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Three-particle equations for the pion-nucleon system

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An analysis is carried out of a π-N-Δ field theory using projection operator techniques developed recently by the author. Relationships are derived between matrix elements of resolvents formed from bare $\pi\pi$N and $\pi\pi\Delta$ states and various three-particle amplitudes. These relations are exactly analogous to those obtained in potential theory between the three-particle Green's function and the transition operators of Alt, Grassberger, and Sandhas. When the three-particle amplitudes are integrated over with appropriate weighting functions, the elastic scattering and production amplitudes are obtained. The weighting functions are the analogs of bound state wave functions for the πN and $\pi\Delta$ subsystems. A separation of the production amplitude into its one-fermion irreducible and reducible parts is derived. It is shown that the three-particle amplitudes are solutions of Alt, Grassberger, and Sandhas type of equations as extended by Kowalski to allow for a three-body interaction. When the Alt, Grassberger, and Sandhas quasiparticle method is applied to these equations, the two-particle equations developed previously by the author are recovered. If the one-fermion irreducible part of the three-body interaction is neglected a closed set of coupled nonlinear integral equations for all of the quantities of interest is obtained.

I. INTRODUCTION

It is a widely held belief that the most general framework for describing the interactions of particles with each other is quantum field theory. In principle with such a theory it is necessary to deal with an infinite variety of particle states in analyzing a physical process, whereas in practice only a finite number of particles is of interest. One of the perpetual problems of theoretical physics is to find ways of reducing a quantum field theory to a system of equations which describe a finite number of physical particles.

In describing the interactions of pions with nuclei it is, in principle, essential to start with a quantum field theory since the number of pions is not conserved. In recent years, effective field theories which treat the π, N, and Δ as elementary particles have been developed[1–3] starting from the MIT bag model[4] or the constituent quark model.[5] There have been several calculations on the pion-nucleon system based on these effective field theories. The calculations employ various techniques, i.e., the summing of a subset of diagrams,[1,6,7] low order perturbation theory,[8,9] or dispersion relations.[10] Recently,[11] the author has carried out an analysis of a π-N-Δ field theory using an extension of the Feshbach[12] projection operator technique.

The use of projection operators in analyzing field theories is quite old. Okubo[13] used them over 30 years ago in a discussion of the Tamm-Dancoff method in meson theory. More recently the Feshbach formalism has been used to analyze the pion-deuteron system,[14,15] pion-nucleus reactions,[16] as well as several sectors of the Lee model.[17–19] The author's use of this formalism[11] is most closely related to the applications given in Refs. 15 and 17—19 in that the projection operators project onto subspaces characterized by a definite number of bare particles. It differs in that for the models in these references each physical sector is spanned by only a few types of bare particle states. For example, in the Stingl and Stelbovics[15] model for the pion-deuteron system the only bare states allowed are $|\mathrm{NN}\rangle$, $|\mathrm{NN}\pi\rangle$, and $|\mathrm{NN}\pi\pi\rangle$.

The projection operator relations developed in Ref. 11 (hereafter referred to as A) are extensions of the equations given in the Green's function formalism of Fonda and Newton.[20,21] In applying these relations to quantum field theory it is necessary to handle the interplay of the creation and annihilation operators and the various Green's functions or resolvents with some finesse otherwise the analysis becomes unmanageable. A set of identities which make this possible is given in A. These identities are relations between resolvents which act in different subspaces, where the projection operators for these subspaces are related by the action of the creation and annihilation operators. For example, if $P_{\pi F}$ projects onto the subspace of bare one-pion—one-fermion states then $a(q)P_{\pi F}=P_F a(q)$, where $a(q)$ is a meson annihilation operator and P_F is a projector for bare one-fermion states. In this case the identities relate a Green's function which acts in the one-meson—one-fermion subspace to one that acts in the one-fermion subspace.

In a π-N-Δ field theory with static fermions the things that are of interest are the propagators and self-energies of the N and Δ, the vertex functions for the various elementary processes ($\mathrm{N}\rightleftarrows\mathrm{N}+\pi$, $\Delta\rightleftarrows\mathrm{N}+\pi$, $\mathrm{N}\rightleftarrows\Delta+\pi$, $\Delta\rightleftarrows\Delta+\pi$), and the various scattering and production amplitudes ($\mathrm{N}+\pi\rightarrow\mathrm{N}+\pi$, $\mathrm{N}+\pi\rightarrow\mathrm{N}+2\pi,\ldots$). Using the techniques referred to above, it has been shown in A that these entities can be obtained from bare matrix elements of resolvent operators of the form $G_P(z)=P(z-PHP)^{-1}P$, where H is the complete Hamiltonian, z is a complex energy parameter, and P is a projection operator onto bare states. For example $\langle \mathrm{N}|G(z)|\mathrm{N}\rangle$,

$\langle \pi N | G(z) | \Delta \rangle$, and $\langle \pi N | G(z) | \pi N \rangle$ are related to the nucleon propagator, the vertex function for $\Delta \rightleftarrows N+\pi$, and the π-N elastic scattering amplitude, respectively. Here $G(z)=(z-H)^{-1}$ and $|N\rangle$, $|\Delta\rangle$, and $|\pi N\rangle$ are bare states. It has also been shown that with projection operator techniques it is straightforward to define and separate out the reducible and irreducible contributions to various amplitudes. For example, $\langle \pi N | G_P(z) | \pi N \rangle$ is simply related to the one-fermion irreducible π-N elastic scattering amplitude if the projection operator P *excludes* all one-fermion states.

The relationships found between the matrix elements of the type $\langle \pi N | G_P(z) | \pi N \rangle$ and the elastic scattering amplitudes are in a sense reduction formulas. They differ from the well-known Lehmann, Symanzik, and Zimmerman (LSZ) reduction formulas[22,23] in that the matrix elements involve bare states rather than physical states. The reduction formulas obtained in A lead naturally to an off-shell extension of the elastic scattering amplitudes which is different from the one on which the well-known Chew-Low analysis[24,25] of π-N scattering is based.

The off-shell amplitudes introduced in A satisfy an exact set of Lippmann-Schwinger equations. If a certain approximation is made for the effective potentials that appear in these equations, it is found that the one-fermion irreducible amplitudes satisfy a set of three-particle equations that are analogous to those that occur in the quasiparticle method for solving nonrelativistic three-particle problems.[26] The starting point of this method is a set of coupled integral equations, the so-called Alt, Grassberger, and Sandhas (AGS) equations, which are exact within the framework of nonrelativistic potential scattering. In the quasiparticle method, the off-shell two-particle amplitudes that appear in the kernels of the integral equations are split into separable terms and remainders. When this is done it is possible to rewrite the AGS three-particle equations as two-particle equations with effective potentials. These potentials can be obtained by solving three-particle equations of the same form as the original equations but with only the remainder terms appearing in the kernels. If the remainders are weak enough the potentials can be calculated by perturbation theory.

There are some differences between the AGS quasiparticle equations and those obtained in A. First of all, the equations developed in A allow for the inclusion of an unstable particle, i.e., the Δ resonance. Second, solving the equations of A does not lead immediately to the physical scattering amplitudes, but rather to their one-fermion irreducible parts. Finally, the off-shell, two-particle amplitudes that appear in the kernels of the AGS-type equations of A are obtained from the solutions of the equations, i.e., the equations are nonlinear. The $N\pi$ and $N\pi\pi$ sectors are coupled and must be solved self-consistently. It has been shown that a first approximation can be obtained by solving a set of linear equations which are of the same type as those obtained with the isobar model.[27–29]

One of the shortcomings of A is that it is not clear what is left out when the approximation is made which leads to the AGS-type, three-particle equations. The motivation for the approximation was that it leads to a *closed* set of coupled, nonlinear integral equations for the coupled $N\pi$-$N\pi\pi$ system. Also, the development of A is incomplete in that no expressions were obtained for the production amplitudes. Here we will remedy these shortcomings by developing an exact three-particle formalism for pion-nucleon scattering using the same quantum field theory model as A.

