Variational formulation of nuclear fluid dynamics

Lucília P. Brito and Constança da Providência
Departamento de Física, Universidade de Coimbra, Coimbra, Portugal
(Received 5 August 1985)

A new fluid dynamical description of giant resonances which avoids the scaling approximation and takes into account the interplay between first sound and zero sound is presented. The new method leads to equations satisfying the sum rules S_{-1}, S_1, and S_3 for the electric modes and the sum rule S_1 for the magnetic modes. The model describes surface modes and separates the low lying modes from the giant resonances. Numerical results for the energy spectrum of ^{208}Pb are presented.

I. INTRODUCTION

Semiclassical methods have been used quite successfully in the interpretation and description of some properties of atomic nuclei with large mass number, which arise from the coherent motion of many nucleons.[1-7] Giant resonances are highly collective excited modes of nuclei which exhaust an appreciable fraction of the appropriate sum rules. A semiclassical description of the many-body nuclear dynamics has the advantage of providing important information about some macroscopic properties such as frequencies and transition amplitudes of the giant resonances without a detailed knowledge of the underlying nucleon-nucleon interaction.

Recently a simple fluid-dynamical model, appropriate to describe the interplay between the zero and first sound modes of finite droplets of nuclear matter and leading to the splitting of the giant resonances and to low-lying modes, has been proposed.[8,9]

In Sec. II we present the model. The equations of motion and their solutions are discussed in Sec. III. In Sec. IV the sum rules verified by the model are proved and Sec. V deals with the results of the model. Finally in the last section some conclusions are taken.

II. CLASSICAL DERIVATION OF THE LAGRANGIAN

The model is based on the classical many-body dynamics in the independent-particle approximation except for effects of quantum statistics which are taken into account by allowing the distribution function to take only the values zero and one. It assumes a spherical atomic nucleus described by the Hamiltonian

$$H=\sum_{k=1}^{A}\frac{p_k^2}{2m}+\tfrac{1}{2}a\sum_{k\neq l=1}^{A}\delta(\mathbf{r}_k-\mathbf{r}_l)+\tfrac{1}{6}b\sum_{k\neq j=l=1}^{A}\delta(\mathbf{r}_j-\mathbf{r}_k)\delta(\mathbf{r}_l-\mathbf{r}_k)\ , \tag{2.1}$$

where p_k denotes the momentum of particle k and a and b are the force parameters of the zero-range two-body and three-body interactions between the nucleons. Let $f_0(\mathbf{r},\mathbf{p})$ be the equilibrium distribution function which minimizes the energy functional

$$E[f]=\int d\Gamma\frac{p^2}{2m}f(\mathbf{r},\mathbf{p})+\frac{a}{2}\int d\Gamma\, d\Gamma'\delta(\mathbf{r}-\mathbf{r}')f(\mathbf{r},\mathbf{p})f(\mathbf{r}',\mathbf{p}')+\frac{b}{6}\int d\Gamma\, d\Gamma' d\Gamma''\delta(\mathbf{r}-\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}'')f(\mathbf{r},\mathbf{p})f(\mathbf{r}',\mathbf{p}')f(\mathbf{r}'',\mathbf{p}'')\ , \tag{2.2}$$

where

$$d\Gamma=\frac{1}{2\pi^3}d^3r\, d^3p$$

and $f(\mathbf{r},\mathbf{p})$ is an arbitrary distribution function which only takes the values zero or one and is subjected to the restriction

$$\int f(\mathbf{r},\mathbf{p})d\Gamma=A\ , \tag{2.3}$$

where A is the number of particles of the system.

The time evolution of the distribution function is given by the Vlasov equation[4,10]

$$\frac{\partial f(\mathbf{r},\mathbf{p},t)}{\partial t}+\{f(\mathbf{r},\mathbf{p},t),h(\mathbf{r},\mathbf{p},t)\}=0 \tag{2.4}$$

where $\{\ ,\ \}$ means the Poisson bracket and the single particle Hamiltonian is given by

$$h(\mathbf{r},\mathbf{p},t)=\frac{p^2}{2m}+a\int d\Gamma'\delta(\mathbf{r}-\mathbf{r}')f(\mathbf{r}',\mathbf{p}',t)+\frac{b}{2}\int d\Gamma' d\Gamma''\delta(\mathbf{r}-\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}'')f(\mathbf{r}',\mathbf{p}',t)f(\mathbf{r}'',\mathbf{p}'',t)\ . \tag{2.5}$$

 ©1985 The American Physical Society

This equation describes a system of particles moving in a mean field and preserves the allowed values of 0 and 1 for $f(\mathbf{r},\mathbf{p},t)$. In order to obtain approximate solutions of the Vlasov equation it is convenient to derive this equation from an action principle.

For this purpose we consider the set of all trial functions $f(\mathbf{r},\mathbf{p})$ related by a canonical transformation to $f_0(\mathbf{r},\mathbf{p})$. If, for all times t, the time dependent distribution function $f(\mathbf{r},\mathbf{p},t)$ belongs to this set a generating function $F(\mathbf{r},\mathbf{p},t)$ may be found such that

$$\frac{\partial f}{\partial t}+\{f,F\}=0 . \tag{2.6}$$

The action principle may now be written as

$$\delta S=\delta \int L\,dt=0 , \tag{2.7}$$

where

$$L=\int d\Gamma\, f(\mathbf{r},\mathbf{p},t)F(\mathbf{r},\mathbf{p},t)-E[f] . \tag{2.8}$$

We wish to study small deviations from equilibrium, so we consider distribution functions which satisfy

$$\begin{aligned} f(\mathbf{r},\mathbf{p},t)=&f_0(\mathbf{r},\mathbf{p})+\{f_0,G\}(\mathbf{r},\mathbf{p},t)\\ &+\tfrac{1}{2}\{\{f_0,G\},G\}(\mathbf{r},\mathbf{p},t)+\cdots , \end{aligned} \tag{2.9}$$

where the infinitesimal generator $G(\mathbf{r},\mathbf{p},t)$ can be decomposed into a time-even and a time-odd part

$$G(\mathbf{r},\mathbf{p},t)=P(\mathbf{r},\mathbf{p},t)+Q(\mathbf{r},\mathbf{p},t) , \tag{2.10}$$

$$\begin{aligned} &Q(\mathbf{r},\mathbf{p},t)=Q(\mathbf{r},-\mathbf{p},t) ,\\ &P(\mathbf{r},\mathbf{p},t)=-P(\mathbf{r},-\mathbf{p},t) . \end{aligned} \tag{2.11}$$

III. APPROXIMATE SOLUTION OF THE VLASOV EQUATION FOR SPHERICAL FERMI LIQUID DROPLETS

We present in this section a fluid-dynamic calculation of the collective states of the model proposed, which illustrate the variational approach. The equilibrium distribution function of this model is

$$f_0(\mathbf{r},\mathbf{p})=\theta[\lambda-h_0(\mathbf{r},\mathbf{p})] \tag{3.1}$$

with

$$\begin{aligned} \theta(x)&=1 \text{ if } x\geq 0 ,\\ &=0 \text{ if } x<0 . \end{aligned}$$

The single-particle Hamiltonian is

$$\begin{aligned} h_0(\mathbf{r},\mathbf{p})&=\frac{p^2}{2m}+U_0(\mathbf{r})\\ &=\frac{p^2}{2m}+a\rho_0(\mathbf{r})+\frac{b}{2}\rho_0^2(\mathbf{r}) , \end{aligned} \tag{3.2}$$

where

$$\rho_0(\mathbf{r})=\int \frac{d^3p}{2\pi^3}f_0(\mathbf{r},\mathbf{p})=\frac{2p_F^3(\mathbf{r})}{3\pi^2} \tag{3.3}$$

is the ground-state density and

$$p_F(\mathbf{r})=[2m(\lambda-U_0)\theta(\lambda-U_0)]^{1/2} \tag{3.4}$$

stands for the Fermi momentum.

