for $\alpha_{1;I}$ and $\alpha'_{1;I}$. We have thus calculated them ourselves, making use of the TTB method. Note that, because of the form assumed by OAI for the boson image, the explicit v (i.e., the $\hat{n}_d$) dependence is lost in the coefficients of (4.4b). On the other hand, this dependence is retained in the OAIT result of (4.4a). Furthermore, the n dependence is different in these two expansions.

It may be worthwhile to note further that, because of the assumed form (4.3a), which could only have been "guessed" from the knowledge of the SR, the OAIT method was, in actuality, to factorize the U factor first as in Eq. (4.1), and then bosonize the remaining part. In this sense, the OAIT formalism lies somewhere in between our method of Sec. II in which both the U and V factors were factorized first, and the OAI formalism in which nothing was factorized out.

In Sec. V, we shall present numerical results for the $\Delta v=2$ matrix element of $C^\dagger_{2\mu}$ and $B^\dagger_{2\mu}$, obtained by using BET, OAI, and OAIT. The needed OAI and OAIT formulas for $(C^\dagger_{2\mu})$ are given in (4.4). For completeness of the presentation, as well as for later convenience, we shall give here all the other boson images relevant to the $\Delta v=2$ case.

The BET image of the scattering operator is given [cf. (2.8) and (2.20)] as

$$(C^\dagger_{2\mu})_{\rm BET}=(\hat{B}^\dagger_{2\mu})_B\sqrt{2}U(n-1,2\hat{n}_d+1)V(n,2\hat{n}_d)\ , \quad (4.5a)$$

$$(\hat{B}^\dagger_{2\mu})_B=d^\dagger_\mu+\sum_I(N_I-1)(\hat{I}/\sqrt{5})[[d^\dagger d^\dagger]_I\tilde{d}]_{2\mu}\ . \quad (4.5b)$$

In the OAI case the $\Delta v=2$ part of the boson image of $B^\dagger_{2\mu}$ can be written as

$$(B^\dagger_{2\mu})_{\rm OAI}=s\left\{d^\dagger_\mu\left[\frac{(2\Omega-n)(2\Omega-n-2)}{4\Omega(\Omega-1)}\right]^{1/2}+\sum_I\frac{\hat{I}}{\sqrt{5}}[[d^\dagger d^\dagger]_I\tilde{d}]_{2\mu}\left\{\left[\frac{(2\Omega-n-4)(2\Omega-n-2)}{4(\Omega-2)(\Omega-3)}\right]^{1/2}N_I-\left[\frac{(2\Omega-n)(2\Omega-n-2)}{4\Omega(\Omega-1)}\right]^{1/2}\right\}\right\}\ . \quad (4.6)$$

The boson expansion of $B^\dagger_{2\mu}$ is identical for BET and OAIT and is given [cf. (2.6)] by

$$(B^\dagger_{2\mu})_B=(\hat{B}^\dagger_{2\mu})_B U(n+1,2\hat{n}_d+1)U(n,2\hat{n}_d)\ , \quad (4.7)$$

where $(B^\dagger_{2\mu})_B$ is given by (4.5b).

Let us conclude this section with a remark on semantics. In their paper, OAI never used the term "expansion." Instead they used terms like "boson mapping" or "boson image." Granted that these terms are quite correct, we may also add that the OAI boson images can also be called "expansions." The reason is that, as we shall argue in the next section, the first terms in (4.4) and (4.6) do not provide, overall, a sufficiently good accuracy, and thus third-order terms are also needed. Therefore, the OAI, as well as the OAIT boson images, are, in fact, expansions (in $\hat{n}_d/\Omega$). In the rest of the present paper, we occasionally refer to the OAI and OAIT formalism as "boson expansion method," and, correspondingly, to the OAI and OAIT boson images as "boson expansion." We believe that the term "expansion" accurately reflects both the spirit of the bosonization technique, as well as that of the final results of all the methods considered in the present paper.

V. COMPARATIVE DISCUSSION OF THE OAI, OAIT, AND BET METHODS

In this section we shall compare our BET formalism, presented in Sec. II, with those of the OAI and OAIT methods, reviewed in Sec. IV. In subsection A we shall present and comment on a few numerical results obtained with the three methods mentioned above for the $1j$-SM with $j=\frac{29}{2}$. In subsection B, we then make a comparative discussion of the various formalisms.

