Application of the Kishimoto-Tamura boson expansion theory to a single-j shell model

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The boson expansion theory of Kishimoto and Tamura is applied to a single-j shell model. It is shown that this theory is quite accurate, giving results that agree very closely with those of the exact fermion calculations. The fast convergence of the boson expansion is also demonstrated. A critical discussion is then made of an earlier paper by Arima, in which he stated that the Kishimoto-Tamura theory gives rise to very poor numerical results. The source of the trouble encountered by Arima is unmasked.

I. INTRODUCTION

The boson expansion theory (BET) of Kishimoto and Tamura (KT), which our group at the University of Texas has been working on for over a decade, was initiated in two papers, i.e., Refs. 1 and 2, which we henceforth refer to as KT1 and KT2, respectively. Based on the formalism developed in these two papers, particularly in KT2, extensive realistic calculations were performed, fitting experimental data quite well for a number of collective even-even nuclei.[3]

More recently, we renewed our investigation of formal aspects of BET. We felt it desirable to put our earlier formalism on a firmer basis. The results of our efforts were presented in Refs. 4 and 5, in which we gave a detailed formal analysis of the KT1 and KT2 expansions, uncovering the true nature of these theories.

In Ref. 5 it was shown, in particular, that the KT1 and KT2 expansions are, in fact, quite different, in that they are defined in different physical spaces. The expansion of KT1 is exact, non-Hermitian, and defined in an overcomplete (and thus unnormalizable) boson subspace. (It is, perhaps, not redundant to point out that the last feature is not something introduced by the bosonization procedure, but is carried over from the underlying fermion description.) The expansion of KT2, on the other hand, is perturbative, Hermitian, and defined in a truncated, but normalizable boson subspace. (In the KT2 fermion system, and hence in the KT2 boson expansion, the sole quadrupole component is retained.)

Because of the way it was derived, the KT1 expansion cannot be truncated (i.e., none of the possible modes can be suppressed) without introducing an unacceptably large amount of error (as we shall see in Sec. III). In practice, the (exact) KT1 formalism is inconvenient to use, because the physical boson states become prohibitively complicated to handle as the number of phonons increases. In order to make realistic calculations possible, one has to introduce a truncation of the modes, and thus the KT1 expansion must be given up. In fact, in all our numerical applications this formalism was never used, in spite of the fact that we did, unfortunately, mention, in constructing the bosonized Hamiltonian, the possibility of a truncation in the KT1 paper. Instead the KT2 theory was used.

As we shall also see later in this paper, the KT2 expansion is not an "approximate" form, nor a "truncated" version of the KT1 expansion, and cannot be obtained from the latter, but must be rederived independently. In Ref. 2 the reason for giving up the KT1 expansion was, perhaps, not stated clearly, but the coefficients of the new expansion were given explicitly, along with the explanation of the procedure used to derive them.

Since the algebra used in KT1 and KT2, as well as in Ref. 5, was very involved, it is possible that the basic mathematical and physical aspects of the BET work of KT still remain obscure to the reader. Therefore, in the present paper we intend to revisit the KT work on BET and reanalyze some of its formal aspects mentioned above in the context of a simple single-j shell model ($1j$-SM). In doing this, we rederive *ab novo* the KT expansions appropriate for this model. This will give us the possibility to discuss, on a practical and simple example, the points mentioned above concerning the nature of the KT1 and KT2 expansions. We then present some numerical results obtained by using the KT formalism, and demonstrate that this BET reproduces rather accurately the original fermion results. We also show the good convergence of the expansion. (Let us note here that, because of the simplicity of the model and of the restricted model space chosen, the calculations remain feasible even when the KT1 expansion is used.)

A few years back, Arima published a paper,[6] in which he presented some numerical results obtained by applying the KT formalism expansion to a $1j$-SM (the same we use in the present paper). He claimed that the $B(E2)$ values for the transitions from two- to one-phonon states calculated in the KT way were far too small compared with the exact results. He thus has cast a serious doubt on our whole work on BET.

It is one of the purposes of this paper to unmask the source of the trouble encountered by Arima and explain the contradiction between our "good" results and Arima's "poor" results. As we shall see, the reason for such contradiction lies in the fact that two totally different expansions were used. We shall show that Arima's results can be reproduced if a "truncated" version of the KT1 expansion is used. As we mentioned above, and will discuss in detail later, the truncation of the KT1 expansion is bound to introduce a large amount of error in all the calculations. It is because the thus obtained boson images of the fermion operators violate badly the original fermion commutation relations. On the other hand, our good results were obtained by using the KT 2 expansion, which was constructed as to satisfy (approximately) the original fermion commutators.

Arima seems to have disregarded completely the KT2 paper, and used an expansion which was, in fact, never used by us for realistic calculations.[3] Therefore, one should not take Arima's results as evidence against the validity of our BET works.[1–3] We may also note here that a recent paper by Kishimoto and Tamura[7] (hereafter referred to as KT3), gave a full formal justification of the KT2 formalism. It was shown that KT2 is nothing but an approximate version of a rigorous and rather general KT3 theory. We thus believe that the mathematical soundness of KT2 and thus the correctness of all our numerical works has been established beyond any doubt. (We may also note that, although the KT1 expansion has to be given up, it does not mean that the whole KT1 paper becomes useless. In fact, the formalism given in Sec. IV and thereafter in KT1 remains valid and useful, if the expansion coefficients, called X_n there, are replaced by the corresponding coefficients obtained in the KT2, or better in the KT3 way.)

In Sec. II we give a brief account of the KT formalism and apply it to the 1j-SM. In Sec. III we discuss Arima's work.[6] In Sec. IV we calculate the $B(E2)$ values for the transition from three- to two-phonon states, demonstrating the fast convergence of our BET. In Sec. V we discuss the accuracy of the Bardeen-Cooper-Schrieffer (BCS) approximation, which was used in practical applications of our BET. Concluding remarks are given in Sec. VI.