The starting point of the analysis to be presented here is the derivation of reduction formulas that relate matrix elements of the type $\langle \pi\pi N | G(z) | \pi\pi N \rangle$ to the three-particle scattering amplitudes. It turns out that these reduction formulas are of exactly the same form as the relations between the three-particle Green's function in potential theory and the three-particle transition operators introduced by Alt, Grassberger, and Sandhas.[26] We will show that when certain of the three-particle amplitudes are integrated over with appropriate weighting functions, we get back the off-shell, two-particle amplitudes of A. These weighting functions are proportional to the projections of off-shell state vectors for the N and Δ onto the bare π-N and π-Δ states, and are analogous to two-particle bound state wave functions in potential theory. We will also show that the production amplitudes can be obtained by integrating over appropriate three-particle amplitudes with these weighting functions. In analyzing the production amplitudes we will obtain a separation of them into their one-fermion irreducible (OFI) and one-fermion reducible (OFR) parts, and in this connection we will introduce one-to-three vertex functions ($N \leftrightarrows N+\pi+\pi$, $\Delta \rightleftarrows N+\pi+\pi$, $N \rightleftarrows \Delta+\pi+\pi$, $\Delta \rightleftarrows \Delta+\pi+\pi$). We will refer to these as production vertex functions.

By starting with an effective Hamiltonian that acts in the subspace of bare $\pi\pi N$ and $\pi\pi\Delta$ states, we will show that the three-particle amplitudes satisfy an extension of the AGS equations which allows for three-particle interactions.[30] We will see that when the quasiparticle method[26] is applied to these AGS-type equations we are led back to the exact two-particle equations of A, and furthermore when the OFI three-particle interaction that appears in the equations for the OFI three-particle amplitudes is dropped, we obtain the approximate three-particle equations of A. This gives the meaning of the approximation discussed above. With the help of the AGS equations, we will obtain two expressions for the production vertex functions. One of these expressions shows that they can be obtained from the one-fermion, one-fermion—one-meson irreducible three-particle amplitudes and the *dressed* one-to-two vertex functions ($N \rightleftarrows N+\pi$, $\Delta \rightleftarrows N+\pi$, $N \rightleftarrows \Delta+\pi$, $\Delta \rightleftarrows \Delta+\pi$), while the other gives them in terms of the OFI three-particle amplitudes and the *bare* one-to-two vertex functions.

The outline of the paper is as follows. Section II summarizes the identities obtained in A which are essential to the analysis presented here and serves to establish notation. Also, in this section we will derive a couple of new identities that are useful in deriving the reduction formulas. These formulas will be derived in Sec. III, where we will also obtain the relations between the AGS-type three-particle amplitudes and the elastic and production amplitudes. Here we will also obtain the separation of the production amplitudes into their OFR and OFI parts, and will introduce the production vertex functions. In Sec. IV we will derive the AGS equations for the three-particle amplitudes and show how we get back to the results of A by using the quasiparticle method. The two alternate expressions for the production vertex functions are also obtained in this section. Section V gives a brief discussion of the results and suggestions for future work.

It will be necessary to refer to several of the equations given in A. Their numbers will always be prefixed with an A, e.g., A(4.86).

II. IMPORTANT IDENTITIES

We begin by summarizing some of the identities developed in A so as to make the present work more self-contained and to establish notation. We will also develop a couple of new identities which will prove very useful in the subsequent analysis.

We introduce projection operators labeled with subscripts and superscripts. The projection operator P_ρ includes the states denoted by the cover index ρ, while P^ρ includes everything but the states labeled by ρ. Obviously the two are related by

$$P_\rho + P^\rho = 1 \ . \tag{2.1}$$

These operators we combine with the Hamiltonian H to define the operators

$$\begin{aligned} &H_{\lambda\rho} = P_\lambda H P_\rho, \quad H_\lambda{}^\rho = P_\lambda H P^\rho \ , \\ &H^\lambda{}_\rho = P^\lambda H P_\rho, \quad H^{\lambda\rho} = P^\lambda H P^\rho \ . \end{aligned} \tag{2.2}$$

We also define Green's functions or resolvents according to

$$G(z) = \frac{1}{z-H} \ , \tag{2.3a}$$

$$G^\lambda(z) = \frac{P^\lambda}{z-H^{\lambda\lambda}} \ , \tag{2.3b}$$

$$G_\lambda(z) = \frac{P_\lambda}{z-H_{\lambda\lambda}} \ , \tag{2.3c}$$

where z is a complex parameter.

We will frequently decompose a projection operator P^α into orthogonal projection operators P_β and P^γ according to

$$P_\beta + P^\gamma = P^\alpha \ , \tag{2.4a}$$

which by (2.1) implies that

$$P^\gamma = 1 - P_\alpha - P_\beta = P^{\alpha\beta} \ . \tag{2.4b}$$

It is shown in Sec. II of A that

$$G^\alpha(z) = G^\gamma(z) + [1 + G^\gamma(z)H] g^\alpha_\beta(z) [1 + HG^\gamma(z)] \ , \tag{2.5}$$

where

$$g^\alpha_\beta(z) = P_\beta G^\alpha(z) P_\beta \ . \tag{2.6}$$

These equations relate the resolvent $G^\alpha(z)$, which acts in the subspace of P^α, to its projection, $g^\alpha_\beta(z)$, onto the smaller subspace β, and to a resolvent $G^\gamma(z)$, which acts in the subspace of P^γ, obtained by removing P_β from P^γ. The operator $g^\alpha_\beta(z)$ can also be written in the form

$$g^\alpha_\beta(z) = \frac{P_\beta}{z - H^\alpha_\beta(z)} \ , \tag{2.7}$$

where

$$H^\alpha_\beta(z) = H_{\beta\beta} + H_\beta{}^\gamma G^\gamma(z) H^\gamma{}_\beta \ . \tag{2.8}$$

We see that $g^\alpha_\beta(z)$ is a resolvent for the pseudo-Hamiltonian $H^\alpha_\beta(z)$. The first term in (2.8) is simply the true Hamiltonian H projected onto the subspace β, while the second term describes a transition from this subspace to the subspace of $P^\gamma = P^\alpha - P_\beta$, propagation there according to the Hamiltonian $H^{\gamma\gamma}$, and return to subspace β. By multiplying (2.7) by $z - H^\alpha_\beta(z)$, we see that $g^\alpha_\beta(z)$ is the solution of an equation in which all of the operators act in the subspace β. Usually this type of equation is only of formal value but here we will see that it leads to useful results.

The results presented so far do not depend on the specific nature of the Hamiltonian. We now assume a quantum field theory of the form

$$H = H_0 + H_1 \ , \tag{2.9}$$

where H_0 is the free Hamiltonian and H_1 contains the interactions. We introduce a set of operators, $a^\dagger(p)$ and $a(p)$, which create and annihilate bosons labeled with the cover index p. These operators satisfy the commutation rule

$$[a(p), a^\dagger(q)] = \delta(p,q) \ , \tag{2.10}$$

with their other commutators zero. Here $\delta(p,q)$ is a product of Dirac delta functions and Kronecker delta symbols. We assume that

$$[H_0, a^\dagger(p)] = \omega_p a^\dagger(p) \ , \tag{2.11}$$

$$[H_1, a^\dagger(p)] = J(p) \ , \tag{2.12}$$

where ω_p is the energy of meson p, and $J(p)$ is an operator which depends only on fermion creation and annihilation operators, and hence commutes with $a(p)$ and $a^\dagger(p)$.