The time-odd generator $P(\mathbf{r},\mathbf{p},t)$, which produces the static deformations, is defined by writing the time-even distribution function

$$\begin{aligned} f_E(\mathbf{r},\mathbf{p},t)&=f_0(\mathbf{r},\mathbf{p})+\{f_0,P\}(\mathbf{r},\mathbf{p},t)+\tfrac{1}{2}\{\{f_0,P\},P\}(\mathbf{r},\mathbf{p},t)+\cdots\\ &=\theta\left[\lambda-h_0(\mathbf{r},\mathbf{p})-W(\mathbf{r},t)-\sum_{\alpha\beta}\frac{1}{2m}p_\alpha p_\beta \chi_{\alpha\beta}(\mathbf{r},t)\right] , \end{aligned} \tag{3.5}$$

where $W(\mathbf{r},t)$ and $\chi_{\alpha\beta}(\mathbf{r},t)$ are scalar and tensor fields which hopefully provide an adequate description of the monopole and quadrupole deformations of the Fermi sphere. For the time-even generator we make the ansatz

$$Q(\mathbf{r},\mathbf{p},t)=\phi(\mathbf{r},t)+\tfrac{1}{2}\sum_{\alpha\beta}p_\alpha p_\beta \phi_{\alpha\beta}(\mathbf{r},t) , \tag{3.6}$$

where $\phi(\mathbf{r},t)$ and $\phi_{\alpha\beta}(\mathbf{r},t)$ are, respectively, scalar and symmetrical tensor fields. This is the simplest choice allowing for the possibility of transverse flow.[11]

Taking into account the approximation (2.9) and the ansatz (3.5) and (3.6) the Lagrangian will be, from expression (2.8) and subjected to the constraint (2.3),

$$\begin{aligned} L=&\int d^3r\left[\left[\rho_1-\frac{\bar\rho_0}{2}\chi_{\alpha\alpha}\right]\left[\dot\phi+\frac{\bar p_F^2}{6}\dot\phi_{\alpha\alpha}\right]-\frac{\bar p_F^2\bar\rho_0}{10}\left[\chi_{\alpha\beta}-\frac{\delta_{\alpha\beta}}{3}\chi_{\gamma\gamma}\right]\left[\dot\phi_{\alpha\beta}-\frac{\delta_{\alpha\beta}}{3}\dot\phi_{\gamma\gamma}\right]\right]\Theta\\ &+\int d\boldsymbol{\Sigma}\cdot\delta\mathbf{R}\bar\rho_0\left[\dot\phi+\frac{\bar p_F^2}{10}\dot\phi_{\alpha\alpha}\right]-E[\rho_1,\chi_{\alpha\beta}]-T[\phi,\phi_{\alpha\beta}] . \end{aligned} \tag{3.7}$$

The first integration is over the whole coordinate space and the field $\rho_1(\mathbf{r},t)$ replaces the field $W(\mathbf{r},t)$,

$$\rho_1(\mathbf{r},t) = -\frac{2m}{\pi^2} p_F(\mathbf{r}) W(\mathbf{r},t) \; . \tag{3.8}$$

The surface integral takes into account the possibility of surface displacements. In fact, if we denote by f' and f two distribution functions corresponding to slightly displaced nuclear boundaries, we find

$$\int d\Gamma (f'-f)(\dot{\phi} + \tfrac{1}{2} p_\alpha p_\beta \dot{\phi}_{\alpha\beta}) = \int d\mathbf{\Sigma}\cdot\delta\mathbf{R}\bar{\rho}_0 \left[\dot{\phi} + \frac{\bar{p}_F^2}{10}\dot{\phi}_{\alpha\alpha}\right] ,$$

where $\delta\mathbf{R}$ is the displacement of the nuclear surface. As the equilibrium self-consistent density is a spherical square density we write

$$\rho_0(\mathbf{r}) = \bar{\rho}_0 \Theta \;\text{ and }\; p_F(\mathbf{r}) = \bar{p}_F \Theta \; . \tag{3.9}$$

In these expressions and in the sequel the factor Θ denotes $\theta(R-r)$, where R is the nuclear radius fixed by the particle number and the equilibrium nuclear matter density $\bar{\rho}_0$. The potential energy functional is given by

$$E[\rho_1,\chi_{\alpha\beta}] = \int d^3r \frac{\bar{p}_F^2}{2m}\left[\frac{G_0}{3\bar{\rho}_0}\left[\rho_1 - \frac{\bar{\rho}_0}{2}\chi_{\alpha\alpha}\right]^2 + \frac{\bar{\rho}_0}{10}\left[\chi_{\alpha\beta} - \frac{\delta_{\alpha\beta}}{3}\chi_{\gamma\gamma}\right]^2\right]\Theta \tag{3.10}$$

with $G_0 = 1+F_0$, F_0 being the Landau parameter

$$F_0 = \frac{3m}{\bar{p}_F^2}(a\bar{\rho}_0 + b\bar{\rho}_0^2) \; . \tag{3.11}$$

The kinetic energy functional is

$$T[\phi,\phi_{\alpha\beta}] = \int d^3r \frac{\bar{\rho}_0}{2m}\left\{\partial_\beta\phi\partial_\beta\phi + \frac{2\bar{p}_F^2}{5}\partial_\beta\phi(\partial_\alpha\phi_{\alpha\beta} + \tfrac{1}{2}\partial_\beta\phi_{\alpha\alpha}) + \frac{\bar{p}_F^4}{35}[(\partial_\alpha\phi_{\alpha\beta} + \tfrac{1}{2}\partial_\beta\phi_{\alpha\alpha})^2 + \tfrac{1}{6}(\partial_\gamma\phi_{\alpha\beta} + \partial_\beta\phi_{\alpha\gamma} + \partial_\alpha\phi_{\beta\gamma})^2]\right\}\Theta , \tag{3.12}$$

where $\partial_\beta \equiv \partial/\partial x_\beta$.