II. ACCURACY OF THE KT BOSON EXPANSION

In this section we discuss the accuracy of the BET of KT. In doing this, we take up the 1j-SM with $j=\frac{23}{2}$ (the same model as in Ref. 6). The use of this simple model allows us to make the whole presentation simple and transparent. For convenience, in our presentation, we shall separate the problem of the accuracy of the KT expansion itself from that of the BCS approximation. Henceforth, through Sec. IV we will be concerned with the accuracy and convergence speed of the expansion, while the discussion of the error introduced by the BCS approximation will be postponed until Sec. V. Thus, up to Sec. V, the terminology "exact fermion results" should be understood within the framework of the quasiparticle description.

A. The fermion problem

We first formulate the fermion problem, having in mind the evaluation of the $B(E2)$ values for the transitions from two- to one-quadrupole phonon states. Since the particle-hole mode description and the BCS theory is consistently used in the KT formalism, we first perform the Bogoliubov transformation

$$a_{jm}^{\dagger}=ud_{jm}^{\dagger}+v(-)^{j-m}d_{j,-m}$$

(where $u^2+v^2=1$), from the particle ($a_{jm}^{\dagger}$) to the quasiparticle ($d_{jm}^{\dagger}$) representation. It is easy to see that the shell model quadrupole operator $P_{2\mu}^{\dagger}$, (as defined also in Ref. 6) is then given (neglecting the noncontributing terms) as

$$P_{2\mu}^{\dagger}=2\sqrt{\Omega}uvB_{2\mu}^{\dagger}\ , \tag{2.1a}$$

$$B_{2\mu}^{\dagger}=\frac{1}{\sqrt{2}}[d_j^{\dagger}d_j^{\dagger}]_{2\mu}\ . \tag{2.1b}$$

In (2.1), $\Omega=j+\frac{1}{2}$ is the pair degeneracy of the 1j-SM, and the bracket denotes the standard angular momentum coupling. As said above, in the following we shall discuss only the BET itself, and shall, therefore, ignore the BCS factor (i.e., the $2\sqrt{\Omega}\ uv$ factor) in (2.1a), and use the $B_{2\mu}^{\dagger}$ operator of (2.1b) as the quadrupole operator.

Using the quasiparticle pair creation operator $B_{2\mu}^{\dagger}$ defined above, we can construct the one- and two-quadrupole phonon states as

$$|j^2;2\mu\rangle=B_{2\mu}^{\dagger}|0\rangle\ , \tag{2.2a}$$

$$|j^4;LM\rangle=\frac{1}{\sqrt{2}N_L^F}[B_2^{\dagger}B_2^{\dagger}]_{LM}|0\rangle\ . \tag{2.2b}$$

Here $|0\rangle$ is the quasiparticle vacuum, and the norm N_L^F is calculated as

$$N_L^F=[1-Y_L(22;22)]^{1/2}\ . \tag{2.3}$$

In (2.3), $Y_L(22;22)$ is a special case of the coefficient $Y_L(ab;cd)$ defined by

$$Y_L(ab;cd)=2\hat{a}\,\hat{b}\,\hat{c}\,\hat{d}\begin{Bmatrix} j & j & a\\ j & j & b\\ c & d & L\end{Bmatrix}, \tag{2.4}$$

where $\hat{a}=(2a+1)^{1/2}$, and the curly bracket quantity is a 9j symbol.

By using (2.1)–(2.4), the reduced matrix elements of the operator $B_{2\mu}^{\dagger}$ between the one- and two- quadrupole phonon states can be easily evaluated as

$$\langle j^4;L||B_2^{\dagger}||j^2;2\rangle=\sqrt{2}\hat{L}N_L^F\ . \tag{2.5}$$

Note that the explicit form of N_L^F was given in (2.3).

B. Bosonization of the fermion problem

In the framework of the KT formalism, the boson image of the pair operator $B_{a\mu}^{\dagger}$ is written as

$$(B_{a\mu}^{\dagger})_B=A_{a\mu}^{\dagger}+\sum_{bcd}\sum_{L}\xi_L(bc;da)\frac{\hat{L}}{\hat{a}}[[A_b^{\dagger}A_c^{\dagger}]_L A_{\tilde{d}}]_{a\mu}\ . \tag{2.6}$$

In (2.6), $A_{a\mu}^{\dagger}$ and $A_{a\mu}$ are ideal boson creation and annihilation operators, with angular momentum a and projection μ; also $A_{d\tilde{\mu}}=(-)^{d-\mu}A_{d,-\mu}$. Note that in the present 1j-SM, all the angular momenta a, b, c, ..., appearing in (2.5) are even integers. Note also that, in general, the expansion $(B_{a\mu}^{\dagger})_B$ involves an infinite series with the nth term being written (schematically) as $(A^\dagger)^n A^{n-1}$. However, within the context of the present and the next section, in which we consider only one- and two-boson states, the two-term expansion in (2.6) makes the theory exact.

The expansion coefficients ξ_L in (2.6) are determined by requiring that the boson images of the various fermion pair operators satisfy the same commutation relations as do the original fermion pair operators. Imposing this condition, we easily find that the equation to be satisfied by ξ_L is given as

$$2\xi_L(bc;da)+\sum_{ef}\xi_L(bc;ef)\xi_L(ef;da)+Y_L(bc;da)=0\ , \qquad (2.7)$$

where Y_L are defined in (2.4). [Note that (2.7) is nothing but a special case of Eq. (3.5c) of KT1. See, however, Ref. 8.]

From now on we shall refer to (2.7) as the *coefficient equation* (for ξ_L.) It is obvious that to satisfy this equation is the same as to let the boson expansion satisfy (to the third order) the original fermion commutation relations.