We now assume that P_ρ and P_σ are two projection operators related by

$$a(p) P_\rho = P_\sigma a(p) \ . \tag{2.13}$$

Here, for example, if P_ρ projects onto an n-meson—one-fermion subspace, P_σ projects onto an $(n-1)$-meson—one-fermion subspace. It is shown in A that

$$a(p) G_\rho(z) = G_\sigma(z-\omega_p)[a(p) + J^\dagger(p) G_\rho(z)] \ , \tag{2.14a}$$

$$G_\rho(z) a^\dagger(p) = [a^\dagger(p) + G_\rho(z) J(p)] G_\sigma(z-\omega_p) \ , \tag{2.14b}$$

which when combined with A(3.8) and A(3.7) leads to the important identity

$$\begin{aligned} a(q) G_\rho(z) a^\dagger(q') = {} & G_\sigma(z-\omega_q)\delta(q,q') \\ & + G_\sigma(z-\omega_q)\Omega_\rho(q,q';z) G_\sigma(z-\omega_{q'}) \ , \end{aligned} \tag{2.15}$$

where

$$\Omega_\rho(q,q';z)=a^\dagger(q')(z-\omega_q-\omega_{q'}-H)a(q)+J^\dagger(q)G_\rho(z)J(q') . \qquad (2.16)$$

We assume that

$$a(p)P_\sigma=P_\tau a(p) . \qquad (2.17)$$

Using this, (2.15), (2.16), (2.14), (2.11), (2.12), and (2.10), it is straightforward to show that

$$a(p)a(q)G_\rho(z)a^\dagger(q')=G_\tau(z-\omega_p-\omega_q)\{(z-\omega_p-\omega_q-H)[a(p)\delta(q,q')+a(q)\delta(p,q')]+a^\dagger(q')(z-\omega_p-\omega_q-\omega_{q'}-H)a(p)a(q)+J^\dagger(q)a(p)G_\rho(z)J(q')+J^\dagger(p)a(q)G_\rho(z)J(q')\}G_\sigma(z-\omega_{q'}) . \qquad (2.18)$$

We will see that the identities (2.15) and (2.18) make it possible to relate bare matrix elements of the Green's functions to physical scattering and production amplitudes.

III. FEW-PARTICLE GREEN'S FUNCTIONS

We assume that our field theory Hamiltonian describes the interaction of P-wave pions with a static nucleon (N) and a static delta (Δ) through the virtual processes

$$\begin{aligned}&N\rightleftarrows N+\pi ,\\&N\rightleftarrows \Delta+\pi ,\\&\Delta\rightleftarrows N+\pi ,\\&\Delta\rightleftarrows \Delta+\pi .\end{aligned} \qquad (3.1)$$

The details of the various vertices involved are irrelevant for the subsequent development. A specific version of the type of model we have in mind is given in Ref. 9.

In a fleshed out notation the meson annihilation operator would be given by $a_{mn}(q)$ where q is the magnitude of the meson's momentum, and m and n are the z components of its angular momentum and isospin, respectively. Here we let q be a cover index for q, m, and n, i.e., $a_{mn}(q)\to a(q)$. The product of delta symbols that appear in the commutation relation (2.10) are abbreviated according to

$$\frac{\delta(p-p')}{p^2}\delta_{mm'}\delta_{nn'}\to\delta(p,p') . \qquad (3.2)$$

In A the index f was used to distinguish N and Δ, and the cover index r was used for the z components of the fermions' angular momentum and isospin. Here we will use f as a cover index for f and r, i.e., $(fr)\to f$.

The bare fermion states satisfy

$$H_0|f\rangle=M_f^{(0)}|f\rangle , \qquad (3.3)$$

and are used to construct the projection operator for the one-fermion subspace according to

$$P_a=\sum_f |f\rangle\langle f| . \qquad (3.4)$$

The states with mesons and a fermion are denoted by

$$|pp'p''\cdots f\rangle=a^\dagger(p)a^\dagger(p')a^\dagger(p'')\cdots|f\rangle . \qquad (3.5)$$

The projection operator for the one-meson—one-fermion subspace is given by

$$P_s=\sum_f\int|pf\rangle dp\langle fp| , \qquad (3.6)$$

where it should be noted that $\int dp$ is actually a shorthand for $\sum_{m,n}\int_0^\infty dp\,p^2$. For notational purposes we also define

$$\begin{aligned}&P_0=0 ,\\&P^0=1-P_0=1 .\end{aligned} \qquad (3.7)$$

In analyzing three-particle Green's functions we will need two-particle Green's functions defined by[11]

$$G^e_{ff'}(p,p';z)=Z_f^{-1/2}\langle fp|G^e(z)|p'f'\rangle Z_{f'}^{-1/2} , \qquad (3.8)$$

where here and in all of the equations in which it appears

$$e=0,a . \qquad (3.9)$$

When $e=0$ all intermediate states are allowed, while the single fermion states are excluded when $e=a$ [see (2.3b), (2.1), (3.7), and (3.4)]. Here Z_N is the wave function renormalization constant for the nucleon, and gives the probability of finding a bare nucleon in the physical nucleon. In principle Z_Δ can be any real, positive number. A practical choice for it is given in Sec. IV of A.

We will need to know the result of acting on a one-meson—one-fermion state $|pf\rangle$ with $G^e(z)$. If in (2.5) we choose $\alpha=e$, $\beta=s$, $\gamma=es$, and use (2.6), (3.6), (3.8), and the fact that $P^{es}H_0|pf\rangle$ is zero, we find

$$G^e(z)=G^{es}(z)+\sum_{f,f'}\int|\phi^e_f(p;z)\rangle dp\,G^e_{ff'}(p,p';z)dp'\langle\phi^e_{f'}(p';z^*)| , \qquad (3.10)$$

where

$$|\phi^e_f(p;z)\rangle=[1+G^{es}(z)H_1]|pf\rangle Z_f^{1/2} . \qquad (3.11)$$

Using the fact that $P^{es}|pf\rangle$ is zero, we obtain from (3.10) and (3.11) the results

$$G^e(z)\,|pf\rangle Z_f^{-1/2}=\sum_{f'}\int |\phi^e_{f'}(p';z)\rangle dp' G^e_{f'f}(p',p;z)\ , \tag{3.12a}$$

$$Z_f^{-1/2}\langle fp\,|\,G^e(z)=\sum_{f'}\int G^e_{ff'}(p,p';z)dp'\langle\phi^e_{f'}(p';z^*)|\ . \tag{3.12b}$$

We now define three-particle Green's functions by

$$G^g_{ff'}(pq;p'q';z)=Z_f^{-1/2}\langle fpq\,|\,G^g(z)\,|\,q'p'f'\rangle Z_{f'}^{-1/2}\ , \tag{3.13}$$

where here and in all of the equations in which it appears

$$g=0,a,as\ . \tag{3.14}$$

We note that when $g=as$, one-fermion, and one-meson—one-fermion intermediate states are excluded. If in (2.15) we let $P_p=P^g$ and use (2.13), (3.8), and (3.12), we obtain

$$G^g_{ff'}(pq;p'q';z)$$
$$=G^e_{ff'}(p,p';z-\omega_q)\delta(q,q')+\sum_{f''f'''}\int G^e_{ff''}(p,p'';z-\omega_q)dp''U^g_{f''f'''}(p''q;p'''q';z)dp'''G^e_{f'''f'}(p''',p';z-\omega_{q'})\ , \tag{3.15}$$

where

$$U^g_{ff'}(pq;p'q';z)=\langle\phi^e_f(p;z^*-\omega_q)\,|\,\Omega^g(q,q';z)\,|\,\phi^e_{f'}(p';z-\omega_{q'})\rangle\ . \tag{3.16}$$

Here, and wherever g and e appear in the same equation, they occur in the combinations

$$(g,e)=(0,0),(a,0),(as,a)\ . \tag{3.17}$$

We will see that when the quantities $U^g_{ff'}$ are integrated with functions analogous to bound state wave functions, we obtain the elastic scattering amplitudes or pieces of them. Keeping this in mind, it is not difficult to see that (3.15) is a relation of the type used by Alt, Grassberger, and Sandhas[26] to define three-particle transition operators.