A. Discussion of the numerical results

In order to test the accuracy and the convergence properties of the three bosonization methods considered here, we have evaluated the reduced matrix elements $\langle v,v;I||C^\dagger_2||v,v-2;I'\rangle$ and $\langle v,v;I||B^\dagger_2||v-2,v-2;I'\rangle$ for $v=6$ and $v=4$. The formulas used for this purpose were given in Sec. IV. The results of the calculations are given in Tables I and II under the headings of BET, OAI, and OAIT. The exact fermion results were given in OAI, and are reproduced in our tables under the heading of "Fermion." (Note the change of sign in Table I. We give the matrix elements of $C^\dagger_{2\mu}$, while OAI gave those of $U_{2\mu}=-C^\dagger_{2\mu}$.)

Some of the matrix elements calculated here were also calculated in OAI and the results were given in their Tables 2 and 9. In making a direct comparison between our tables and those of OAI, however, the reader should keep in mind that, as mentioned in Sec. IV, the boson results given in OAI as OAI results are actually those of OAIT. Thus, the entries in our tables under the OAIT heading are reproductions of the entries in the OAI tables. To our knowledge, the results given under the OAI heading in our tables have not been reported anywhere else previously.

In both Tables I and II, there are two entries, called 0(1) and 0(3), under each boson heading. The 0(1) values were obtained by using the first-order term, i.e., only the term with $d^\dagger$ in the various expansions. On the other hand, the 0(3) values were obtained by including the third term as well, i.e., the term with $d^\dagger d^\dagger d$. Note that all the 0(3) results are new in the present paper, since OAI reported only 0(1) values.

Let us look at the performance of each boson method separately, beginning with our SR + BET. From both Tables I and II we see that the error in 0(1) is as high as 19% for $v=4$ and 30% for $v=6$ for both the $C^\dagger_{2\mu}$ and $B^\dagger_{2\mu}$ operators. (Note, incidentally, that, with BET, the error is the same for both operators. This is not the case with either OAI or OAIT, as seen below.) This large 0(1) error does not bother us too much, however, since the real test for both accuracy and convergence of the expansion lies in the 0(3) results. From our tables we see that the error in 0(3) is within 1% for all the $v=6$ entries, except

TABLE I. Boson and fermion values for the matrix element $\langle v,v:I||C^{\dagger}_{2\mu}\,|\,v,v-2;I'\rangle$ ($j=\frac{29}{2}$).

		BET		OAI		OAIT		
I	I'	0(1)	0(3)	0(1)	0(3)	0(1)	0(3)	Fermion
$v=4$								
4	2	1.66	1.57	1.49	1.57	1.55	1.57	1.57
2	2	1.24	1.17	1.11	1.17	1.15	1.17	1.17
0	2	0.55	0.45	0.50	0.45	0.52	0.45	0.45
$v=6$								
0	2	0.74	0.65	0.59	0.64	0.63	0.65	0.66
2	0	1.13	0.94	0.89	0.93	0.97	0.94	0.93
2	2	0.72	0.50	0.57	0.51	0.62	0.51	0.52
2	4	0.97	0.68	0.77	0.68	0.83	0.68	0.68
3	2	1.65	1.46	1.31	1.43	1.41	1.45	1.46
3	4	−1.04	−0.92	−0.83	−0.91	−0.89	−0.92	−0.92
4	2	1.60	1.42	1.27	1.39	1.37	1.41	1.41
4	4	1.52	1.35	1.21	1.33	1.31	1.34	1.34
6	4	2.66	2.36	2.11	2.31	2.28	2.34	2.35

one, in which it is 2%. [For $v=4$ the 0(3) results are exact by construction.] This shows that the accuracy and the convergence of our expansion are remarkably good.