As done in KT1, we solve Eq. (2.7) by first making for ξ_L the ansatz

$$\xi_L(bc;ad)=2r\Delta_{bc;da}+2sY_L(bc;da)\ , \qquad (2.8a)$$

$$\Delta_{bc;da}=\tfrac{1}{2}(\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd})\ . \qquad (2.8b)$$

We also note that the Y's satisfy the following *completeness relation:*

$$\sum_{ef}Y_L(bc;ef)Y_L(ef;da)=2\Delta_{bc;da}-Y_L(bc;da)\ , \qquad (2.9)$$

which can be easily proved. [Equation (2.9) is nothing but a special case of Eq. (3.12) of KT1.] Inserting (2.8) into (2.7), and making use of (2.9), we obtain the following set of equations for the parameters r and s:

$$r+r^2+2s^2=0\ , \qquad (2.10a)$$

$$4s-4s^2+8rs+1=0\ . \qquad (2.10b)$$

As done in KT1, the set of roots of (2.10) that we shall use for the boson expansion (2.6) is

$$r=\frac{1}{2\sqrt{3}}(1-\sqrt{3})\ \text{and}\ s=-\frac{1}{2\sqrt{3}}\ . \qquad (2.11)$$

[Note, however, that any set of solutions of (2.10) gives rise to a perfectly legitimate boson expansion. See also Refs. 4 and 5.] This, together with (2.8), allows us to rewrite (2.7) as

$$(B^\dagger_{a\mu})_{\mathrm{KT1}}=A^\dagger_{a\mu}+\sum_{bcd}\sum_{L}\frac{\hat{L}}{\hat{a}}\left[\left(\frac{1}{\sqrt{3}}-1\right)\Delta_{bc;da}-\frac{1}{\sqrt{3}}Y_L(bc;da)\right][[A^\dagger_b A^\dagger_c]_L A_{\tilde{d}}]_{a\mu}\ . \qquad (2.12)$$

This is the KT1 boson expansion for $B^\dagger_{2\mu}$, up to the third-order term.

Since the original fermion commutation relations [i.e., the coefficient equations (2.7)] are satisfied exactly by the KT1 boson expansions, (2.12) allows one to reproduce the fermion result given in (2.5) exactly; see Eq. (2.23a), below.

However, the KT1 expansion is not very convenient to use in practice, because, for one thing, the bosonized states are not given as simple ideal boson states; see Eq. (2.20b) below. Actually, the physics underlying realistic problems of the nuclear collective motions indicates that we do not need to use such a sophisticated theory as KT1. In most cases of practical interest, the quadrupole collective component plays a dominant role, and the truncation to this single component does make sense.[2] In fact, our successful realistic calculations[3] were performed using the KT2 formalism, in which such a truncation was made. We now turn to the derivation of the KT2-type expansion for the $1j$-SM.

We want to stress the fact that *the truncation is done in the fermion stage of the formulation, and, consequently, that the bosonization procedure must be carried out completely anew.* In practice, however, we can still use Eqs. (2.6)—(2.8) just as they stand, if we set all the angular momenta $a, b, \ldots, f$ in these equations equal to 2. Equations (2.7) and (2.8) are thus replaced by

$$2\xi_L(22;22)+\xi_L^2(22;22)+Y_L(22;22)=0\ , \qquad (2.13)$$

$$\xi_L(22;22)=2r+2sY_L(22;22)\ . \qquad (2.14)$$

If (2.14) is inserted into (2.13), the latter will contain terms linear and quadratic in Y_L. The KT2 approximation (apart from the truncation discussed above) is to suppress the Y_L^2 term, on the grounds that it is of a higher order. [Note that Y_L^2 cannot be linearized, as done in (2.9), because the completeness relation does not hold any more. This is what, formally, discriminates KT2 from KT1.] Once the Y_L^2 term is suppressed, it is easy to see that (2.13), with the ansatz (2.14), is reduced to the following two equations for r and s:

$$r+r^2=0\ , \qquad (2.15a)$$

$$4s+8rs+1=0 \qquad (2.15b)$$

which replace Eqs. (2.10a) and (2.10b). One set of solutions of (2.15) is given by

$$r=0\ \text{and}\ s=-\tfrac{1}{4}\ , \qquad (2.16)$$

which is nothing but the choice made in KT2 [as explicitly given in Eq. (7.9) of Ref. 2]. With this particular choice, one has the relation that $\xi_L(22;22)=-\frac{1}{2}Y_L(22;22)$, and (2.6) can be rewritten as

$$(B^\dagger_{2\mu})_{\rm KT2}=A^\dagger_{2\mu}-\tfrac{1}{2}\sum_L \frac{\hat{L}}{\sqrt{5}}Y_L(22;22)[[A^\dagger_2A^\dagger_2]_L A_{\tilde{2}}]_{2\mu} \,. \tag{2.17}$$

This is the KT2 expansion for $B^\dagger_{2\mu}$, up to the third-order term. By comparing the form of (2.12) and (2.17), it is obvious that, as we remarked above, the KT2 expansion *cannot be obtained* from the KT1 expansion by simply suppressing the summation over the angular momenta in the latter.

Having derived the KT1 and KT2 formalisms for the 1j-SM, we are now ready to test their numerical accuracy. To do this, we first construct the one- and two-quadrupole boson states as

$$|1;2\mu)=(B^\dagger_{2\mu})_B\,|0)\,, \tag{2.18a}$$

$$|2;LM)=\frac{1}{\sqrt{2}N^B_L}[(B^\dagger_2)_B(B^\dagger_2)_B]_{LM}\,|0)\,, \tag{2.18b}$$

where $|0)$ is the boson vacuum, and $(B^\dagger_{2\mu})_B$ represents the boson expansions, $(B^\dagger_{2\mu})_{\rm KT1}$ or $(B^\dagger_{2\mu})_{\rm KT2}$, given by (2.12) or (2.17), depending on whether the KT1 or the KT2 formalism is used.

By inserting $(B^\dagger_{2\mu})_{\rm KT1}$ and $(B^\dagger_{2\mu})_{\rm KT2}$ for $(B^\dagger_{2\mu})_B$ in (2.18), it is straightforward to obtain the boson norm N^B_L in the KT1 and KT2 frameworks. They are given, respectively, as

$$(N^B_L)_{\rm KT1}=[1-Y_L(22;22)]^{1/2}\,, \tag{2.19a}$$

$$(N^B_L)_{\rm KT2}=[1-\tfrac{1}{2}Y_L(22;22)]\,. \tag{2.19b}$$

Note that in obtaining (2.19a) the completeness relation (2.9) was used.

By substituting the $(B^\dagger_{2\mu})_{\rm KT1}$ and $(B^\dagger_{2\mu})_{\rm KT2}$ in (2.18) once again, it is trivial to see that the one-boson state in (2.18a) is given as a simple ideal boson state $A^\dagger_{2\mu}\,|0)$ for both KT1 and KT2. In order to construct the two-boson states we may carry out a "step-by-step" operation.[9] For the KT2 case we find that

$$|2;LM)=\frac{1}{\sqrt{2}}[A^\dagger_2A^\dagger_2]_{LM}\,|0)\,, \tag{2.20a}$$

i.e., that the two-boson state is again an ideal boson state (and this feature persists for all the states with higher numbers of bosons as well).