We will now show that when $g=as$ in (3.16), we can use $e=0$ on the right-hand side of the equation. According to A(4.78), (3.11), and (2.12), we have

$$|\phi^0_f(p;z)\rangle=|\phi^a_f(p;z)\rangle+\sum_{f'}|f'\rangle\frac{\bar{\xi}^{(0)}_{f'f}(p)Z_f^{1/2}}{z-M^{(0)}_{f'}}\ , \tag{3.18}$$

where we have defined

$$\xi^{(0)}_{ff'}(p)=\langle f\,|\,J^\dagger(p)\,|\,f'\rangle\ , \tag{3.19a}$$

$$\bar{\xi}^{(0)}_{ff'}(p)=\xi^{(0)*}_{f'f}(p)\ . \tag{3.19b}$$

The functions given by (3.19) are bare vertex functions and were introduced in A in a slightly different way. If we flesh out the notation we can write

$$\xi^{(0)}_{fr,f'r'}(p\mu)=\gamma^{(0)}_{ff'}(p)\langle fr\mu\,|\,f'r'\rangle,\ \ \mu=(mn)\ , \tag{3.20}$$

where $\langle fr\mu\,|\,f'r'\rangle$ is a product of Clebsch-Gordan coefficients defined by A(4.28). In A, we called $\gamma^{(0)}_{ff'}(p)$ the bare vertex function. It has the virture of depending only on p and the fermion labels N and Δ. If in (3.16) we consider $(g,e)=(as,a)$, and use (2.16) and (3.18), we find that

$$U^g_{ff'}(pq;p'q';z)$$
$$=\langle\phi^0_f(p;z^*-\omega_q)\,|\,\Omega^g(q,q';z)\,|\,\phi^0_{f'}(p';z-\omega_{q'})\rangle\ . \tag{3.21}$$

We now wish to relate the $U^g_{ff'}$'s to the elastic scattering amplitudes. In order to do so it is convenient to derive an alternative expression for the $|\phi^e_f\rangle$'s. According to (3.11) and A(3.6b), we can write

$$|\phi^e_f(p;z)\rangle=[a^\dagger(p)+G^{es}(z)J(p)]\,|\,F(z-\omega_p)\rangle\ , \tag{3.22}$$

where

$$|F(z)\rangle=[1+G^a(z)H_1]\,|\,f\rangle Z_f^{1/2}\ . \tag{3.23}$$

It is shown in A that $|\mathrm{N}(M_{\mathrm{N}}\pm i\epsilon)\rangle$ is the physical nucleon state. Using (3.22), A(4.66), and A(4.29), we have

$$|F(z)\rangle=|f\rangle Z_f^{1/2}+\sum_{f'}\int|\phi^a_{f'}(p';z)\rangle dp'F_{f'}(p';z)\ , \tag{3.24}$$

with

$$F_{f'}(p';z)=Z_{f'}^{-1/2}\langle f'p'\,|\,F(z)\rangle\ , \tag{3.25a}$$

$$=\tilde{\Gamma}_{f'}(z-\omega_{p'})\xi_{f'f}(p';z)\ . \tag{3.25b}$$

Here $\tilde{\Gamma}_f(z)$ is a renormalized fermion propagator which, according to A(4.5) and A(4.15), is given by

$$\tilde{\Gamma}_f(z)\delta_{ff'}=Z_f^{-1/2}\langle f\,|\,G(z)\,|\,f'\rangle Z_{f'}^{-1/2}\ , \tag{3.26}$$

while $\xi_{ff'}(p;z)$ is a dressed vertex function defined by

$$\xi_{ff'}(p;z)=\langle F(z^*-\omega_p)\,|\,J^\dagger(p)\,|\,F'(z)\rangle\ , \tag{3.27a}$$

where we have used A(4.26a). We also define

$$\bar{\xi}_{ff'}(p;z)=\xi^*_{f'f}(p;z^*)\ . \tag{3.27b}$$

If we flesh out the notation we can write

$$\xi_{fr,f'r'}(p\mu;z)=\gamma_{ff'}(p,z-\omega_p,z)\langle fr\mu\,|\,f'r'\rangle\,,\quad \mu=(mn)\,. \tag{3.28}$$

In A we called $\gamma_{ff'}$ the dressed vertex function. It is independent of the quantum numbers r and μ.

According to (3.25a) and A(4.17), $N_f(p;M_N\pm i\epsilon)$ is proportional to the projection of the physical nucleon state onto a one-meson—one-fermion state. We will see that it plays a role analogous to a two-particle bound state wave function. Of course, its norm is not one unless all of the other Fock space components vanish.

If we put (3.18) into (3.24), use (3.25b) and the orthogonality relations for the Clesbsch-Gordan coefficients in $\xi^{(0)}_{ff'}$ and $\xi_{ff'}$, as well as A(4.33), A(4.11), and A(4.15), we find

$$|F(z)\rangle=\sum_{f'}\int |\phi^0_{f'}(p';z)\rangle dp' F_{f'}(p';z) + |f\rangle\frac{\widetilde{\Gamma}_f^{-1}(z)Z_f^{-1/2}}{z-M_f^{(0)}}\,. \tag{3.29}$$

We note that $\widetilde{\Gamma}_N^{-1}(M_N)=0$, i.e., the nucleon propagator has a pole at the physical nucleon mass, therefore the second term on the right-hand side of (3.29) vanishes when $f=N$ and $z=M_N$. According to A(4.38), A(4.50), A(4.19), and (2.16)

$$X^e_{ff'}(q,q';z) =\langle F(z^*-\omega_q)\,|\,\Omega^e(q,q';z)\,|\,F'(z-\omega_{q'})\rangle\,, \tag{3.30}$$

$$V^e_{ff'}(q,q';z) =\langle F(z^*-\omega_q)\,|\,\Omega^{es}(q,q';z)\,|\,F'(z-\omega_{q'})\rangle\,. \tag{3.31}$$

Here the $X^0_{ff'}$'s are off-shell extensions of the pion-nucleon elastic scattering amplitudes, while the $X^a_{ff'}$'s are their one-fermion irreducible parts. In A it is shown that these off-shell amplitudes satisfy Lippmann-Schwinger equations with potentials given by (3.31) and propagators given by $\widetilde{\Gamma}_f(z-\omega_p)$. According to A(4.40), (3.28), (3.27b), and A(4.27), the two types of amplitudes are related by

$$X^0_{ff'}(q,q';z)=X^a_{ff'}(q,q';z)+\sum_{f''}\xi_{ff''}(q;z)\widetilde{\Gamma}_{f''}(z)\bar{\xi}_{f''f'}(q';z)\,. \tag{3.32}$$

If we put (3.29) into (3.30) and (3.31) with $e=a$, and use (2.16), $P^aJ(q)\,|\,f\rangle=0$, as well as (3.21), we find

$$X^a_{ff'}(q,q';z)=\sum_{f''f'''}\int F^*_{f''}(p'';z^*-\omega_q)dp''U^a_{f''f'''}(p''q;p'''q';z)dp'''F_{f'''}'(p''';z-\omega_{q'})\,, \tag{3.33}$$

$$V^a_{ff'}(q,q';z)=\sum_{f''f'''}\int F^*_{f''}(p'',z^*-\omega_q)dp''U^{as}_{f''f'''}(p''q;p'''q';z)dp'''F_{f'''}'(p''';z-\omega_{q'})\,. \tag{3.34}$$

Equation (3.33) is also true if we replace the index a by 0 and assume that we have the on-shell relations $z=M_N+\omega_q+i\epsilon$, $f=f'=Nr$, and $q=q'$. Off shell, additional terms come in. These equations are important in that they relate the two-particle amplitudes and potentials treated in A to the three-particle quantities introduced here.