According to their definitions the expressions for the density $\rho(\mathbf{r},t)$ and the current density $\mathbf{j}(\mathbf{r},t)$ are

$$\rho(\mathbf{r},t) = \int \frac{d^3p}{2\pi^3} f(\mathbf{r},\mathbf{p},t) = \bar{\rho}_0 + \rho_1 - \tfrac{1}{2}\bar{\rho}_0\chi_{\alpha\alpha} - \tfrac{1}{2}\rho_1\chi_{\alpha\alpha} + \tfrac{1}{8}\bar{\rho}_0(\chi_{\alpha\alpha}^2 + 2\chi_{\alpha\beta}^2) \tag{3.13}$$

and

$$j_k(\mathbf{r},t) = \int \frac{d^3p}{2\pi^3}\frac{p_k}{m} f(\mathbf{r},\mathbf{p},t) = -\frac{\bar{\rho}_0}{m}\left[\partial_k\phi + \frac{\bar{p}_F^2}{10}(\partial_k\phi_{\alpha\alpha} + 2\partial_j\phi_{jk})\right] - \frac{1}{5m}\phi_{kj}\partial_j(\bar{p}_F^2\bar{\rho}_0) \; . \tag{3.14}$$

In order to ensure that the current density is not singular at the surface, the following boundary condition on the field $\phi_{\alpha\beta}$ is imposed

$$x_\alpha\phi_{\alpha\beta}\,|_{r=R} = 0 \; . \tag{3.15}$$

The equations of motion and the additional boundary conditions that are needed to specify the dynamical fields arise naturally from the variation of the action subjected to the restriction (3.15). Arbitrary variations of the fields lead, in the interior, to the set of equations

$$\dot{\rho}_1 - \frac{\bar{\rho}_0}{2}\dot{\chi}_{\alpha\alpha} = -\frac{\delta T}{\delta\phi} , \tag{3.16a}$$

$$\delta_{\alpha\beta}\left[\frac{\bar{p}_F^2\dot{\rho}_1}{6} - \frac{\bar{p}_F^2\bar{\rho}_0}{20}\dot{\chi}_{\alpha\alpha}\right] - \frac{\bar{p}_F^2\bar{\rho}_0}{10}\dot{\chi}_{\alpha\beta} = -\frac{\delta T}{\delta\phi_{\alpha\beta}} , \tag{3.16b}$$

$$\dot{\phi} + \frac{\bar{p}_F^2}{6}\dot{\phi}_{\beta\beta} = \frac{\delta E}{\delta\rho_1} , \tag{3.16c}$$

$$\delta_{\alpha\beta}\left[-\frac{\bar{\rho}_0}{2}\dot{\phi} - \frac{\bar{p}_F^2\bar{\rho}_0}{20}\dot{\phi}_{\gamma\gamma}\right] - \frac{\bar{p}_F^2\bar{\rho}_0}{10}\dot{\phi}_{\alpha\beta} = \frac{\delta E}{\delta\chi_{\alpha\beta}} \; . \tag{3.16d}$$

We emphasize that Eqs. (3.16) are valid only in the interior. Free variations at the surface give the boundary conditions

$$\hat{\mathbf{x}}_\beta\left[\partial_\beta\phi + \frac{\bar{p}_F^2}{5}(\partial_\alpha\phi_{\alpha\beta} + \tfrac{1}{2}\partial_\beta\phi_{\alpha\alpha})\right] + m(\delta\dot{\mathbf{R}}\cdot\hat{\mathbf{n}})\Big|_{r=R} = 0 , \tag{3.17a}$$

$$\delta_{\alpha\beta}\left\{\hat{\mathbf{x}}_\eta\left[\partial_\eta\phi + \frac{\bar{p}_F^2}{7}(\partial_\alpha\phi_{\alpha\eta} + \tfrac{1}{2}\partial_\eta\phi_{\gamma\gamma}) + m(\delta\dot{\mathbf{R}}\cdot\hat{\mathbf{n}})\right]\right\} + \hat{\mathbf{x}}_\alpha\left[\partial_\beta\phi + \frac{\bar{p}_F^2}{7}(\partial_\gamma\phi_{\beta\gamma} + \tfrac{1}{2}\partial_\beta\phi_{\gamma\gamma}) - \xi_\beta\right] + \alpha\leftrightarrow\beta + \hat{\mathbf{x}}_\gamma\frac{p_F^2}{7}(\partial_\gamma\phi_{\alpha\beta} + \partial_\beta\phi_{\alpha\gamma} + \partial_\alpha\phi_{\beta\gamma})\Big|_{r=R} = 0 , \tag{3.17b}$$

and

$$\dot{\phi}+\frac{\bar{p}_F^2}{10}\dot{\phi}_{\alpha\alpha}\bigg|_{r=R}=0\,. \tag{3.17c}$$

In the second boundary condition the vector ξ is the Lagrange multiplier that takes into account the restriction (3.15)

We seek solutions of the form

$$A(\mathbf{r},t)=A_0(\mathbf{r})+A_1(\mathbf{r})t+\sum_n A^{(n)}(\mathbf{r})\sin(\omega_n t+\alpha_n) \tag{3.18}$$

for the fields $\rho_1(\mathbf{r},t)$ and $\chi_{\alpha\beta}(\mathbf{r},t)$ and the displacement $\delta\mathbf{R}(\mathbf{r},t)$, and of the form

$$B(\mathbf{r},t)=B_0(\mathbf{r})+\sum_n B^{(n)}(\mathbf{r})\cos(\omega_n t+\alpha_n) \tag{3.19}$$

for the fields $\phi(\mathbf{r},t)$ and $\phi_{\alpha\beta}(\mathbf{r},t)$.

Equations (3.16) lead to the random phase approximation (RPA) equations for the eigenmodes $A^{(n)}$ and $B^{(n)}$ [we recall that Eqs. (3.20a) and (3.20b) are valid only in the interior]

$$\omega_n\left[\rho_1^{(n)}-\frac{\bar{\rho}_0}{2}\chi_{\alpha\alpha}^{(n)}\right]=-\frac{\delta T}{\delta\phi^{(n)}}\,, \tag{3.20a}$$

$$\omega_n\left[\delta_{\alpha\beta}\left[\frac{\bar{p}_F^2\rho_1^{(n)}}{6}-\frac{\bar{p}_F^2\bar{\rho}_0}{20}\chi_{\alpha\alpha}^{(n)}\right]-\frac{\bar{p}_F^2\bar{\rho}_0}{10}\chi_{\alpha\beta}^{(n)}\right]=-\frac{\delta T}{\delta\phi_{\alpha\beta}^{(n)}}\,, \tag{3.20b}$$

$$\omega_n\left[\phi^{(n)}+\frac{\bar{p}_F^2}{6}\phi_{\gamma\gamma}^{(n)}\right]=-\frac{\delta E}{\delta\rho_1^{(n)}}\,, \tag{3.20c}$$

$$\omega_n\left[\delta_{\alpha\beta}\left[\frac{\bar{\rho}_0}{2}\phi^{(n)}+\frac{\bar{p}_F^2\bar{\rho}_0}{20}\phi_{\alpha\alpha}^{(n)}\right]+\frac{\bar{p}_F^2\bar{\rho}_0}{10}\phi_{\alpha\beta}^{(n)}\right]=\frac{\delta E}{\delta\chi_{\alpha\beta}^{(n)}}\,. \tag{3.20d}$$

For the zero frequency modes the fields ρ_1 and $\chi_{\alpha\beta}$ are zero and the fields ϕ and $\phi_{\alpha\beta}$ satisfy the following equations:

$$\nabla^2\phi^{(0)}+\frac{p_F^2}{5}(\partial_\alpha\partial_\beta\phi_{\alpha\beta}^{(0)}+\tfrac{1}{2}\nabla^2\phi_{\alpha\alpha}^{(0)})=0 \tag{3.21}$$

and

$$\delta_{\alpha\beta}\left[\nabla^2\phi^{(0)}+\frac{\bar{p}_F^2}{7}(\partial_\alpha\partial_\beta\phi_{\alpha\beta}^{(0)}+\tfrac{1}{2}\nabla^2\phi_{\alpha\alpha}^{(0)})\right]+2\partial_\alpha\partial_\beta\phi^{(0)}$$
$$+\frac{\bar{p}_F^2}{7}(\nabla^2\phi_{\alpha\beta}^{(0)}+\partial_\alpha\partial_\beta\phi_{\gamma\gamma}^{(0)}+2\partial_\gamma\partial_\beta\phi_{\alpha\gamma}^{(0)}+2\partial_\gamma\partial_\alpha\phi_{\beta\gamma}^{(0)})=0\,. \tag{3.22}$$

Zero frequency modes appear because the model does not account for the surface tension that opposes the free expansion of the surface. Obviously, these modes should not be forgotten. Indeed they will be shown to be very important in the development of the model.