Using $(B^\dagger_{2\mu})_{\rm KT1}$, on the other hand, we find that

$$|2;LM)=\frac{1}{\sqrt{2}N^B_L}\frac{1}{\sqrt{3}}\sum_{ab}[\Delta_{22;ab}-Y_L(22{:}ab)]\times[A^\dagger_aA^\dagger_b]_{LM}\,|0)\,, \tag{2.20b}$$

which is clearly not a simple ideal boson state, but rather a quite complicated superposition of such states. This is why the use of the KT1 expansion is not convenient for realistic many-boson calculations. In the present simple 1j-SM case, however, states such as (2.20b) are tractable and calculations involving these states can still be carried out without much trouble. Actually, (2.19a) was obtained by using (2.20b).

One will easily recognize that the $B(E2)$ matrix element $\langle j^4;LM\,|B^\dagger_{2\mu}\,|j^2;2\mu'\rangle$ we have calculated in subsection A, and are trying to copy here in the boson way, is nothing but an element of a large norm matrix, the elements of which can be written as $\langle 0\,|[B_aB_b]_{LM}[B^\dagger_cB^\dagger_d]_{LM}\,|0\rangle$. This matrix is singular, in general,[4,5,7] and this is another way of looking at the difficulty of using the BET of KT1 in practice. In the present section, however, we do not consider the whole norm matrix, but only a few elements of it. This is why the use of the KT1 expansion does not cause any trouble here.

We can now turn to the calculation of the reduced matrix elements of $B^\dagger_{2\mu}$ that interest us here. In terms of N^B_L, it is easy to show that

$$(2;L\,||(B^\dagger_2)_B||\,1;2)=\sqrt{2}\hat{L}N^B_L\,. \tag{2.21}$$

The boson matrix element given in (2.21) is to be compared with the original fermion matrix element in (2.5). As a direct measure of the accuracy of the KT expansions, we therefore define the ratio

$$R_L=\frac{(2;L\,||(B^\dagger_2)_B||\,1;2)}{\langle j^4;L\,||B^\dagger_2||\,j^2;2\rangle}=\frac{N^B_L}{N^F_L}\,. \tag{2.22}$$

Using the fermion norm in (2.3) and the boson norms in (2.19), we finally have the following two ratios, one for the KT1 and the other for the KT2 boson expansions:

$$(R_L)_{\rm KT1}=1 \text{ (for all } L)\,, \tag{2.23a}$$

$$(R_L)_{\rm KT2}=\frac{[1-\frac{1}{2}Y_L(22;22)]}{[1-Y_L(22;22)]^{1/2}}\,. \tag{2.23b}$$

In Table I, numerical values of the squares of the ratios $(R_L)_{\rm KT1}$ and $(R_L)_{\rm KT2}$ are given in the first and second columns, respectively, for $L=0$, 2, and 4. The fact that $(R^2_L)_{\rm KT1}=1$ means that the KT1 is an exact BET in the context of the present paper, as it should be from its derivation. [For convenience, we list, in the fourth column of Table I, also the values of $Y_L(22;22)$.]

Although the commutation relations, or, equivalently, the coefficient equations (2.13), are satisfied only approximately by the KT2 boson expansion [being violated by a quantity of the order $O(Y^2_L)$], the ratios $(R^2_L)_{\rm LT2}$ given in Table I are quite close to 1. This means that the KT2 formalism is actually very accurate, in sharp contrast to what was claimed in Arima's work.[6] For comparison, the results obtained by Arima are also given in the third column of Table I and are denoted by $(R^2_L)_{\rm Arima}$. They are clearly much smaller (by a factor of more than 3) than the correct ratios $(R^2_L)_{\rm KT2}$. In the next section we shall point out and discuss the source of the trouble encountered by

TABLE I. Ratios of the boson and fermion $B(E2)$ values for the two- to one-phonon transitions.

L	$(R^2_L)_{\rm KT1}$	$(R^2_L)_{\rm KT2}$	$(R^2_L)_{\rm Arima}$	$Y_L(22;22)$
0	1	1.060	0.20	0.3908
2	1	1.004	0.29	0.1227
4	1	1.004	0.29	0.1199

Arima in his use of the KT formalism, which led him to the incorrect and gravely misleading results given in Table I.

As an aside, we may make here an additional remark. As we stated above, we suppressed the Y_L^2 term, in solving (2.13) [together with (2.14)]. We did this, because we wanted to follow faithfully what had been done in our previous work.[2,3] Actually, however, this *perturbative* procedure of solving the coefficient equations was not really necessary. As seen, (2.13) is a very simple quadratic equation in $\xi_L(22;22)$, and thus can be solved directly without making the ansatz of (2.14). In fact, we obtain $\xi_L(22;22)$ as

$$\xi_L(22;22)=-1+[1-Y_L(22;22)]^{1/2}\ , \tag{2.24}$$

which is nothing but the KT3 result and is exact, within the truncated space. (For a detailed discussion of the nonperturbative solution of the coefficient equations within the framework of the commutator method see also Ref. 5.)

If (2.24) is used in (2.6) (with $a=b=c=d=2$), the boson $B(E2)$'s reproduce the exact fermion $B(E2)$'s.