We now obtain the relation between the three-particle Green's functions defined by (3.13) and the production amplitudes. In order to do this we make the identifications $P_\rho=P^g$, $P_\sigma=P^e$, $P_\tau=1$ in (2.18) and use A(4.18), (3.12a), (3.8), A(4.19), (3.17), (3.18), and $P^{as}J(q')\,|\,f\rangle=0$. We find

$$G^g_{ff'}(pq;p'q';z)=\widetilde{\Gamma}_f(z-\omega_p-\omega_q)\sum_{f''}\int Y^g_{ff''}(pq;p''q';z)dp''G^e_{f''f'}(p'',p';z-\omega_{q'})\,, \tag{3.35}$$

where

$$Y^g_{ff'}(pq;p'q';z)=\widetilde{\Gamma}_f^{-1}(z-\omega_p-\omega_q)\delta_{ff'}[\delta(p,p')\delta(q,q')+\delta(q,p')\delta(p,q')] + \langle F(z^*-\omega_p-\omega_q)\,|\,[a(p)a(q)+J^\dagger(q)a(p)G^g(z)+J^\dagger(p)a(q)G^g(z)]J(q')\,|\,\phi^0_{f'}(p';z-\omega_{q'})\rangle\,. \tag{3.36}$$

From (3.25b), (3.36), and (3.29), it follows that

$$B^g_{ff'}(pq;q';z)\equiv\sum_{f''}\int Y^g_{ff''}(pq;p''q';z)dp''F_{f''}'(p'';z-\omega_{q'})\,, \tag{3.37a}$$

$$\begin{aligned}&=\xi_{ff'}(p;z-\omega_q)\delta(q,q')+\xi_{ff'}(q;z-\omega_p)\delta(p,q')\\ &\quad+\langle F(z^*-\omega_p-\omega_q)\,|\,[a(p)a(q)+J^\dagger(q)a(p)G^g(z)+J^\dagger(p)a(q)G^g(z)]\\ &\quad\times J(q')\left[\,|\,F'(z-\omega_{q'})\rangle-|\,f'\rangle\frac{\widetilde{\Gamma}_{f'}^{-1}(z-\omega_{q'})Z_{f'}^{-1/2}}{z-\omega_{q'}-M_{f'}^{(0)}}\right].\end{aligned} \tag{3.37b}$$

It should be noted that the second term in the second square bracket vanishes when $g=a,as$, or when $g=0$, $f'=N$, and $z=M_N+\omega_{q'}+i\epsilon$. Using A(4.17) and Eq. (14) of Ref. 31, it is straightforward to show that

$$B^0_{NN'}(pq;q';E+i\epsilon)={}_-\langle Npq\,|\,J(q')\,|\,N'\rangle_+\,,\quad \text{on shell}\,, \tag{3.38a}$$

where on shell means

$$E=M_N+\omega_p+\omega_q=M_N+\omega_{q'}\,. \tag{3.38b}$$

Here $|\mathrm{N}'\rangle_+$ is a physical nucleon state, and $|\mathrm{N}pq\rangle_-$ is an *out* state of two pions and a nucleon which is defined by Eq. (14) of Ref. 31. Using the results of Sec. II of Ref. 31 it is straightforward to show that (3.38) gives the amplitude for $\pi_{q'}+\mathrm{N}'\to\pi_p+\pi_q+\mathrm{N}$.

From (3.37), A(4.24), and (3.27), it follows that the relation between $B^0_{ff'}$ and $B^a_{ff'}$ is given by

$$B^0_{ff'}(pq;q';z)=B^a_{ff'}(pq;q';z)+\sum_{f''}\zeta_{ff''}(pq;z)\widetilde{\Gamma}_{f''}(z)\left[\xi_{f''f'}(q';z)-\langle F''(z^*)\,|\,J(q')\,|\,f'\rangle\frac{\widetilde{\Gamma}_{f'}^{-1}(z-\omega_{q'})Z_{f'}^{-1/2}}{z-\omega_{q'}-M^{(0)}_{f'}}\right], \tag{3.39}$$

with

$$\zeta_{ff'}(pq;z)=\langle F(z^*-\omega_p-\omega_q)\,|\,[J^\dagger(q)a(p)+J^\dagger(p)a(q)]\,|\,F'(z)\rangle\ , \tag{3.40a}$$

$$=\langle F(z^*-\omega_p-\omega_q)\,|\,J^\dagger(q)G(z-\omega_p)J^\dagger(p)+J^\dagger(p)G(z-\omega_q)J^\dagger(q)\,|\,F'(z)\rangle\ , \tag{3.40b}$$

where the second form of $\zeta_{ff'}$ is obtained by using A(4.19). We note that (3.39) and (3.40) are analogous to (3.32) and (3.27), respectively. We will refer to $\zeta_{ff'}(pq;z)$ as the *production vertex function.*

According to (2.14a), (3.40), and A(4.18), we have

$$\langle fpq\,|\,G(z)\,|\,f'\rangle=Z_f^{1/2}\widetilde{\Gamma}_f(z-\omega_p-\omega_q)\zeta_{ff'}(pq;z)\widetilde{\Gamma}_{f'}(z)Z_{f'}^{1/2} \tag{3.41}$$

and

$$\zeta_{ff'}(pq;z)=\widetilde{\Gamma}_f^{-1}(z-\omega_p-\omega_q)Z_f^{-1/2}\langle fpq\,|\,F'(z)\rangle\ ; \tag{3.42}$$

thus the production vertex function is related to a bare matrix element of the complete resolvent and to the projections of $|F(z)\rangle$ onto the two-meson—one-fermion states. Clearly (3.42) is analogous to (3.25). Putting (3.24) into (3.42) and using (3.18), we obtain

$$\zeta_{ff'}(pq;z)=\widetilde{\Gamma}_f^{-1}(z-\omega_p-\omega_q)Z_f^{-1/2}\sum_{f''}\int\langle fqp\,|\,\phi^e_{f''}(p'';z)\rangle dp''F'_{f''}(p'';z)\ . \tag{3.43}$$

We note that according to (3.22) and A(4.85) $|\phi^a_f(p;z)\rangle$ is the same as the vector $|\phi(f,p;z)\rangle$ in A. Keeping this in mind and using A(4.88), (3.25), and (3.12b), we find

$$Z_f^{-1/2}\langle fpq\,|\,\phi^a_{f'}(p';z)\rangle=\widetilde{\Gamma}_f(z-\omega_p-\omega_q)\xi_{ff'}(q,z-\omega_p)\delta(p,p')+\sum_{f''}\int G^a_{ff''}(q,q'';z-\omega_p)dq''A^{as}_{f''f'}(q''p;p';z)\ , \tag{3.44}$$

where

$$A^{as}_{ff'}(qp;p';z)=\langle\phi^a_f(q;z^*-\omega_p)\,|\,J^\dagger(p)\,|\,\phi^a_{f'}(p';z)\rangle\ . \tag{3.45}$$

We note that according to (3.18) and (3.11), we can replace $\langle\phi^a_f|$ in (3.45) by $\langle\phi^0_f|$. Using this, as well as (3.22), A(4.19), (3.29), (3.21), and (2.16), we find that the quantity defined by

$$A^g_{ff'}(pq;q';z)\equiv\sum_{f''}\int U^g_{ff''}(pq;p''q';z)dp''F'_{f''}(p'';z-\omega_{q'}) \tag{3.46}$$

agrees with (3.45) when $g=as$. In comparing results obtained here with those of A, it should be noted that $A^{as}_{ff'}(pq;q';z)$ is there written as [see (4.87)] $A_{ff'}(q,p;q';z)$.

We will see in Sec. IV that the combination of (3.43), (3.44), and (3.46) leads to useful relations for the production vertex function. There we will derive a set of equations for operators that act only in the two-meson—one-fermion space. Matrix elements of these operators lead back to the entities encountered in the present section.

IV. THREE-PARTICLE EQUATIONS

We begin by introducing a projection operator for the two-meson—one-fermion subspace, as well as three pseudo-Hamiltonians that act in this subspace. We write

$$P_t=\sum_f\int|pqf\rangle\frac{dp\,dq}{2}\langle fqp\,| \tag{4.1}$$

and

$$H^g_t(z)=P_t[H_0+H_1G^{gt}(z)H_1]P_t\ , \tag{4.2}$$

where we have used (2.8), (2.2), and (2.4). In writing (4.2) we have used the fact that we are dealing with a quantum field theory Hamiltonian for which $P_tH_1P_t=0$ and $P^tH_0P_t=0$.