Let us next consider the 0(1) results given by the OAI method, starting from Table I for $C^{\dagger}_{2\mu}$. As seen, the overall error there is consistently about 10%, thus being consistently smaller than the 0(1) error with BET. For the $B^{\dagger}_{2\mu}$ operator in Table II, we see that the 0(1) results of OAI are much better overall than those in Table I, although it must be noted that there are few cases in which the error is rather large; 13% for $v=4$ and 18% for $v=6$. We thus see that the zeroth-order (first-order, in our terminology) OAI theory performs much better for the $B^{\dagger}_{2\mu}$ operator than it does for the $C^{\dagger}_{2\mu}$ operator. Still, overall, the error level in Table I, and the appearance of large errors in a few results in Table II seem to lead us to conclude that the zeroth-order OAI expansion cannot be reckoned satisfactory. [Note that an 18% error in a matrix element translates into about a 32% error, e.g., in $B(E2)$.] The 0(3) results given by OAI are just about the same as those of BET, showing good convergence, as well as the necessity of the second term in the boson expansion.

For the OAIT results we can repeat exactly what we said about OAI, except that what pertained to the $C^{\dagger}_{2\mu}$ in OAI now applies to the $B^{\dagger}_{2\mu}$ operator, and *vice versa*. In other words, neither OAI nor OAIT performs consistently for the two operators at the first-order level.

The OAI theory was developed as an attempt to provide a microscopic version of the IBA. In order to achieve this, OAI had to show that the resultant boson images were in the IBA form, maintaining, at the same time, a reasonably high accuracy in the zeroth order. However, these two goals turned out to be difficult to achieve simultaneously. On the one hand, they were able to derive a theory looking exactly like IBA (namely boson images with v-independent coefficients), but whose first-order accuracy for the $C^{\dagger}_{2\mu}$ operator (the case they chose

TABLE II. Boson and fermion values for the matrix element $\langle v,v:I||B^{\dagger}_{2}||v-2,v-2;I'\rangle$ ($j=\frac{29}{2}$).

		BET(=OAIT)		OAI		
I	I'	0(1)	0(3)	0(1)	0(3)	Fermion
$v=4$						
4	2	4.24	4.00	3.95	4.00	4.00
2	2	3.16	2.98	2.94	2.98	2.98
0	2	1.41	1.15	1.32	1.15	1.15
$v=6$						
0	2	1.73	1.53	1.49	1.51	1.55
2	0	2.64	2.20	2.28	2.18	2.18
2	2	1.69	1.18	1.46	1.19	1.20
2	4	2.27	1.57	1.95	1.59	1.59
3	2	3.87	3.43	3.34	3.38	3.42
3	4	−2.45	−2.17	−2.11	−2.14	−2.16
4	2	3.76	3.33	3.24	3.28	3.31
4	4	3.58	3.17	3.09	3.13	3.14
6	4	6.24	5.54	5.38	5.46	5.51

as a guide) was found not sufficiently good. On the other hand, they had the OAIT theory, which gave a much better 0(1) result for the $C^{\dagger}_{2\mu}$ operator, but which did not have the desired IBA-type form (in that the boson images had v-dependent coefficients). In their paper OAI presented their new (OAI) formalism, but gave numerical results pertaining to OAIT without any explicit mention of it. As seen in Sec. IV formally, and confirmed numerically by the results in Tables I and II, OAI and OAIT are two different theories and should not be used interchangeably.

We may also add that, having chosen the $C^{\dagger}_{2\mu}$ operator as a guide (on the ground that it conserves the particle number), OAI had to explain the large error present in the OAIT results for the $B^{\dagger}_{2\mu}$ operator. The explanation OAI gave in their Sec. 3.5 is that $B^{\dagger}_{2\mu}$ does not conserve the particle number, and hence leads to a large error (because the difference in the fermion and boson norms now comes in directly). In the light of the opposing performance of OAI and OAIT, as we have seen just above, however, this explanation is puzzling at best: It is contradicted by the OAI results.