III. ASSESSMENT OF ARIMA'S APPLICATION OF THE KT FORMALISM

In his application of the KT formalism,[6] Arima started by writing down the boson expansion $(B_{2\mu}^\dagger)_B$ as

$$(B_{2\mu}^\dagger)_B=A_{2\mu}^\dagger+\sum_{bcd}\sum_L \xi[bc(L)d;2][(A_b^\dagger A_c^\dagger)_L A_{\tilde{d}}]_{2\mu}\ . \tag{3.1}$$

In (3.1) b, c, and d again represent all the angular momenta of the nucleon pairs permitted by the $1j$-SM. Arima did not give the explicit form of the expansion coefficients ξ in (3.1), but he did state that these coefficients should be determined so that the boson expansion series satisfies the original fermion commutation relations. For the untruncated case, we may thus safely assume that he took ξ as

$$\xi[bc(L)d;2]=\frac{\hat{L}}{\hat{2}}\xi_L(bc;d2)\ , \tag{3.2}$$

where ξ_L was defined in (2.8). The use of (2.11) then gives

$$\xi[bc(L)d;2]=\frac{\hat{L}}{\hat{2}}\left[\left[\frac{1}{\sqrt{3}}-1\right]\Delta_{bc;d2}-\frac{1}{\sqrt{3}}Y_L(bc;d2)\right]\ . \tag{3.3}$$

With the coefficient ξ of (3.3), the expansion (3.1) is nothing but the KT1 expansion (2.12) with $a=2$. As we showed in Sec. II, this expansion reproduces the exact fermion result, and is thus correct.

For his calculations in the truncated case, Arima *appears to have assumed* that the *truncated* expansion can be obtained from (3.1) [with the coefficients ξ still being given by (3.3)], by simply dropping the summation over b, c, and d. [Again, he did not give explicitly the coefficients ξ for the truncated expansion, but kept using the same notation as in (3.1) with $b=c=d=2$.]

By using the thus obtained expansion, Arima proceeded to calculate the $B(E2)$ values. It is easy to see that the reduced matrix element of $(B_{2\mu}^\dagger)_B$ is now obtained as $\sqrt{2}/\sqrt{3}[1-Y_L(22;22)]\hat{L}$, which results in

$$(R_L^2)_{\text{Arima}}=\tfrac{1}{3}[1-Y_L(22;22)]\ . \tag{3.4}$$

It is now trivial to confirm that (3.4) reproduces exactly Arima's numerical results given in the third column of Table I. (Note that, as in the KT2 case, the boson space is spanned by ideal boson states, which can be easily checked by performing the step-by-step operation with Arima's truncated expansion.)

In the previous section we went through a detailed derivation of the KT1 (exact) and KT2 (truncated) boson expansions using consistently the principle that the commutation relations must be preserved. We also stressed there, in particular, that KT2 expansion cannot be derived from the KT1 expansion, by simply suppressing the summation over the angular momenta. The key point is that, the use of the completeness relation (2.9) is permitted in deriving the KT1 expansion, but is not permitted in obtaining the KT2 expansion.

As described above, Arima obtained his truncated version of the boson expansion from the untruncated KT1 expansion, thus making an implicit (and illegitimate) use of the completeness relation (2.9). (See also the concluding remarks to this section.) Thus the expansion used by Arima is most likely to violate the original commutation relations. To check this, let us insert Arima's ξ_L [Eq. (3.3) with $b=c=d=2$] into the coefficient equation (2.13). Denoting the left-hand side of (2.13) by G_L^2, we find that

$$G_L^2=\tfrac{1}{3}[2-Y_L(22;22)-Y_L^2(22;22)]\ . \tag{3.5}$$

In order for the commutation relations to be satisfied, G_L^2 must vanish (at least approximately). The actual numerical values are $G_0^2=0.4855$, $G_2^2=0.6207$, and $G_4^2=0.6219$, and represent the amount of error introduced in $(N_L^B)^2$ by using the erroneously truncated BET. Since the ratios of the $B(E2)$'s calculated in Sec. II are actually simply given as $(N_L^B)^2/(N_L^F)^2$, it is a reasonable guess that the above large values of G_L^2 are directly responsible for the too small $(N_L^B)^2$, and hence of the too small $(R_L^2)_{\text{Arima}}$. We can easily confirm that the following relation is satisfied exactly

$$(R_L^2)_{\text{Arima}}=1-\frac{G_L^2}{(N_L^F)^2}\ , \tag{3.6}$$

where N_L^F is the exact fermion norm given in (2.3). Note that $G_L^2/(N_L^F)^2\simeq\frac{2}{3}$, as seen from (3.5) and (2.3). Hence (3.6) makes $(R_L^2)_{\text{Arima}}\simeq\frac{1}{3}$. This is in fact very close to the values given in the third column of Table I.

In concluding this section, let us say that in Ref. 1 we did make a truncation of the modes in constructing the Hamiltonian. (See Sec. IV and its sequel in KT1. See also the remark we made about KT1 towards the end of the Introduction of the present paper.) However, when performing numerical calculations, based on it, we soon discovered that the error was far too large to make any physical sense. This was, basically, the reason that led us to search for a different approach, the result of which was the KT2 expansion. We find it very unfortunate that Arima disregarded the work of KT2, and failed to detect the above trouble in his work of Ref. 6, especially in view of the large negative influence that his paper has had on the nuclear community with regard to our work in BET.

IV. FIFTH-ORDER CALCULATIONS WITH THE KT2 EXPANSION

In Sec. II, we showed that the KT expansion is quite accurate, by considering one- and two-phonon states. In the present section we intend to go one step further and take the three-phonon states into consideration. This will allow us to estimate the accuracy further, as well as the convergence rate of the expansion. Furthermore, this will give us the opportunity to discuss a specific problem pertaining to the truncation of the fermion space to a single quadrupole phonon, a problem that does not show up if only one- and two-phonon states are considered.

Let us first present the fermion formulation. The orthonormal three-phonon states can be written as

$$|3;IM\rangle=\frac{1}{\sqrt{6}N_3^F(I)}\sum_i C_i^I[[B_2^\dagger B_2^\dagger]_i B_2^\dagger]_{IM}|0\rangle\ , \tag{4.1}$$

where C_i^I denotes the coefficient of fractional parentage (CFP), and $N_3^F(I)$ is the total norm given by

$$N_3^F(I)=[1-(Y_3)_I]^{1/2}=\left[\frac{1}{6}\sum_{i'i}C_{i'}^I C_i^I N_3^2(i,i';I)\right]^{1/2}, \tag{4.2}$$

with

$$N_3^2(i,i'I)=[1-3Y_i(22;22)]\delta_{ii'}+2\hat{i}\,\hat{i}\,'\sum_\lambda W(2ii'2;\lambda I)Y_i(22;2\lambda)Y_{i'}(22;2\lambda). \tag{4.3}$$

In (4.3) the notation W stands for the Racah coefficient, and the quantity Y was defined in Sec. II. The summation over λ, also in (4.3b), has its origin in the commutation relation $[C^\dagger,B^\dagger]$, where $C^\dagger$ is the scattering operator, and it can, in principle, take on all the values permitted by the $1j$-SM, i.e., $\lambda=j+j$. We shall refer to the case in which the summation over λ is retained as the *exact* (or, untruncated) fermion calculation, and to the case in which this summation is limited to, as the truncated fermion calculations.