We are going to use (2.7) to derive integral equations for the three-particle Green's functions defined by (3.13). To this end we write

$$Z_f^{1/2}\langle fqp\,|\,z-H^g_t(z)\,|\,p'q'f'\rangle Z_{f'}^{1/2}=\widetilde{\Gamma}_f^{-1}(z-\omega_p-\omega_q)\delta_{ff'}(p,p')\delta(q,q')-V^e_{ff'}(p,p';z-\omega_q)\delta(q,q')$$
$$-V^e_{ff'}(q,q';z-\omega_p)\delta(p,p')-V^g_{3ff'}(pq;p'q';z)+(p'\leftrightarrow q')\ , \tag{4.3}$$

where $\widetilde{\Gamma}_f$ and $V^e_{ff'}$ are given by (3.26), (3.31), and (2.16). The symbol $(p'\leftrightarrow q')$ represents the adding on of all of the previous terms with p' and q' interchanged. The quantity $V^g_{3ff'}$ is a three-particle potential, and, in a sense, (4.3) is its definition. Of course, it is necessary to verify that it is truly such a potential, i.e., it must not contain delta functions such as appear in the other terms. This can be done by using the techniques employed in A for analyzing the two-particle potentials. It is important to note that (4.3) implies that $V^g_{3ff'}(pq;p'q';z)$ is invariant under the interchanges $p\leftrightarrow q$ or $p'\leftrightarrow q'$.

We shall now show that $V^a_{3ff'}=V^{as}_{3ff'}$. In (2.5) and (2.6), we identify $P^\alpha=P^{at}$, $P_\beta=P_s$, $P^\gamma=P^{ast}$ and use $H_0P_s=P_sH_0P_s$ and $H_1P_s=(P_a+P_t)H_1P_s$ to show that

$$G^{at}(z)-G^{ast}(z)=P_sG^{at}(z)P_s\ . \tag{4.4}$$

It is easy to verify that

$$G^\rho(z)=G^\rho_0(z)+G^\rho(z)H_1G^\rho_0(z)\ , \tag{4.5a}$$

where

$$G^\rho_0(z)=\frac{P^\rho}{z-P^\rho H_0P^\rho}\ . \tag{4.5b}$$

If we use this in (4.4), we find

$$G^{at}(z)-G^{ast}(z)=P_sG^{at}_0(z)P_s\ . \tag{4.6}$$

From (2.12), (3.6), and (3.19), it follows that

$$P_sH_1\,|pqf\rangle=\sum_{f'}[\,|pf'\rangle\bar{\xi}^{(0)}_{f'f}(q)+|qf'\rangle\bar{\xi}^{(0)}_{f'f}(p)]\ . \tag{4.7}$$

Now if we use (4.3) and (4.2) to calculate $V^a_{3ff'}-V^{as}_{3ff'}$, we find with the help of (3.17), (4.6), (4.7), (3.20), A(4.79), and A(4.76) that

$$V^a_{3ff'}(pq;p'q';z)=V^{as}_{3ff'}(pq;p'q';z)\ . \tag{4.8}$$

Thus we have shown that there are only two three-particle potentials defined by (4.3), i.e., $V^0_{3ff'}$ and $V^a_{3ff'}$. It is not difficult to show that $V^a_{3ff'}$ can be obtained from $V^0_{3ff'}$ by deleting all intermediate processes in which only one fermion is present, so $V^a_{3ff'}$ is one-fermion irreducible.

According to (2.7), (2.6), (4.1), (4.3), and (3.13), we have

$$\sum_{f''}\int[\widetilde{\Gamma}_f^{-1}(z-\omega_p-\omega_q)\delta_{ff''}\delta(p,p'')\delta(q,q'')-V^e_{ff''}(p,p'';z-\omega_q)\delta(q,q'')-V^e_{ff''}(q,q'';z-\omega_p)\delta(p,p'')$$
$$-V^g_{3ff''}(pq;p''q'';z)]dp''dq''G^g_{f''f'}(p''q'';p'q';z)=\delta_{ff'}[\delta(p,p')\delta(q,q')+\delta(p,q')\delta(q,p')]\ , \tag{4.9}$$

where we have also used the invariance of $G^g_{f''f}$ under the interchange $p''\leftrightarrow q''$. We will now reformulate (4.9), which is our basic three-particle equation, as an operator equation in which the operators are to be thought of as the type of entities which appear in three-particle potential theories. To this end we introduce free three-particle states $|pqf\rangle_\alpha$ and ${}_\alpha\langle fqp\,|$ where $\alpha=1,2$. In these states pion α is labeled by q, while pion $\beta\neq\alpha$ is labeled by p. In contrast to Fock space states, the order of the pion labels is significant. We have

$$|pqf\rangle_\alpha=|qpf\rangle_\beta,\quad \alpha\neq\beta\ . \tag{4.10}$$

We assume that these states are normalized so that

$${}_\alpha\langle fqp\,|p'q'f'\rangle_\alpha=\delta_{ff'}\delta(p,p')\delta(q,q')\ , \tag{4.11a}$$

$$\sum_f\int|pqf\rangle_\alpha dp\,dq\,{}_\alpha\langle fqp\,|=1_t\ , \tag{4.11b}$$

and define operators according to

$${}_\alpha\langle fqp\,|G_0(z)|p'q'f'\rangle_\alpha=\widetilde{\Gamma}_f(z-\omega_p-\omega_q)\delta_{ff'}\delta(p,p')\delta(q,q')\ , \tag{4.12a}$$

$${}_\alpha\langle fqp\,|V^g_\alpha(z)|p'q'f'\rangle_\alpha=V^e_{ff'}(p,p';z-\omega_q)\delta(q,q'),\quad \alpha=1,2\ , \tag{4.12b}$$

$${}_\alpha\langle fqp\,|V^g_3(z)|p'q'f'\rangle_\alpha=V^g_{3ff'}(pq;p'q';z)\ , \tag{4.12c}$$

$${}_\alpha\langle fqp\,|G^g_t(z)|p'q'f'\rangle_\alpha=G^g_{tff'}(pq;p'q';z)\ . \tag{4.12d}$$

Here we are using the standard three-particle notation in which, e.g., $V^g_1(z)$ is the potential for *subsystem* 1 which consists of pion 2 and a fermion. The operator $G^g_t(z)$ is defined by

$$G^g_t(z)=[G_0^{-1}(z)-V^g(z)]^{-1}\ , \tag{4.13}$$

where

$$V^g(z)=\sum_{\alpha=1}^{3}V^g_\alpha(z)\ . \tag{4.14}$$

According to (4.10)—(4.14), $G^g_{tff'}$ obeys an equation just like (4.9), but with only the first term on the right-hand side. Using (4.10) we have

$$G^g_{ff'}(pq;p'q';z)=\sum_{\alpha=1}^{2}{}_\beta\langle fqp\,|G^g_t(z)|p'q'f'\rangle_\alpha\ , \tag{4.15a}$$

$$=\sum_{\beta=1}^{2}{}_\beta\langle fqp\,|G^g_t(z)|p'q'f'\rangle_\alpha\ , \tag{4.15b}$$

where (4.15a) can be verified by substituting into (4.9), and (4.15b) follows from the fact that $G^g_t(z)$ must be symmetric in the pion labels, 1 and 2, which implies that

$$G^g_{tff'}(pq;q'p';z)=G^g_{tff'}(qp;p'q';z)\ . \tag{4.16}$$

It is convenient to introduce

$$G^g_0(z)=G_0(z)\ , \tag{4.17a}$$

$$G^g_\alpha(z)=[G_0^{-1}(z)-V^g_\alpha(z)]^{-1},\quad \alpha=1,2,3\ , \tag{4.17b}$$

where (4.17a) is only for notational convenience, while (4.17b) defines resolvents for three-particle systems in which only one of the potentials act. If we write out the integral equation that follows from (4.17b) when $\alpha=1,2$, we obtain an equation like (4.9) but with only the first two terms in the square bracket on the left-hand side and the first term on the right-hand side. Upon comparison with A(4.51), we find

$${}_\alpha\langle fqp \mid G^g_\alpha(z) \mid p'q'f'\rangle_\alpha = G^e_{ff'}(p,p';z-\omega_q)\delta(q,q'), \quad \alpha=1,2 . \quad (4.18)$$