Note that the use of the strategy, in which one starts with the equality, e.g., of (4.2), does not by itself determine uniquely the resulting boson image. As we saw in Sec. IV, different forms can be subsequently assumed, and the boson expansion coefficients are then determined accordingly. (In this regard we may note that Otsuka[13] took advantage of this "ambiguity" and "made up" still another expansion, which can be viewed as an "average" between the OAI and OAIT formalisms, and which gives, in fact, better overall first-order results than either OAI or OAIT. We may also note here that, for the purpose of making the first-order expansion good, the best job done so far seems to be that of Bonatsos *et al.*,[14] in which the relevant commutation relations are used as a guide to determine the expansion coefficients.) On the other hand, in our procedure of Sec. II, since we carry out the SR completely prior to bosonization, there is no room for ambiguity in the resulting expansion. It is clear that the difference in the expansions in the OAI and OAIT approaches is rooted in the approximate treatment of the U and V factors, a fact which is also responsible for the role that s bosons play in these theories. We shall discuss this aspect more in detail in the next subsection.

B. Discussion on the OAI formalism

As was explicitly state in OAI, and we have mentioned above, their work was done with the purpose of providing a microscopic version of IBA. As is well known, IBA is characterized by the appearance of s bosons, in addition to d bosons, in the Hamiltonian, as well as in the pair operators. As we saw in Sec. IV, the OAI and OAIT approaches do produce a formalism that contains s bosons. Because of the appearance of s bosons, OAI stated that their boson theory is drastically different from the conventional BET. We shall now argue, however, that the OAI theory is not that much different from the other BET's.

A simple way to look at the matter is to note that our SR + BET (and equivalently NCQP + BET), which is exact, does not contain s bosons. In other words, we have shown that it is, at least, not mandatory to have s bosons, in order to construct an exact BET, even in the seniority scheme. Of course if the (S,D) fermion space of (2.5) is demanded to be mapped onto the corresponding (s,d) boson space, as done in Eq. (4.2), the appearance of s bosons in the image of any operator is inevitable. However, the choice of such a mapping is by no means mandatory. A question then naturally arises. Does the s boson have any physical meaning, or is it purely a mathematical artifice?

The presence of the s bosons is related to the presence of the S_+ operators, which represent the Cooper pairs in the fermion description. The use of the SR, however, allows one to eliminate (completely or partially) the S_+ operators from the fermion formalism. Their original presence is then accurately remembered by the U and V factors. In other words the U and V factors are the most important carriers of the information pertaining to the Cooper pairs. Therefore, if one lets the U and V factors carry all of the information (which is the same as to carry out the SR completely, as we did in Sec. II), the S_+ operators disappear completely from the fermion description, and, consequently, the s bosons do not appear in the boson expansion.

The BCS approximation, which we have been using in all our previous realistic calculations, is exactly the same as the SR + BET in this regard. The only, but somewhat troublesome, difference is that the U and V factors in BCS are approximate, since they do not carry any v dependence. This in turn means that they do not count correctly the number of S_+ in the original fermion system, which is the same as to say that BCS fails to take into account the blocking effect.

We may now go back to the question about the significance of the s bosons in OAI. Neither OAI nor OAIT permit the U and V to carry the full information on S_+, but instead let s bosons share part of the burden. Therefore, s bosons do have a physical meaning in OAI as carriers of part of the information about Cooper pairs. However, at the same time, we may also add that the explicit presence of s bosons in these theories is more the result of a choice made to suit some specific purpose, rather than the inevitable consequence of physical requirements that could not be satisfied otherwise. As we discussed above, Cooper pairs, and consequently s bosons, can be totally "absorbed away" either exactly, or approximately.

In conclusion, the important point we want to get through to the reader in this section is that we do not see the presence of s either as indispensable, or as the "signature" of a dramatically different approach to bosonization. Furthermore, as we remarked at the end of Sec. IV, the OAI method is also a boson expansion method, with $\hat{n}_d/\Omega$ as the smallness parameter. Therefore, the OAI statement that their theory is drastically different from the conventional BET does not seem to be justified. The valuable contribution to boson theory made by OAI is, we believe, in initiating the use of the seniority reduction in the derivation of the boson expansion, a technique which allows one to use n-particle states directly to derive the boson coefficients. We have ourselves used this technique to full advantage in our formulation presented in Sec. II, albeit in a different spirit and, consequently, with different results.