The following normalization conditions are satisfied

$$\int d^3r\left[-\left[\rho_1^{(n)}-\frac{\bar{\rho}_0}{2}\chi_{\alpha\alpha}^{(n)}\right]\left[\phi^{(m)}+\frac{\bar{p}_F^2}{6}\phi_{\alpha\alpha}^{(m)}\right]+\frac{\bar{p}_F^2\bar{\rho}_0}{10}\left[\chi_{\alpha\beta}^{(n)}-\frac{\delta_{\alpha\beta}}{3}\chi_{\gamma\gamma}^{(n)}\right]\left[\phi_{\alpha\beta}^{(m)}-\frac{\delta_{\alpha\beta}}{3}\phi_{\gamma\gamma}^{(m)}\right]\right]\Theta$$
$$-\int d\boldsymbol{\Sigma}\cdot\delta\mathbf{R}^{(n)}\rho_0\left[\phi^{(m)}+\frac{p_F^2}{10}\phi_{\alpha\alpha}^{(m)}\right]\delta_{m0}=\delta_{mn}\,. \tag{3.23}$$

We introduce now the solutions for the tensor fields and the coupled scalar field both for electric and magnetic modes.

A. Electric modes

The general solution is a linear combination of the eigenmode solutions. The angular dependence of the fields $\chi_{\alpha\beta}$ and $\phi_{\alpha\beta}$ is a linear combination of the four linearly independent angular tensor functions with even parity for angular momentum l,

$$\partial_\alpha\partial_\beta Y_{lm}\,,$$
$$\delta_{\alpha\beta}Y_{lm}\,,$$
$$(x_\alpha\partial_\beta+x_\beta\partial_\alpha)Y_{lm}\,,$$
$$x_\alpha x_\beta Y_{lm}\,.$$

From these tensors we construct three traceless solutions

$$[\phi_{\alpha\beta}]_{1E}=\{(\partial_\alpha\partial_\beta-\delta_{\alpha\beta}\nabla^2)l^2-[\partial_\alpha(\nabla\times\mathbf{l})_\beta+\partial_\beta(\nabla\times\mathbf{l})_\alpha]$$
$$-[(\nabla\times\mathbf{l})_\alpha(\nabla\times\mathbf{l})_\beta$$
$$+(\nabla\times\mathbf{l})_\beta(\nabla\times\mathbf{l})_\alpha]\}j_l(k_1r)Y_{l0}\,, \tag{3.24a}$$

$$[\phi_{\alpha\beta}]_{2E}=\left[\partial_\alpha\partial_\beta-\frac{\delta_{\alpha\beta}}{3}\nabla^2\right]j_l(k_2r)Y_{l0}\,, \tag{3.24b}$$

$$[\phi_{\alpha\beta}]_{3E}=[\partial_\alpha(\nabla\times\mathbf{l})_\beta+\partial_\beta(\nabla\times\mathbf{l})_\alpha]j_l(k_3r)Y_{l0}\,. \tag{3.24c}$$

The radial dependence of these solutions is given by the spherical Bessel functions $j_l(k_ir)$. The fields $[\phi_{\alpha\beta}]_{1E}$ and $[\phi_{\alpha\beta}]_{3E}$ verify the relations

$$\partial_\alpha[\phi_{\alpha\beta}]_{1E}=0 \text{ and } \partial_\alpha\partial_\beta[\phi_{\alpha\beta}]_{3E}=0\,,$$

and are, therefore, transverse fields which are not coupled to scalar fields. To each one of these solutions corresponds a different sound velocity c_{1T} and c_{3T},

$$c_{1T}^2=\frac{\bar{p}_F^2}{7m^2}\,,$$

$$c_{3T}^2=\frac{3\bar{p}_F^2}{7m^2}\,. \tag{3.25}$$

The longitudinal solution $[\phi_{\alpha\beta}]_{2E}$ yields two different sound velocities

$$c_{2{a \atop b}}^2=\frac{\bar{p}_F^2}{2m}\left\{\frac{6}{7}+\frac{F_0}{3}\pm\left[\left[\frac{F_0}{3}\right]^2+\tfrac{1}{35}\left(\tfrac{96}{7}+8F_0\right)\right]^{1/2}\right\}, \tag{3.26}$$

and the scalar fields coupled to $[\phi_{\alpha\beta}]_{2a}$ and $[\phi_{\alpha\beta}]_{2b}$ (one tensor field for each sound velocity) are

$$\phi_i=\frac{2\bar{p}_F^2}{15}C_i k_{2i}^2 j_l(k_{2i}r)Y_{l0}\,, \tag{3.27}$$

where the constants

$$C_i=\frac{\bar{p}_F^2 G_0}{\bar{p}_F^2 G_0-3m^2c_{2i}^2}\,,\quad (i=a,b)$$

contain the dependence on the longitudinal sound velocities.

A fifth nontraceless solution for $\phi_{\alpha\beta}$ is chosen taking advantage of the invariance of the equations of motion and boundary conditions with respect to the transformation

$$\phi'=\phi+\frac{\bar{p}_F^2}{2}V(\mathbf{r})\,,$$

$$\phi'_{\alpha\beta}=\phi_{\alpha\beta}-\delta_{\alpha\beta}V(\mathbf{r})\,, \tag{3.28}$$

where $V(\mathbf{r})$ is an arbitrary function.

We then choose

$$[\phi_{\alpha\beta}]_{5E}=\delta_{\alpha\beta}F(r)Y_{l0} \tag{3.29}$$

with the scalar field

$$\phi_5=-\frac{\bar{p}_F^2}{2}F(r)Y_{l0}\,, \tag{3.30}$$

$F(r)$ being an arbitrary function. This solution is not trivial because (3.15) is not invariant under the transformation (3.28) since we have

$$x_\alpha\phi'_{\alpha\beta}=x_\alpha\phi_{\alpha\beta}-x_\beta F(r)Y_{l0}\,.$$

The fields ρ_1 and $\chi_{\alpha\beta}$ are obtained from the fields $\phi_{\alpha\beta}$ and ϕ through Eqs. (3.16).