Following Alt, Grassberger, and Sandhas,[26] as well as Kowalski,[30] we introduce transition operators $U^g_{\beta\alpha}(z)$ by

$$G^g_t(z)=\delta_{\beta\alpha}G^g_\beta(z)+G^g_\beta(z)U^g_{\beta\alpha}(z)G^g_\alpha(z); \quad \beta,\alpha=0,1,2,3 . \quad (4.19)$$

If we take matrix elements of this equation when $\beta=1,2$ and $\alpha=1,2$, and use (4.18), (4.11b), and (4.15), we find upon comparison with (3.15) that

$$U^g_{ff'}(pq;p'q';z)=\sum_{\beta/\alpha=1}^{2} {}_\beta\langle fqp \mid U^g_{\beta\alpha}(z) \mid p'q'f'\rangle_\alpha , \quad (4.20)$$

where $\sum_{\beta/\alpha=1}^2$ means $\sum_{\beta=1}^2$ *or* $\sum_{\alpha=1}^2$. If we take matrix elements of (4.19) with $\beta=0$ and $\alpha=1,2$, and use (4.17a), (4.12a), (4.18), (4.11b), and (4.15), we find upon comparison with (3.35) that

$$Y^g_{ff'}(pq;p'q';z)=\sum_{\beta/\alpha=1}^{2} {}_\beta\langle fqp \mid U^g_{0\alpha}(z) \mid p'q'f'\rangle_\alpha . \quad (4.21)$$

Thus we have shown that the matrix elements of the transition operators taken with respect to the free states lead to the quantities from which the scattering and production amplitudes can be calculated.

It can be shown[26,30] that the transition operators defined by (4.19) satisfy the equations

$$U^g_{\beta\alpha}(z)=\bar\delta_{\beta\alpha}G_0^{-1}(z)+\sum_{\gamma=1}^{3}\bar\delta_{\beta\gamma}T^g_\gamma(z)G_0(z)U^g_{\gamma\alpha}(z) , \quad (4.22a)$$

where

$$\bar\delta_{\beta\alpha}=1-\delta_{\beta\alpha} \quad (4.22b)$$

and

$$T^g_\gamma(z)=V^g_\gamma(z)+V^g_\gamma(z)G^g_\gamma(z)V^g_\gamma(z) , \quad (4.23a)$$

$$=V^g_\gamma(z)+V^g_\gamma(z)G_0(z)T^g_\gamma(z), \quad \gamma=1,2,3 . \quad (4.23b)$$

If we take matrix elements of (4.23a) when $\gamma=1,2$, and use (4.12b), (4.11b), and (4.18), we find upon comparison with A(4.53), A(4.57), and A(4.52) that

$${}_\gamma\langle fqp \mid T^g_\gamma(z) \mid p'q'f'\rangle_\gamma = X^e_{ff'}(p,p';z-\omega_q)\delta(q,q'), \quad \gamma=1,2 , \quad (4.24)$$

thus the operators $T^g_\gamma(z)$ with $\gamma=1,2$ are simply related to the two-particle, off-shell scattering amplitudes.

We now consider solving (4.22) with $g=a$ by the AGS quasiparticle method.[26,30] If each of the transition operators $T^a_\gamma(z)$ are split into two parts according to

$$T^a_\gamma(z)=T^{(1)}_\gamma(z)+T^{(2)}_\gamma(z) , \quad (4.25)$$

it can be shown that

$$U^a_{\beta\alpha}(z)=\bar U_{\beta\alpha}(z)+\sum_{\gamma=1}^{3}\bar U_{\beta\gamma}(z)G_0(z)T^{(1)}_\gamma(z)G_0(z)U^a_{\gamma\alpha}(z) , \quad (4.26)$$

where

$$\bar U_{\beta\alpha}(z)=\bar\delta_{\beta\alpha}G_0^{-1}(z)+\sum_{\gamma=1}^{3}\bar\delta_{\beta\gamma}T^{(2)}_\gamma(z)G_0(z)\bar U_{\gamma\alpha}(z) , \quad (4.27a)$$

$$=\bar\delta_{\beta\gamma}G_0^{-1}(z)+\bar\delta_{\beta3}T^{(2)}_3(z)\bar\delta_{3\alpha}+\sum_{\gamma=1}^{2}[\bar\delta_{\beta\gamma}+\bar\delta_{\beta3}T^{(2)}_3(z)G_0(z)\bar\delta_{3\gamma}]T^{(2)}_\gamma(z)G_0(z)\bar U_{\gamma\alpha}(z) . \quad (4.27b)$$

According to (4.24) and (3.17), $T^a_\gamma(z)$ with $\gamma=1,2$ is related to the complete off-shell amplitudes $X^0_{ff'}$ which in turn can be split into their one-fermion irreducible and reducible parts by (3.32). These splittings suggest that we define

$$T^{(1)}_\gamma(z)=\sum_f\int \mid \xi_f(z-\omega_q)q\rangle_\gamma dq\,\tilde\Gamma_f(z-\omega_q)_\gamma\langle q\xi_f(z^*-\omega_q)\mid, \quad \gamma=1,2$$
$$=0, \quad \gamma=3 , \quad (4.28)$$

where

$${}_\gamma\langle fqp \mid \xi_{f'}(z)q'\rangle_\gamma=\xi_{ff'}(p;z)\delta(q,q') . \quad (4.29)$$

From (4.25), (4.24), (4.28), (4.29), (3.27b), (3.32), (4.8), and (4.23b), it then follows that

$$T^{(2)}_\gamma(z)=T^{as}_\gamma(z) . \quad (4.30)$$

Upon comparing (4.27a) with (4.22), we find that

$$\bar U_{\beta\alpha}(z)=U^{as}_{\beta\alpha}(z) . \quad (4.31)$$

If we use (4.11b), (4.12a), (4.29), (3.27b), (3.25), (4.20), (3.33), and (3.34), we can show that

$$X^a_{ff'}(q,q';z)=\sum_{\beta/\alpha=1}^{2} {}_\beta\langle q\xi_f(z^*-\omega_q) \mid G_0(z)U^a_{\beta\alpha}(z)G_0(z) \mid \xi_{f'}(z-\omega_{q'})q'\rangle_\alpha , \quad (4.32)$$

$$V^{a}_{ff'}(q,q';z)=\sum_{\beta/\alpha=1}^{2}{}_{\beta}\langle q\xi_f(z^*-\omega_q)\,|\,G_0(z)U^{as}_{\beta\alpha}(z)G_0(z)\,|\,\xi_{f'}(z-\omega_{q'})q'\rangle_\alpha\ . \tag{4.33}$$

These relations provide the connections between the three-particle formalism presented here and the two-particle approach in A. In particular, if we put (4.28) into (4.26) and use (4.31)—(4.33), we find

$$X^{a}_{ff'}(q,q';z)=V^{a}_{ff'}(q,q';z)+\sum_{f''}\int V^{a}_{ff''}(q,q'';z)dq''\widetilde{\Gamma}_{f''}(z-\omega_{q''})X^{a}_{f''f'}(q'',q';z)\ , \tag{4.34}$$

which could also be obtained from A(4.58), A(4.53), and A(4.52).