The reason for separating these two cases is that, as stressed in Secs. II and III, the truncation of the components has to be done before carrying out the bosonization of the problem. Below we shall perform the boson calculations by using the KT2 expansion, which was derived after truncating the fermion system to a single quadrupole component ($\lambda=2$). The fermion calculations, on the other hand, will be performed both for the *exact* and the *truncated* cases. This will allow us to estimate separately the amount of two kinds of errors, one due to the truncation (i.e., the violation of the commutation relation) made in the fermion stage, and the other due to the use of the KT2 approximation. The error in the final boson results is, of course, a combination of these two types of errors.

In terms of the partial norms, defined in (4.2b), the fermion reduced matrix elements of the pair operator $B_{2\mu}^\dagger$ between the three- and two-phonon states are given as

$$\langle j^6;I||B_2^\dagger||j^4;L\rangle=\hat{I}\frac{1}{\sqrt{6}N_3^F(I)}\times 6\sum_i C_i^I N_3^2(i,L;I)\frac{1}{\sqrt{2}N_2^F(L)}\ . \tag{4.4}$$

(For clarity, we denote here by $N_2^F(L)$ the two-phonon fermion norm, rather than by N_L^F used in the previous two sections.) The same expression given in (4.4) is valid for the fermion calculations in both the exact and truncated cases, as long as the summation over λ is included or suppressed consistently in the calculation of the partial, and hence of the total norms.

For the boson calculations we shall use the perturbative expansion of KT2 in which the terms up to the order of Y^2 are retained consistently. This is needed, because in the calculations involving three-boson states, the fifth-order term in the expansion will also contribute. Here we shall not go through the derivation of the Y^2 terms in the expansion. We shall only say that the expansion coefficients can be obtained in a manner totally analogous to that described in Sec. II, starting from the coefficient equations given in KT1 (but setting all angular momenta equal to 2), and using the ansatz $\xi=a+bY+cY^2$, for both the third- and fifth-order terms. One can also obtain the same expansion, starting from the KT3 expansion or the square-root representation of Ref. 5, and expanding the norm factors, retaining terms up to Y^2. The boson expansion has thus the form

$$(B_{2\mu}^\dagger)_B=A_{2\mu}^\dagger+(B_{2\mu}^\dagger)_B^{(3)}+(B_{2\mu}^\dagger)_B^{(5)}\ , \tag{4.5}$$

where

$$(B_{2\mu}^\dagger)_B^{(3)}=-\sum_L\frac{\hat{L}}{\sqrt{5}}[\tfrac{1}{2}Y_L(22;22)+\tfrac{1}{8}Y_L(22;22)Y_L(22;22)][[A_2^\dagger A_2^\dagger]_L A_{\tilde{2}}]_{2\tilde{\mu}}\ , \tag{4.6a}$$

$$(B_{2\mu}^\dagger)_B^{(5)}=-\tfrac{1}{8}\sum_I\sum_{L_1L_2L_3}Y_I(22;22)Y_{L_1}(22;22)W(2L_1I2;2L_2)W(2L_2;IL_3)\times\frac{1}{\sqrt{5}}\hat{L}_1\hat{L}_2\hat{L}_3\hat{I}^2[[[A_2^\dagger A_2^\dagger]_{L_1}A_2^\dagger]_{L_2}[A_2A_2]_{\tilde{L}_3}]_{2\mu}\ . \tag{4.6b}$$

Let us point out here that the second term in Eq. (4.6a) was omitted in the previous sixth-order calculations (see first paper of Ref. 3). For consistency this term should have been retained throughout. However, as we shall see, the Y^2-type terms in the expansion introduce, in almost every case, only a small correction to the third-order results (see also Ref. 5).

We calculate the reduced matrix elements of the pair operator for the exact and truncated fermion cases, by using Eq. (4.4), and for the boson case by using (4.5). As done in Sec. II, we then calculate the ratios of the boson results to the fermion results. The squares of these ratios are presented in Table II. In part A of the table the ratios are given for the exact fermion calculations, and in part B for the truncated fermion calculations. In part C we give similar ratios for the $B(E2)$ values between the truncated fermion results and the exact fermion results. In the first column of parts A and B we give the values obtained with only the first-order term in the expansion (4.5). In the second column the ratios are given for the boson results that include the first-order term and the Y term in the third-order term of the expansion. Finally, in the third column the contributions coming from all the terms in the expansion are included.

Let us first look at part B of Table II. We can see that for all the transitions the convergence rate is quite fast (i.e., that the difference between the second and third columns is very small), and the accuracy is extremely good (i.e., that the ratios are very close to 1) for all, except for the cases of the $(2\rightarrow2)$ and $(2\rightarrow4)$ transitions. The reason for these two exceptional cases can be traced back to the fact that for $I=2$ the Y_3 term in Eq. (4.2a) is rather large, and thus higher powers of Y also will give a sizable contribution. Even so, the error in these two cases does not exceed 14 percent. Overall, we may say that the boson transcription of the fermion problem works quite satisfactorily.

If we look now at part A of the same table, we see that the convergence rate is as fast as in the truncated case. The error in the final results is within 8 percent. We must stress, however, that the results of part A should be interpreted carefully, because the termination of the expansion at terms of order Y^2 occasionally *helps,* instead of *hurting* the results, thus giving a somewhat misleading impression. To have a better estimate of the error due to the truncation of the fermion systems, we must look at part C of Table II. There we can see that the violation of the Pauli principle introduces errors of up to 16 percent. The same amount of error would also be present in the results of part A, had we performed the boson calculations by using the expansion obtained by solving the coefficient equations exactly,[5] instead of treating them perturbatively, as we did in the present paper.