For zero-frequency modes there are five linearly independent solutions of the local equations for the fields ϕ and $\phi_{\alpha\beta}$. These solutions are written as follows

$$[\phi_{\alpha\beta}^{(0)}]_1=[\partial_\alpha(\nabla\times\mathbf{l})_\beta+\partial_\beta(\nabla\times\mathbf{l})_\alpha-(l+1)\partial_\alpha\partial_\beta]r^{l+2}Y_{l0}\,, \tag{3.31a}$$

$$[\phi^{(0)}]_1=0\,,$$

$$[\phi_{\alpha\beta}^{(0)}]_l=\partial_\alpha\partial_\beta r^l Y_{l0}\,, \tag{3.31b}$$

$$[\phi^{(0)}]_2=0\,,$$

$$[\phi_{\alpha\beta}^{(0)}]_3=\left[\partial_\alpha\partial_\beta-\frac{\delta_{\alpha\beta}}{3}\nabla^2\right]r^{l+2}Y_{l0}\,, \tag{3.31c}$$

$$[\phi^{(0)}]_3=-\frac{11\bar{p}_F^2}{21}(2l+3)r^l Y_{l0}\,,$$

$$[\phi_{\alpha\beta}^{(0)}]_4=(\partial_\alpha\partial_\beta-\delta_{\alpha\beta}\nabla^2)r^{l+2}Y_{l0}\,, \tag{3.31d}$$

$$[\phi^{(0)}]_4=\frac{\bar{p}_F^2}{7}(2l+3)r^l Y_{l0}\,,$$

$$\begin{aligned}[\phi_{\alpha\beta}^{(0)}]_5=\{&(\partial_\alpha\partial_\beta-\delta_{\alpha\beta}\nabla^2)l^2\\ &-[(\nabla\times\mathbf{l})_\alpha(\nabla\times\mathbf{l})_\beta+(\nabla\times\mathbf{l})_\beta(\nabla\times\mathbf{l})_\alpha]\\ &+(2l+1)/3[\partial_\alpha(\nabla\times\mathbf{l})_\beta+\partial_\beta(\nabla\times\mathbf{l})_\alpha]\\ &-5(l+1)(l+2)/3\partial_\alpha\partial_\beta\}r^{l+4}Y_{l0}\,,\end{aligned} \tag{3.31e}$$

$$[\phi^{(0)}]_5=2\bar{p}_F^2(l+1)(l+2)(2l+5)r^{l+2}Y_{l0}\,.$$

The solution for a given normal mode is a linear combination of the five particular solutions fixed by the boundary conditions and the normalization conditions.

B. Magnetic modes

For the purely transverse magnetic modes the equations obtained in Sec. III become simpler because the fields $\rho_1(\mathbf{r},t)$ and $\phi(\mathbf{r},t)$ vanish and the fields $\phi_{\alpha\beta}(\mathbf{r},t)$ and $\chi_{\alpha\beta}(\mathbf{r},t)$ are traceless tensors.

To construct the tensor fields $\phi_{\alpha\beta}$ and $\chi_{\alpha\beta}$ we use the two linearly independent transverse angular functions with angular momentum l,

$$(\partial_\alpha l_\beta+\partial_\beta l_\alpha)Y_{l0}$$

and

$$(x_\alpha l_\beta+x_\beta l_\alpha)Y_{l0}\,.$$

From these, two linearly independent solutions are built

$$[\phi_{\alpha\beta}]_{1M}=[(\nabla\times\hat{\mathbf{u}}_\gamma)_\beta l_\alpha l_\gamma+(\nabla\times\hat{\mathbf{u}}_\gamma)_\alpha l_\beta l_\gamma]\times j_l(k_1r)Y_{l0} \tag{3.32a}$$

and

$$[\phi_{\alpha\beta}]_{2M}=(l_\alpha\partial_\beta+l_\beta\partial_\alpha)j_l(k_2r)Y_{l0}\,, \tag{3.32b}$$

where $\hat{\mathbf{u}}$ is a unit vector, and $j_l(k_ir)$ are spherical Bessel functions. These solutions correspond to two different sound velocities, c_{1T} and c_{2T},

$$c_{1T}^2=\frac{\bar{p}_F^2}{7m^2} \tag{3.33a}$$

and

$$c_{2T}^2=\frac{3\bar{p}_F^2}{7m^2}\,, \tag{3.33b}$$

and the following relations are verified

$$\partial_\alpha[\phi_{\alpha\beta}]_{1M}=0 \text{ and } \partial_\alpha\partial_\beta[\phi_{\alpha\beta}]_{2M}=0\,.$$

The general solution is

$$\phi_{\alpha\beta}=F_{1M}[\phi_{\alpha\beta}]_{1M}+F_{2M}[\phi_{\alpha\beta}]_{2M}\,, \tag{3.34}$$

where F_{1M} and F_{2M} are constants fixed by the boundary conditions and normalization conditions.

IV. SUM RULES

This model satisfies several important sum rules: for the electric modes, the sum rules S_1, S_{-1}, and S_3 and for the magnetic modes the sum rule S_1. We will derive these sum rules using an adequate set of initial conditions for each case.

In order to prove the energy-weighted sum rule for the electric modes we consider the following initial conditions

$$\phi(\mathbf{r},0)=D(\mathbf{r}) ,$$
$$\phi_{\alpha\beta}(\mathbf{r},0)=\rho_1(\mathbf{r},0)=\chi_{\alpha\beta}(\mathbf{r},0)=\delta\mathbf{R}(\mathbf{r},0)=0 . \quad (4.1)$$

We expand the fields $\phi(\mathbf{r},0)$ and $\phi_{\alpha\beta}(\mathbf{r},0)$,

$$[\phi(\mathbf{r},0),\phi_{\alpha\beta}(\mathbf{r},0)]=\sum_n c_n(\phi^{(n)}(\mathbf{r}),\phi_{\alpha\beta}^{(n)}(\mathbf{r})) , \quad (4.2)$$

where, from the orthogonality relations, we have

$$c_n=-\int d^3r \left[\rho_1^{(n)}-\frac{\bar{\rho}_0}{2}\chi_{\alpha\alpha}^{(n)}\right] D(\mathbf{r})\Theta-\int d\mathbf{\Sigma}\cdot\delta\mathbf{R}^{(n)}\rho_0 D(\mathbf{r}) . \quad (4.3)$$

The coefficients c_n are closely related to the probability amplitude for a transition from the ground state to the excited state $|n\rangle$,

$$c_n=\sqrt{2}\langle n\,|\,D(\mathbf{r})\,|\,0\rangle . \quad (4.4)$$

The density field $\rho_1(\mathbf{r},t)$, the tensor field $\chi_{\alpha\beta}(\mathbf{r},t)$, the surface displacement $\delta\mathbf{R}(\mathbf{r},t)$ and the respective derivatives are then

$$[\rho_1(\mathbf{r},t),\chi_{\alpha\beta}(\mathbf{r},t)]=\sum_{n\neq 0} c_n(\rho_1^{(n)}(\mathbf{r}),\chi_{\alpha\beta}^{(n)}(\mathbf{r}))\sin\omega_n t , \quad (4.5a)$$

$$[\dot{\rho}_1(\mathbf{r},t),\dot{\chi}_{\alpha\beta}(\mathbf{r},t)]=\sum_{n\neq 0} \omega_n c_n(\rho_1^{(n)}(\mathbf{r}),\chi_{\alpha\beta}^{(n)}(\mathbf{r}))\cos\omega_n t \quad (4.5b)$$

and

$$\delta\mathbf{R}(\mathbf{r},t)=c_0\delta\mathbf{R}^{(0)}t+\sum_{n\neq 0} c_n\delta\mathbf{R}^{(n)}(\mathbf{r})\sin\omega_n t ,$$

$$\delta\dot{\mathbf{R}}(\mathbf{r},t)=c_0\delta\mathbf{R}^{(0)}+\sum_{n\neq 0} \omega_n c_n\delta\mathbf{R}^{(n)}(\mathbf{r})\cos\omega_n t . \quad (4.5c)$$