If we use (4.11b), (4.12a), (4.29), (3.25), and (4.20), we find upon comparison with (3.46) that

$$A^{g}_{ff'}(pq;q';z)=\sum_{\beta/\alpha=1}^{2}{}_{\beta}\langle fqp\,|\,U^{g}_{\beta\alpha}(z)G_0(z)\,|\,\xi_{f'}(z-\omega_{q'})q'\rangle_\alpha\ . \tag{4.35}$$

From (3.34) and (3.46), it follows that

$$V^{a}_{ff'}(q,q';z)=\sum_{f''}\int F^{*}_{f''}(p'';z-\omega_q)dp''A^{as}_{f''f'}(p''q;q';z)\ , \tag{4.36}$$

which can also be obtained from (4.33) and (4.35), with the help of (4.11b), (4.12a), (4.29), and (3.25b). Equation (4.36) is the same as A(4.86). By combining (4.35), (4.31), (4.27), (4.29), (4.10), (4.11b), (4.30), (4.24), (3.17), and (4.12a), we find

$$A^{as}_{ff'}(pq;q';z)=\xi_{ff'}(q,z-\omega_p)\delta(p,q')+\sum_{f''}\int X^{a}_{ff''}(q,q'';z-\omega_p)dq''\widetilde{\Gamma}_{f''}(z-\omega_p-\omega_{q''})A^{as}_{f''f'}(q''p;q';z)+R^{as}_{ff'}(pq;q';z)\ , \tag{4.37}$$

where

$$R^{g}_{ff'}(pq;q';z)=\sum_{\beta/\alpha=1}^{2}{}_{\beta}\langle fqp\,|\,T^{g}_{3}(z)G_0(z)U^{g}_{3\alpha}(z)G_0(z)\,|\,\xi_{f'}(z-\omega_{q'})q'\rangle_\alpha\ . \tag{4.38}$$

The quantity $R^{g}_{ff'}$ comes from the $\gamma=3$ term in (4.27a). An alternative expression for it can be obtained by comparing (4.27a) and (4.27b). Equation (4.37) agrees with A(4.91a). In A it was pointed out that a practical approximation scheme is obtained by neglecting $R^{as}_{ff'}$ in (4.37), for by so doing closed set of equations for the quantities of interest is obtained. In A it was not clear what this approximation means. Here we see that according to (4.38), dropping $R^{as}_{ff'}$ in (4.37) is equivalent to assuming that the effects of the three-particle potential $V^{as}_{3}(z)$ are negligible.

We now turn our attention to the production amplitudes and the associated vertex functions. According to (4.11b), (4.12a), (4.29), (4.21), (3.25b), and (3.37a), the production amplitudes are given by

$$B^{g}_{ff'}(pq;q';z)=\sum_{\beta/\alpha=1}^{2}{}_{\beta}\langle fqp\,|\,U^{g}_{0\alpha}(z)G_0(z)\,|\,\xi_{f'}(z-\omega_{q'})q'\rangle_\alpha\ . \tag{4.39}$$

By combining (4.26), (4.31), (4.28), and (4.32), we find the following relation between the $g=a$ and $g=as$ amplitudes,

$$B^{a}_{ff'}(pq;q';z)=B^{as}_{ff'}(pq;q';z)+\sum_{f''}\int B^{as}_{ff''}(pq;q'';z)dq''\widetilde{\Gamma}_{f''}(z-\omega_{q''})X^{a}_{f''f'}(q'',q';z)\ . \tag{4.40}$$

From (4.22) it follows that

$$U^{g}_{0\alpha}(z)=\sum_{\beta=1}^{2}U^{g}_{\beta\alpha}(z)-T^{g}_{3}(z)G_0(z)U^{g}_{3\alpha}(z),\quad \alpha=1,2\ . \tag{4.41}$$

If we put this into (4.39) and use (4.10), (4.38), and (4.35), we obtain

$$B^{g}_{ff'}(pq;q';z)=A^{g}_{ff'}(pq;q';z)+A^{g}_{ff'}(qp;q';z)-R^{g}_{ff'}(pq;q';z)\ . \tag{4.42}$$

If we combine A(4.37), (3.44), (4.37), and (4.42), we find

$$Z_f^{-1/2}\langle fqp\,|\,\phi^{a}_{f'}(p';z)\rangle=\widetilde{\Gamma}_f(z-\omega_p-\omega_q)B^{as}_{ff'}(pq;p';z)\ , \tag{4.43}$$

which when put into (3.43) leads to the following expression for the production vertex function,

$$\zeta_{ff'}(pq;z)=\sum_{f''}\int B^{as}_{ff''}(pq;p'';z)dp''F'_{f''}(p'';z)\ . \tag{4.44}$$

From A(4.59a), A(4.28), A(4.68), (3.20), (3.28), A(4.71a), A(4.52), A(4.53b), and A(4.53c), it follows that

$$\xi_{ff'}(p;z)=\sum_{f''}\int [\delta_{ff''}\delta(p,p'')+X^{a}_{ff''}(p,p'';z)\widetilde{\Gamma}_{f''}(z-\omega_{p''})]dp''Z^{1/2}_{f''}\xi^{(0)}_{f''f'}(p'')Z^{1/2}_{f'}\ . \tag{4.45}$$

If we put this and (3.25b) into (4.44), and use (4.40), we find that the production vertex function can also be obtained from

$$\zeta_{ff'}(pq;z)=\sum_{f''}\int B^{a}_{ff''}(pq;p'';z)\widetilde{\Gamma}_{f''}(z-\omega_{p''})dp''Z_{f''}^{1/2}\xi^{(0)}_{f''f'}(p'')Z_{f'}^{1/2} . \tag{4.46}$$

According to (4.44) and (3.25b), the production vertex functions can be obtained from the *one-fermion*, *one-fermion—one-meson* irreducible production amplitudes, the fermion propagators, and the *dressed* vertex functions, while in (4.46) the *one-fermion* irreducible production amplitudes, the fermion propagators, and the *bare* vertex functions appear explicitly.

V. DISCUSSION

The main result of the analysis presented here is that standard three-particle equations[26,30] can be used to describe pion-nucleon scattering when the underlying model is one in which static fermions interact with pions through the virtual processes (3.1). In fact, according to (4.22) the standard equations[26,30] can be used for the complete three-particle amplitudes ($g=0$), the one-fermion irreducible (OFI) amplitudes ($g=a$), and the OFI, one-fermion, one-meson irreducible amplitudes ($g=as$). As (4.24) and (3.17) show, the two-particle amplitudes that appear in the kernels of the three-particle integral equations are the complete two-particle amplitudes when $g=0,a$ and their OFI parts when $g=as$. The $g=0$ and $g=a$ forms of the three-particle equations (4.22) differ only in the T matrices that arise from the three-particle interactions $V_3^g(z)$ [see (4.23)]. The interaction $V_3^a(z)$ is the OFI part of $V_3^0(z)$, and $V_3^a(z)=V_3^{as}(z)$.

The analysis shows that it is not necessary to solve (4.22) for the complete three-particle amplitudes, as it is possible to obtain everything of interest from the OFI amplitudes. For example, (3.39), (3.37a), (4.21), and (4.46) show that it is possible to obtain the on-shell production amplitudes from the $g=a$ solutions of (4.27), the one-to-two vertex functions, and the fermion propagators. Furthermore, according to A, the one-to-two vertex functions and the fermion propagators can be obtained from the OFI two-particle amplitude, which in turn can be obtained from the $g=a$ solutions of (4.22) by means of (4.32).

Of course, it should be kept in mind that the equations are inherently nonlinear in that the solutions of the integral equations are needed to construct their kernels. The usual way to deal with such nonlinearities is to solve the equations by iteration. In A a case is made for a set of linear integral equations whose solutions should give a good first approximation. These linear equations are of the type[27–29] obtained with the isobar idea and produce solutions which satisfy two-particle and three-particle unitarity and have reasonable analytic structure.

In order for the system of equations to be closed it is necessary to neglect the effects of the OFI part of the three-particle interaction. It will be necessary to analyze this interaction in some detail to see if its effects can be included, at least in some approximate way. It may be possible to calculate the three-particle potential by perturbation theory starting from (4.3), which is its definition. It should be emphasized that the effect of the OFR part of the three-particle potential is taken into account in the closed system of nonlinear equations.

As far as other future work is concerned, it will be highly desirable to see if it is possible to develop three-particle equations for the pion-nucleon system starting from a covariant field theory. This is so for at least two related reasons; the pions are relativistic over much of the energy range of interest, and it is impossible to develop a completely consistent theory in which relativistic pions are mixed with nonstatic but nonrelativistic baryons. Preliminary work indicates that the techniques developed here and in A are adequate for this task.

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