As we said in Secs. II and III, the truncation of the fermion system is something that one has to do. Whether the truncation introduces acceptable errors or not depends on the concrete problem at hand. In our realistic calculations,[3] where Y is of the order of 0.1, the error is within the range of 10 percent. For the same reason, the amount of error due to the bosonization is small, and therefore the overall error is in the range of 15 percent.

V. ADEQUACY OF THE BCS APPROXIMATION

So far we have examined only the accuracy of the KT boson expansion, and found that it is indeed quite good (Secs. II and IV). However, the KT formalism, in practical applications, uses the BCS approximation. We shall now assess the accuracy of this approximation, by comparing its results with those obtained by using the number-conserving quasiparticle (NCQP) approach, recently developed by Li.[10]

In the NCQP approach, one constructs "effective operators," which can be used directly in the quasiparticle space (of the BCS theory). Thus the simplicity of the BCS calculations is maintained. Yet, the NCQP results are much more accurate than are the BCS results. This is achieved because the effective operators are constructed in such a way as to incorporate the effect of the number projection for a given N-particle system, and to take into account the blocking effect. (In the $1j$-SM case, the NCQP approach gives exact results.)

TABLE II. Ratios of the $B(E2)$ values for the three- to two-phonon states transitions for the case of: (A) boson and untruncated fermion calculations; (B) boson and truncated fermion calculations; (C) truncated and untruncated fermion calculations.

			Part A			Part B		Part C
I_i	I_f	0(1)	$+0(Y)$	$+0(Y^2)$	0(1)	$+0(Y)$	$+0(Y^2)$	
0	2	1.2213	0.9399	0.9245	1.3252	1.0199	1.0032	0.9216
2	0	1.5378	1.0470	1.0046	1.5349	1.0450	1.0027	1.0019
2	2	2.2144	1.0577	0.9664	2.6121	1.2476	1.1400	0.8477
2	4	2.2216	1.0567	0.9595	2.6312	1.2516	1.1365	0.8443
3	2	1.3762	1.0621	1.0550	1.3210	1.0195	1.0127	1.0418
3	4	1.3807	1.0621	1.0504	1.3259	1.0200	1.0088	1.0413
4	2	1.2724	0.9838	0.9736	1.3182	1.0192	1.0086	0.9652
4	4	1.2765	0.9838	0.9735	1.3231	1.0197	1.0091	0.9648
6	4	1.3097	1.0145	1.0039	1.3153	1.0189	1.0082	0.9957

For the purpose of the present paper, we need to obtain explicitly the "effective operator" $(P^\dagger_{2\mu})_N$ that corresponds to the shell model operator $P^\dagger_{2\mu}$. It can be obtained readily from Eqs. (3.37) and (4.6) of Ref. 10, and it is found to be

$$(P^\dagger_{2\mu})_N=\hat{P}[2\sqrt{\Omega}U(N-1,\tilde{n}-1)V(N,\tilde{n}-2)B^\dagger_{2\mu}]\hat{P}\ , \quad (5.1)$$

where

$$U(N,\tilde{n})=\left[\frac{2\Omega-N-\tilde{n}}{2(\Omega-\tilde{n})}\right]^{1/2}$$

and (5.2)

$$V(N,\tilde{n})=\left[\frac{N-\tilde{n}}{2(\Omega-\tilde{n})}\right]^{1/2}.$$

In (5.1), N is the total number of (valence) nucleons, while $\tilde{n}=\sum_m d^\dagger_{jm}d_{jm}$ is the quasiparticle number operator. Further, $\hat{P}$ is a projection operator that eliminates any spurious component from the quasiparticle states, and U and V are the new number-conserving amplitudes as defined by Eq. (5.2). Because of the presence of N and $\tilde{n}$ in the U and V factors, the number conservation as well as the blocking effect are properly taken into account. [As done in the previous sections, we have omitted in (5.1) all the terms that do not contribute to the present $B(E2)$ calculations.]

Note that, in the present $1j$-SM, $\tilde{n}$ and $\hat{P}$ commute. Thus, by comparing (5.1) with (2.1a), it is easy to see that the use of the NCQP approach means to replace u, v, and $B^\dagger_{2\mu}$, that appear in (2.1), by $U(N-1,\tilde{n}-1)$, $V(N,\tilde{n}-2)$, and $\hat{P}B^\dagger_{2\mu}\hat{P}$, respectively. (Below, we consider only the BCS errors due to the number nonconservation and the neglect of the blocking effect, and will disregard the correction by the projection operator. The latter gives rise to correction of the order of $1/\Omega^2$.)

In the $1j$-SM, the BCS amplitudes u and v are given as

$$u=\left[1-\frac{N}{2\Omega}\right]^{1/2} \text{ and } v=\left[\frac{N}{2\Omega}\right]^{1/2}.$$

TABLE III. Squares of the ratios of the BCS uv and number projected uv amplitudes.

N	4	6	8	10	12
r_N^2, $\tilde{n}=4$	1.389	1.055	0.952	0.911	0.900
r_N^2, $\tilde{n}=6$		1.500	1.037	0.907	0.875

As a measure of the accuracy of the BCS approximation in the calculations of the $B(E2)$ values, we thus find it convenient to introduce the ratio

$$r_N^2=\frac{u^2v^2}{[U(N-1,\tilde{n}-1)V(N,\tilde{n}-2)]^2}=\frac{\left[\frac{(2\Omega-N)N}{4\Omega^2}\right]}{\left[\frac{(2\Omega-N-\tilde{n}+2)}{2(\Omega-\tilde{n}+1)}\frac{(N-\tilde{n}+2)}{2(\Omega-\tilde{n}+2)}\right]}. \quad (5.3)$$

In Table III we give the numerical values of r_N^2 for $N=4$–12, and for $\tilde{n}=4$ and 6. (The cases with $\tilde{n}=4$ and 6 are, respectively, for the transitions from two- to one-phonon and the three- to two-phonon states, as considered in the previous sections.) As seen, the BCS error is at most 12 percent, except for the cases with $N=\tilde{n}$. (Note that, for a given N-particle system, the BCS theory is known to work poorly when $\tilde{n}\simeq N$.) This indicates that, for most cases, the BCS approximation is quite adequate.