Observing that $T[\phi,\phi_{\alpha\beta}]$ is a quadratic functional in the fields we may write, from the equations of motion

$$-\int D\left[\dot{\rho}_1-\frac{\bar{\rho}_0}{2}\dot{\chi}_{\alpha\alpha}\right]\Theta\, d^3r-\int d\mathbf{\Sigma}\cdot\delta\dot{\mathbf{R}}\bar{\rho}_0 D$$
$$=\int D\frac{\delta T}{\delta D}d^3r$$
$$=2T[D,0] . \quad (4.6)$$

Finally from (4.3), (4.5), and (4.6) we obtain

$$c_0^2+\sum_n \omega_n c_n^2=2T[D,0] . \quad (4.7)$$

In a similar way we may derive the inverse energy-weighted sum rule S_{-1} from the following initial conditions

$$\phi(\mathbf{r},0)=\phi_{\alpha\beta}(\mathbf{r},0)=0 ,$$
$$[\rho_1(\mathbf{r},0),\chi_{\alpha\beta}(\mathbf{r},0),\delta\mathbf{R}(\mathbf{r},0)]$$
$$=\sum_n g_n(\rho_1^{(n)}(\mathbf{r}),\chi_{\alpha\beta}^{(n)}(\mathbf{r}),\delta\mathbf{R}^{(n)}(\mathbf{r})) , \quad (4.8)$$

where

$$g_n=\int d^3r \left[-\left[\rho_1-\frac{\bar{\rho}_0}{2}\chi_{\alpha\alpha}\right]\left[\phi^{(n)}+\frac{p_F^2}{6}\phi_{\alpha\alpha}^{(n)}\right]+\frac{\bar{p}_F^2\bar{\rho}_0}{10}\left[\chi_{\alpha\beta}-\frac{\delta_{\alpha\beta}}{3}\chi_{\gamma\gamma}\right]\left[\phi_{\alpha\beta}^{(n)}-\frac{\delta_{\alpha\beta}}{3}\phi_{\gamma\gamma}^{(n)}\right]\right]\Theta$$
$$-\int d\mathbf{\Sigma}\cdot\delta\mathbf{R}\bar{\rho}_0\left[\phi^{(n)}+\frac{\bar{p}_F^2}{10}\phi_{\alpha\alpha}^{(n)}\right]\delta_{n0} . \quad (4.9)$$

Thus, it may be written for an arbitrary time t

$$(\phi,\phi_{\alpha\beta})=-\sum_n g_n\sin\omega_n t(\phi^{(n)}(\mathbf{r}),\phi_{\alpha\beta}^{(n)}(\mathbf{r})) ,$$
$$(\rho_1,\chi_{\alpha\beta},\delta\mathbf{R})=\sum_n g_n\cos\omega_n t$$
$$\times(\rho_1^{(n)}(\mathbf{r}),\chi_{\alpha\beta}^{(n)}(\mathbf{r}),\delta\mathbf{R}^{(n)}(\mathbf{r})) . \quad (4.10)$$

From the equations of motion we have

$$\int d^3r\left[\left[\rho_1-\frac{\bar{\rho}_0}{2}\chi_{\alpha\alpha}\right]\left[\dot{\phi}+\frac{\bar{p}_F^2}{6}\dot{\phi}_{\beta\beta}\right]-\frac{\bar{p}_F^2\bar{\rho}_0}{10}\left[\chi_{\alpha\beta}-\frac{\delta_{\alpha\beta}}{3}\chi_{\gamma\gamma}\right]\left[\dot{\phi}_{\alpha\beta}-\frac{\delta_{\alpha\beta}}{3}\dot{\phi}_{\gamma\gamma}\right]\right]\Theta$$
$$+\int d\mathbf{\Sigma}\cdot\delta\mathbf{R}\bar{\rho}_0\left[\dot{\phi}+\frac{\bar{p}_F^2}{10}\dot{\phi}_{\alpha\alpha}\right]=\int d^3r\left[\rho_1\frac{\delta E}{\delta\rho_1}+\chi_{\alpha\beta}\frac{\delta E}{\delta\chi_{\alpha\beta}}\right] . \quad (4.11)$$

Finally, from Eqs. (3.23), (4.10), and (4.11) the following sum rule is easily derived

$$\sum_n \omega_n g_n^2=2E[\rho_1,\chi_{\alpha\beta}] . \quad (4.12)$$

Now we consider the system perturbed by an external static field

$$D(\mathbf{r})=\sum_{j=1}^{A} D(\mathbf{r}_j) . \quad (4.13)$$

The perturbed energy of the system is

$$E'=E[\rho_1,\chi_{\alpha\beta}]+\int d^3r\,D(\mathbf{r})\left[\rho_1-\frac{\bar{\rho}_0}{2}\chi_{\alpha\alpha}\right]\Theta + \int d\boldsymbol{\Sigma}\cdot\delta\mathbf{R}\bar{\rho}_0 D(\mathbf{r})\ . \tag{4.14}$$

In order to determine the fields ρ_1 and $\chi_{\alpha\beta}$ we minimize E' and obtain the following equations

$$D(\mathbf{r})\Theta=-\frac{\delta E}{\delta\rho_1},\ \ \delta_{\alpha\beta}\frac{\bar{\rho}_0}{2}D(\mathbf{r})\Theta=\frac{\delta E}{\delta\chi_{\alpha\beta}}\ , \tag{4.15}$$

in the interior. The perturbing field $D(\mathbf{r})$ must satisfy $D(\mathbf{R})=0$ at the boundary. The solution of these equations is

$$\chi_{\alpha\beta}=0,\ \ \rho_1=-\frac{3m\bar{\rho}_0}{\bar{p}_F^2 G_0}D(\mathbf{r})\ . \tag{4.16}$$

We now expand the field $D(\mathbf{r})$ in the normal modes

$$D(\mathbf{r})=\sum_n c_n\phi^{(n)}(\mathbf{r}) \tag{4.17}$$

with

$$c_n=-\int d^3r\,D(\mathbf{r})\rho_1^{(n)}(\mathbf{r})\Theta\ . \tag{4.18}$$

There is no surface contribution to c_n because $D(\mathbf{r})$ is zero at the boundary. Considering Eqs. (4.10) and (4.15) and the fact that E is a quadratic functional we have

$$\begin{aligned} c_n&=\int d^3r\left[\rho_1^{(n)}(\mathbf{r})\frac{\delta E}{\delta\rho_1}+\chi_{\alpha\beta}^{(n)}(\mathbf{r})\frac{\delta E}{\delta\chi_{\alpha\beta}}\right]\\ &=\int d^3r\left[\rho_1(\mathbf{r})\frac{\delta E}{\delta\rho_1^{(n)}}+\chi_{\alpha\beta}(\mathbf{r})\frac{\delta E}{\delta\chi_{\alpha\beta}^{(n)}}\right]\\ &=-\int d^3r\,\omega_n\left[\rho_1\phi^{(n)}-\frac{\bar{p}_F^2\bar{\rho}_0}{10}\chi_{\alpha\beta}\phi_{\alpha\beta}^{(n)}\right]=\omega_n g_n\ . \end{aligned} \tag{4.19}$$