VI. SUMMARY AND CONCLUSIONS

In this paper we presented two different techniques to derive a boson formalism for the $1j$-SM problem. The first method was described in Sec. II. We started there by constructing the basis states in the (S,D) space in the seniority scheme. We then carried out a *complete* SR of the matrix elements of the nucleon pair operators, and finally, we bosonized the reduced matrix elements by using the BET technique.[1,4] The resultant boson formalism, which we called SR + BET, is *exact* (up to the order to which the boson expansion is obtained) and is characterized by the fact that it contains only d bosons.

The second bosonization method was presented in Sec. III and is based on the NCQP approach.[6] This method makes use of the quasiparticle description of BCS, but removes (approximately, in general) the errors due to the number nonconservation and the presence of spurious components in the BCS states. This is achieved by introducing a projection operator, which is effectively incorporated into the pair operators themselves. These new "effective" pair operators were then bosonized by using the BET technique resulting in the boson formalism, which we called NCQP + BET. In the $1j$-SM case, the NCQP method is exact, and the NCQP + BET was found to coincide with the SR + BET.

It is clear that the SR and NCQP approaches are nothing but two different ways of treating the fermion part of the problem, before the actual bosonization is performed. In this sense, they perform the same function as the BCS approximation does in BCS + BET, i.e., they take care of the effect of the Cooper pairs present in the original fermion problem. This is why the Cooper-type s boson is not present, in these three boson theories. (Note, incidentally, that in the many-j case monopole pairs exist, other than Cooper's, that are responsible, e.g., for pairing vibrations. Monopole pairs of this type must be treated explicitly in the boson description if the underlying physics requires it.) The accuracy of the resulting boson formalism depends, of course, on the approximation made in the fermion stage and on the truncation of the boson expansion. We want to stress here that the bosonization procedure itself is the same in all three cases, namely the KT3 and TTB methods.

In Sec. IV we recapitulated OAI work. We emphasized there that the reason why in the OAI theory both s and d bosons are present is related to the fact the OAI map the original (S,D) fermion space onto the (s,d) boson space. We stressed that, although such procedure is perfectly legitimate and correct, it does not represent the only way to derive a number conserving boson expansion, as the results of the present paper have shown. The choice of the mapping made by OAI was dictated by the need of making contact with the phenomenological IBA in an attempt to provide a microscopic justification of this theory. The OAI theory seems to have achieved the goal.

We remarked, however, that this theory is not sufficiently accurate at the first order (zeroth in OAI terminology). The numerical results presented in Sec. V show that OAI does require higher-order terms in order to guarantee an acceptable overall accuracy. The OAIT theory, which, strictly speaking, cannot be viewed as a justification of IBA due to the explicit n_d dependence in the expansion coefficients, suffers from the same accuracy problem as OAI. As for the s boson, we argued that while it does carry physical meaning in OAI and OAIT, its presence in a boson formalism is not mandatory in order to guarantee that the theory is number conserving, as SR + BET and NCQP + BET are proof of. One should not, therefore, overestimate the importance of the role played by the s boson in the various boson theories that do have it, including IBA.

All the results and discussion presented in this paper pertained to the $1j$-SM, which albeit instructive, is too simple a model to serve as a basis for an effective appraisal of the different methods. The real testing ground for the various boson expansion approaches is the realistic many-j cases. Up to now only the BCS + BET has been extensively applied to describe collective motions in nuclei.[12] In this paper we presented two boson formalisms, the SR + BET and NCQP + BET. However, as far as we know, the extension of the SR method to many-j cases, i.e., the SR in the generalized seniority scheme,[15] appears to be a quite difficult task. On the other hand such an extension of the NQCP + BET is rather straightforward, and is currently in progress. In this regard, the equivalence of this formalism to the SR + BET in the $1j$-SM limit is a pleasing and encouraging fact. It indicates that the NCQP + BET might be the answer to the problem on how to handle the generalized seniority scheme. In any case, it provides us with a viable alternative to it.

We finally want to remark that the NCQP + BET method shares some features with the method of Suzuki *et al.*,[16] in that they both make use of the quasiparticle language. The two methods differ in the treatment of the s degree of freedom. This difference is reflected in the fact that while NCQP + BET does not have s bosons, the boson formalism of Ref. 16 does.

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