For completeness, we give in Table IV the squares of the ratios of the $B(E2)$'s, obtained by using the BCS and the KT2-type BET, over those obtained by performing number-projected (NCQP) fermion calculations, which may thus be called the exact fermion $B(E2)$'s. (Note that, for the transitions from three- to two-phonon states, the untruncated fermion norms are taken.) As seen, except for the cases with $\tilde{n}=N$, the total error does not exceed 20 percent. The best results are obtained for $N=8$, in which the total error is within five percent. It is

TABLE IV. Ratios of the $B(E2)$ values of the BCS plus BET to the (number-conserving) exact fermion results.

I_i	N / I_f	4	6	8	10	12
0	2	1.4723	1.1183	1.0091	0.9656	0.9540
2	2	1.3945	1.0592	0.9558	0.9146	0.9036
4	2	1.3945	1.0592	0.9558	0.9146	0.9036
0	2		1.3867	0.9587	0.8385	0.8089
2	0		1.5069	1.0417	0.9112	0.8790
2	2		1.4496	1.0021	0.8765	0.8456
2	4		1.4392	0.9950	0.8703	0.8395
3	2		1.5825	1.0940	0.9569	0.9231
3	4		1.5756	1.0893	0.9527	0.9191
4	2		1.4604	1.0096	0.8830	0.8519
4	4		1.4603	1.0095	0.8830	0.8518
6	4		1.5058	1.0410	0.9105	0.8784

well known that the BCS theory works poorly when the number of particles is small, while it does quite well around the middle of the shell. Thus, the results of Table IV do not come as a surprise. In our realistic calculations the number of particles is usually large, so that we can be confident that total error does not exceed 15 percent.

As a final remark, let us note that Li's method,[10] used above to obtain improved (exact for the 1j-SM case) fermion results, can be incorporated into the boson mapping technique. In a forthcoming paper[11] it will be shown how to carry out the bosonization of the "effective operators," which can then be used in the usual boson space. This will cure most of the error due to the BCS approximation in the boson calculations. Thus, in the 1j-SM case considered in the present paper, for the ratios of Table III we will obtain $r_N^2=1$ for all $\tilde{n}$ and N, and consequently, the ratios for the $B(E2)$ values in Table IV will coincide with the ratios given in Tables I and II. This treatment will be most important for those cases in which the particle number N and/or the effective degeneracy Ω is small, such as those given in the first and second columns of Tables III and IV.

VI. CONCLUDING REMARKS

We have applied the boson expansion theory of Kishimoto and Tamura[1,2] to a 1-j shell model. In Sec. II we derived the KT1 and KT2 expansions and used them to calculate the $B(E2)$ values for the transitions from the two- to one-phonon states. There we also calculated the corresponding fermion $B(E2)$'s exactly (in the BCS framework; see the beginning of Sec. II), and compared the results with the boson results (Table I). As seen, the KT1 expansion is exact, and the (perturbative) expansion of KT2 gives results very close to the fermion values. In Sec. III we then rebutted the results presented in a paper by Arima,[6] in which he claims that very poor numerical results were obtained when the KT formalism was used. We showed that the source of the trouble encountered by Arima lies in the fact that he used a "truncated" version of KT1, which, by construction, is bound to badly violate the fermion commutation relations. Such expansion was never used in our practical applications. In Sec. IV we calculated the $B(E2)$'s for the three- to two-phonon states transitions, that required, in the KT2 expansion, also the presence of the fifth-order term (Table II). The results again confirmed that the accuracy and the speed of convergence of the expansion are quite good. In Sec. IV we also discussed a problem pertaining to the truncation of the fermion space to a single component. As seen in part C of Table II, this truncation is the principal source of error; the boson mapping introduces only a very small amount of error.

In Sec. V we discussed the BCS approximation and estimated the BCS error by comparing the results with those obtained by using the number conserving approach of Li.[10] The results are summarized in Table III. As seen, the error is never very large, except for the case in which the number of particles is small. The results of Secs. II, IV, and V are then combined and are given in Table IV. There are the results obtained by using the BCS plus BET are compared with the exact (number-projected) fermion results. As seen, the error does not exceed 20 percent, except for cases with too small particle numbers.

Overall, we feel that the presentation made in this paper confirmed the mathematical and physical soundness of the BET of KT. Furthermore, we showed that the accuracy of the numerical results and the speed of convergence of the expansion can be considered more than satisfactory for practical purposes, if the theory is used correctly and within the bounds of its applicability.

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[1]T. Kishimoto and T. Tamura, Nucl. Phys. **A192**, 246 (1972).

[2]T. Kishimoto and T. Tamura, Nucl. Phys. **A270**, 317 (1976).

[3]T. Tamura, K. J. Weeks, and T. Kishimoto, Phys Rev. C **20**, 307 (1979); Nucl. Phys. **A347**, 359 (1980); K. J. Weeks and T. Tamura, Phys. Rev. Lett. **44**, 533 (1980); Phys. Rev. C **22**, 888 (1980); **22**, 1312 (1980); K. J. Weeks, T. Tamura, T. Udagawa, and F. J. W. Hahne, *ibid.* **24**, 703 (1981).

[4]V. G. Pedrocchi and T. Tamura, Prog. Theor. Phys. **68**, 820 (1982).

[5]V. G. Pedrocchi and T. Tamura, Phys. Rev. C **28**, 410 (1983).

[6]A. Arima, Nucl. Phys. **A347**, 339 (1980).

[7]T. Kishimoto and T. Tamura, Phys. Rev. C **27**, 341 (1983).

[8]In KT1 an additional parameter, called x, was introduced, with a value always larger than unity. However, it was later recognized that x must, in fact, be equal to 1 in order to let the scattering operator annihilate the vacuum; see, e.g., Ref. 4. Throughout this paper we set $x=1$. In our previous practical applications the value of x was determined through a self-consistent search for every individual case (see Ref. 2), and the obtained values were very close to unity. We may also note that the final results were not very sensitive to the value of x.

[9]See, e.g., T. Tamura, K. J. Weeks, and V. G. Pedrocchi, Phys. Rev. C **27**, 1297 (1981).

[10]C. T. Li, Nucl. Phys. **A417**, 37 (1983).

[11]C. T. Li, V. G. Pedrocchi, and T. Tamura (unpublished).