The desired sum rule is finally obtained from Eq. (4.12)

$$\sum_n\omega_n^{-1}c_n^2=2E[\rho_1,0]=\int d^3r\frac{3m\bar{\rho}_0}{\bar{p}_F^2G_0}D^2(\mathbf{r})\Theta\ . \tag{4.20}$$

In order to derive the sum rule S_1 for the magnetic modes we observe that the magnetic transition operator

$$\tfrac{1}{2}\sum_i[\mathbf{p}_i\cdot\mathbf{A}(\mathbf{r}_i)+\mathbf{A}(\mathbf{r}_i)\cdot\mathbf{p}_i],\ \operatorname{div}\mathbf{A}=0\ ,$$

induces a deformation in the system which is described by the tensor $\chi_{\alpha\beta}$ of the particular form

$$\chi_{\alpha\beta}=\partial_\alpha A_\beta+\partial_\beta A_\alpha\ . \tag{4.21}$$

We consider the initial conditions

$$\begin{aligned} &\phi_{\alpha\beta}(\mathbf{r},0)=0\ ,\\ &\chi_{\alpha\beta}(\mathbf{r},0)=\sum_n d_n\chi_{\alpha\beta}^{(n)}(\mathbf{r})=\partial_\alpha A_\beta+\partial_\beta A_\alpha\ , \end{aligned} \tag{4.22}$$

where d_n is determined by the normalization relations

$$d_n=\int d^3r\frac{\bar{\rho}_0\bar{p}_F^2}{10}\chi_{\alpha\beta}\phi_{\alpha\beta}^{(n)}\Theta\ . \tag{4.23}$$

From the expansion

$$\dot{\phi}_{\alpha\beta}(\mathbf{r},0)=-\sum_n\omega_n d_n\phi_{\alpha\beta}^{(n)}(\mathbf{r})\ , \tag{4.24}$$

and the equations of motion which lead to

$$-\int\frac{\bar{p}_F^2\bar{\rho}_0}{10}\chi_{\alpha\beta}\dot{\phi}_{\alpha\beta}\Theta\,d^3r=\int\chi_{\alpha\beta}\frac{\delta E}{\delta\chi_{\alpha\beta}}d^3r=2E[0,\chi_{\alpha\beta}] \tag{4.25}$$

with

$$E[0,\chi_{\alpha\beta}]=\int\frac{\bar{p}_F^2\bar{\rho}_0}{20m}\chi_{\alpha\beta}^2\Theta\,d^3r\ , \tag{4.26}$$

we obtain the sum rule

$$\sum_n\omega_n d_n^2=2E[0,\chi_{\alpha\beta}]\ . \tag{4.27}$$

The coefficient d_n is related to the transition amplitude

$$d_n=\sqrt{2}\langle n\,|\,\mathbf{p}\cdot\mathbf{A}\,|\,0\rangle\ , \tag{4.28}$$

where $\mathbf{p}\cdot\mathbf{A}$ denotes the sum $\frac{1}{2}\sum[\mathbf{p}_i\cdot\mathbf{A}(\mathbf{r}_i)+\mathbf{A}(\mathbf{r}_i)\cdot\mathbf{p}_i]$.

From this last sum rule we may easily derive the sum rule S_3 for the electric modes observing that

$$\{H,D\}=-\frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla}D\ . \tag{4.29}$$

We choose the following initial conditions

$$\begin{aligned} &\phi(\mathbf{r},0)=\phi_{\alpha\beta}(\mathbf{r},0)=0\ ,\\ &[\rho_1(\mathbf{r},0),\delta\mathbf{R}(\mathbf{r},0)]=\sum_n d_n(\rho_1^{(n)}(\mathbf{r}),\delta\mathbf{R}^{(n)}(\mathbf{r}))=0\ ,\\ &\chi_{\alpha\beta}(\mathbf{r},0)=\sum_n d_n\chi_{\alpha\beta}^{(n)}(\mathbf{r})=\frac{2}{m}\partial_\alpha\partial_\beta D(\mathbf{r})\ , \end{aligned} \tag{4.30}$$

where the coefficients d_n, $n\neq 0$, are given by

$$d_n=\int\left[\frac{\bar{\rho}_0}{2}\chi_{\alpha\alpha}\left[\phi^{(n)}+\frac{\bar{p}_F^2}{6}\phi_{\alpha\alpha}^{(n)}\right]+\frac{\bar{p}_F^2\bar{\rho}_0}{10}\left[\chi_{\alpha\beta}-\frac{\delta_{\alpha\beta}}{3}\chi_{\gamma\gamma}\right]\left[\phi_{\alpha\beta}^{(n)}-\frac{\delta_{\alpha\beta}}{3}\phi_{\gamma\gamma}^{(n)}\right]\right]\Theta\,d^3r \tag{4.31}$$

and $d_0=0$. The sum rule (4.27) is still verified. From the initial conditions and the definition of the d_n it may be shown by partial integration that

$$\begin{aligned} d_n&=\int d^3r\,D\frac{\delta T}{\delta\phi^{(n)}}\\ &=-\int\omega_n\rho_1^{(n)}D\Theta\,d^3r-\int d\boldsymbol{\Sigma}\cdot\delta\mathbf{R}^{(n)}\omega_n\bar{\rho}_0 D\\ &=\omega_n c_n\ . \end{aligned} \tag{4.32}$$

The coefficients c_n have been defined by expression (4.3).

TABLE I. For the states indicated, energies and percentages of the EWSR are given. The following excitation operators have been used: r^2Y_{00} for $l=0$ and r^lY_{l0} for $l\geq 1$.

$l^\pi=0^+$		$l^\pi=1^-$		$l^\pi=2^+$		$l^\pi=3^-$		$l^\pi=4^+$	
$E=\hbar\omega$ (MeV)	% of EWSR	$E=\hbar\omega$ (MeV)	% of EWSR	$E=\hbar\omega$ (MeV)	% of EWSR	$E=\hbar\omega$ (MeV)	% of EWSR	$E=\hbar\omega$ (MeV)	% of EWSR
18.11	90.94	0	100	2.96	32.26	0	36.7	0	37.2
20.03	5.11	6.07		11.00	63.11	8.05	3×10^{-1}	11.75	1.96
29.72	0.278	9.24		17.49	4.06×10^{-3}	17.68	50.36	22.96	34.98
38.95	0.357	24.5		19.82	2.45×10^{-3}	22.75	6.95×10^{-1}	26.58	5.2×10^{-2}
		25.1		21.12	0.022	23.89	3.31	28.61	10.38
		29.2		28.88	3.29×10^{-4}	26.82	6.97	33.03	10.77
		33.9		29.05	8.81×10^{-4}	32.77	8.61×10^{-3}	36.45	4.7×10^{-2}
		34.1		37.21	1.16×10^{-4}	33.14	6.58×10^{-1}	37.31	2.43

This result also relies on the assumption that

$$r\partial_r D(\mathbf{r})\,|_{r=R}=0\ . \tag{4.33}$$

The result (4.32) is in agreement with the following relation

$$\left\langle n\left|-\frac{\mathbf{p}}{m}\cdot\nabla D\right|0\right\rangle=\langle n\,|\,[H,D]\,|\,0\rangle=\omega_n\langle n\,|\,D\,|\,0\rangle\ . \tag{4.34}$$

Finally the sum rule S_3 is obtained from (4.27) if we replace d_n by $\omega_n c_n$,

$$\sum_n \omega_n^3 c_n^2=2E[0,\partial_\alpha\partial_\beta D]\ . \tag{4.35}$$

V. NUMERICAL RESULTS

The results presented in this section refer to the nucleus ^{208}Pb. We consider a zero-range effective interaction which corresponds to a single particle potential of the form $U=a\rho^2+b\rho^3$. The parameters a and b are adjusted in order to reproduce the saturation properties of the nuclear matter,

$$k_F=1.26\ \text{fm}^{-1}\ (\bar{\rho}_0=0.135\ \text{fm}^{-3})\ ;$$

$$E/A=-13.8\ \text{MeV}\ ,$$

which yield

$$a=-799.13\ \text{MeV fm}^3\ ,$$

$$b=6711.23\ \text{MeV fm}^6\ .$$

In Table I the energy levels and the corresponding percentages of the EWSR for the electric modes are given for several multipolarity states.

$l^\pi=0^+$. For the monopole modes, which are purely longitudinal, we obtain a very collective state at 18 MeV which exhausts about 91% of the EWSR.

$l^\pi=1^-$. The lowest 1^- state is a zero frequency mode ($\omega=0$), amounting to the uniform translation of the nucleus; for the transition operator chosen in the present formulation it exhausts the total EWSR.

$l^\pi=2^+$. For the quadrupole modes, the strength is divided between a low state which occurs at $\sim$3 MeV and carries 32.26% of the EWSR and a very collective state (giant vibration) at 11 MeV with 63.11% of the EWSR.

$l^\pi=3^-$. In the case of the octupole vibrations we get one state at 17.68 MeV which exhausts a considerable fraction of the EWSR (50.36%); a zero frequency mode occurs due to the free expansion of the surface and carries about 36.7% of the EWSR; the remaining part of the EWSR lies in the range of 23—26 MeV ($\sim$10%).

$l^\pi=4^+$. The results for these modes are similar to the 3^- case.

Table II shows the numerical results obtained for the magnetic modes.

For the magnetic modes, since the model does not allow for the spin-flip mechanism, only isoscalar spin zero

TABLE II. For the states indicated, energies and percentages of EWSR are given, for a transition operator of the form lr^lY_{l0}.

$l^\pi=1^+$		$l^\pi=2^-$		$l^\pi=3^+$	
$E=\hbar\omega$ (MeV)	% of EWSR	$E=\hbar\omega$ (MeV)	% of EWSR	$E=\hbar\omega$ (MeV)	% of EWSR
0		7.84	80.12	15.44	62.63
27.52		15.19	8.45	19.04	18.41
43.43		24.48	0.57	28.21	0.58
58.85		33.5	8.3×10^{-3}	37.36	0.049
		34.07	3.95	42.47	5.97
		42.42	5.8×10^{-2}	46.33	6.3×10^{-2}
		50.19	1.76	55.2	0.55×10^{-2}
		51.23	5.47×10^{-2}	56.72	2.92

modes are taken into account. The energy levels obtained are in good agreement with other calculations.[3,12] It is remarkable that the transition strength splits over several levels for the states 2^- and 3^+.

VI. CONCLUSIONS

A variational formulation of nuclear fluid dynamics has been proposed which is valid for zero temperature. The equations for small amplitude oscillations around a stationary state and the boundary conditions which must be satisfied by the fields arise naturally from this variational formulation. As an illustration of the variational approach, a fluid-dynamical calculation which avoids the so-called scaling approximation but takes duely into account distortions of the Fermi sphere related to the first sound and zero sound modes has been discussed.

The model presented satisfies some important sum rules: the energy-weighted sum rule, S_1, both for electric and magnetic modes and also the inverse energy-weighted sum rule, S_{-1}, and the cubic energy-weighted sum rule, S_3, for electric modes.

The results obtained for the energy levels of the nucleus ^{208}Pb exhibit a remarkable splitting of the giant resonances and allow for the separation between the giant resonances and the low-lying modes, as we can see, for example, for the quadrupole states where we obtain the giant resonance state at about 11 MeV and a low-lying state at 3 MeV. For the monopole case, there are no low-lying states and the "breathing mode" appears in good agreement with the corresponding RPA results.[13] Our model accounts for surface modes. These modes occur at zero frequency due to the use of zero range forces which leads to the absence of a surface tension. Although the description of these modes is not realistic we may emphasize that the fraction of the EWSR absorbed by these states is essential to obtain good agreement with experimental data. Similar modes are discussed in Ref. 2.

ACKNOWLEDGMENTS

The authors are very grateful to Professor J. da Providência for suggesting this problem and for his continuous guidance during the course of the work, and they would also like to thank Professor Holzwarth for his valuable suggestions. This work was supported by Instituto Nacional de Investigacão Científica and Fundação Calouste Gulbenkian, Lisboa, Portugal.

[1]G. Holzwarth and G. Eckart, Nucl. Phys. **A325**, 1 (1979).

[2]G. Eckart, G. Holzwarth, and J. P. da Providência, Nucl. Phys. **A364**, 1 (1981).

[3]J. P. de Providência and G. Holzwart, Nucl. Phys. **A398**, 59 (1983).

[4]D. M. Brink and M. di Toro, Nucl. Phys. **A372**, 151 (1981).

[5]A. Bonasera, M. di Toro, and U. Lombardo, Proceedings of the Nuclear Physics Workshop, Trieste, Italy, 1981, edited by C. H. Dasso (North-Holland, Amsterdam, 1982).

[6]S. Stringari, Ann. Phys. (N.Y.) **151**, 35 (1983).

[7]J. P. da Providência, J. Phys. (Paris), Colloq. **C6**, 333 (1984).

[8]L. P. de Brito and C. da Providência, Phys. Lett. **143B**, 36 (1984).

[9]J. da Providência, L. P. Brito, and C. da Providência, Nuovo Cimento **87A**, 248 (1985).

[10]P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer, Berlin, 1980).

[11]S. Stringari, Nucl. Phys. **A325**, 199 (1979).

[12]G. Holzwarth, in *Density Functional Methods in Physics,* Proceedings of the NATO AS1, Alcabideche, Portugal, 1983, edited by R. Dreizler and J. da Providência (Plenum, New York, 1985); H. Thorn, Diplomarbeit, Siegen University, 1984 (unpublished).

[13]F. E. Serr, G. F. Bertsch, and J. Borysowicz, Phys. Lett. **92B**, 241 (1980).