General Lorentz-invariant representation of NN scattering amplitudes

J. A. Tjon
Institute for Theoretical Physics, 3508 TA Utrecht, The Netherlands

S. J. Wallace
Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742
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A Lorentz-invariant representation for NN scattering amplitudes is derived. The general NN representation assuming parity invariance involves 128 amplitudes for a given isospin, all but 8 of which involve negative energy projection operators and therefore possess vanishing matrix elements in positive energy states. When charge symmetry and time-reversal invariance are taken into account, the number of independent amplitudes reduces to 56 for a given isospin, and this number further reduces to 44 when all particles are on mass shell. Relativistic meson theory is used to determine the negative energy terms since they cannot be determined from physical scattering data. The formalism is developed to determine the complete set of invariant amplitudes starting from partial wave t-matrix elements which arise from solving quasipotential equations for NN scattering.

I. INTRODUCTION

Recent work has shown that a relativistic impulse approximation[1] provides interesting predictions for intermediate energy proton-nucleus scattering.[2–5] Although the construction of an optical potential requires knowledge of the fully-off-shell relativistic NN amplitudes, successful predictions have been made by choosing a "good" set of five relativistic covariants.[1] The associated invariant amplitudes are determined by equating positive energy matrix elements to known physical amplitudes.[6] Once the five NN amplitudes and covariants are fixed, one can predict negative energy matrix elements which control pair effects in the relativistic description of proton-nucleus scattering.[3] However, negative energy matrix elements are ambiguous because it is possible to alter them without changing the positive energy matrix elements.[7] For example, one may add new covariants whose positive energy matrix elements vanish. The amplitude of such new covariants cannot be fixed without theoretical input. Positive energy NN data are necessary but not sufficient input to construct the relativistic NN amplitudes and optical potential unambiguously.

In this work, a general representation of NN amplitudes in terms of Lorentz covariants is developed consistent with parity invariance. The representation chosen clearly embeds the Fermi covariants used previously but also contains terms which only affect negative energy matrix elements. The invariant amplitudes are obtained by equating matrix elements to c.m. frame helicity amplitudes. The latter may be obtained by solving dynamical equations for NN scattering based on meson theory. Previous work by Tjon and collaborators has developed a satisfactory description of NN phase shifts in the 0—1000 MeV range by solving coupled-channel (NN,NΔ,$\Delta\Delta$) Bethe-Salpeter equations based on meson exchange.[8] This formalism provides partial wave t matrices. In this paper we show how to construct the c.m. frame helicity amplitudes from the partial wave t matrices based upon a quasipotential reduction[9] of the meson exchange dynamics.

For purposes of constructing an optical potential, it is essential to be able to boost the NN amplitudes to the p + nucleus c.m. frame. By using a covariant representation, the boost and its associated Wigner rotation of spin operators are automatically taken into account in a simple and straightforward manner. Using the formalism of this paper and meson theoretical calculations of Ref. 8 (slightly extended), the first calculations of a complete set of Lorentz-invariant amplitudes have been used in Refs. 7 and 10 to develop the proton-nucleus optical potential.

This paper presents the detailed formalism necessary to determine NN amplitudes. Section II derives a general covariant representation for NN scattering and Sec. III analyzes the role of symmetries in reducing the number of independent amplitudes. Section IV develops the relations between invariant amplitudes and the c.m. frame helicity amplitudes. A matrix which provides the necessary link between relativistic covariants and helicity amplitudes is explicitly constructed in Appendix A. The construction of c.m. frame helicity amplitudes for positive and negative energy states is discussed in Sec. V. Our analysis closely follows some parts of the paper by Kubis,[11] except that the NN dynamics is assumed to be adequately described in the quasipotential reduction of Ref. 9. A transformation from the Dirac spinors of Kubis to the standard spinors of Bjorken and Drell[12] is given in Appendix B. Details concerning how the partial wave t-matrix elements obtained by solving the quasipotential equation are used in the partial wave expansion of helicity amplitudes are given in Appendix C. Some concluding remarks are given in Sec. VI.

II. GENERAL LORENTZ-INVARIANT REPRESENTATION

In order to unambiguously expand the relativistic NN amplitude, one first needs a complete set of Lorentz covariants which can describe transitions between all possible two-fermion states. The two-fermion system has four spin states and four energy states: $++$, $+-$, $-+$, and $--$. Thus, there are 16 states in all and a 16×16 matrix is needed to describe all matrix elements for a given isospin. However, not all 256 elements of this matrix are independent. Parity invariance reduces the number of independent elements to 128. Other symmetries to be discussed further on, reduce the independent elements to 56. For the case that the particles are on mass shell, i.e., $p_i^2=m_i^2$, or symmetrically off mass shell, the number of independent amplitudes further reduces to 44.

A general parity invariant representation of the NN amplitudes can be developed as follows. Following Ref. 11, consider the complete set of NN helicity amplitudes, $\phi^{(\rho_1'\rho_2'\rho_1\rho_2)}_{\lambda_1'\lambda_2'\lambda_1\lambda_2}$ where $\lambda=\pm\frac{1}{2}$ denotes helicity and $\rho=\pm$ denotes energy, 1 and 2 are particle labels, and primes denote final state quantum numbers. The helicity amplitudes are Dirac matrix elements of $2ip\hat{F}$ as follows in the c.m. frame:

$$\phi^{(\rho_1'\rho_2'\rho_1\rho_2)}_{\lambda_1'\lambda_2'\lambda_1\lambda_2}=(2ip)\bar{u}^{(\rho_1')}_{\lambda_1'}(\mathbf{p}')\bar{u}^{(\rho_2')}_{-\lambda_2'}(-\mathbf{p}')\times\hat{F}u^{(\rho_1)}_{\lambda_1}(\mathbf{p})u^{(\rho_2)}_{-\lambda_2}(-\mathbf{p})\;, \tag{2.1}$$

where the convention for particle 2 helicity states follows Refs. 11, 13, and 14. Furthermore, the γ matrices and spinors follow the conventions of Bjorken and Drell except in Sec. V where spinors of Kubis are used. For the present, the problem is how to obtain invariant amplitudes which can be used in an arbitrary frame from helicity amplitudes which have a simple partial wave expansion only in the NN center-of-mass frame. Construction of the optical potential is easily done in terms of Lorentz covariants although other methods can also be used. The helicity amplitudes are not covariants for $\theta\neq0$ due to the well-known Wigner rotation[15] which arises in Lorentz boosts.

A Lorentz-invariant representation of the NN amplitude constructed directly from the c.m. frame helicity amplitudes using projection operators takes the form

$$\hat{F}=(2ip)^{-1}\sum_{\{\lambda\}\{\rho\}}\Lambda^{(\rho_1')}_{\lambda_1'}(\mathbf{p}')\Lambda^{(\rho_2')}_{-\lambda_2'}(-\mathbf{p}')\times\phi^{(\rho_1'\rho_2'\rho_1\rho_2)}_{\lambda_1'\lambda_2'\lambda_1\lambda_2}\Lambda^{(\rho_1)}_{\lambda_1}(\mathbf{p})\Lambda^{(\rho_2)}_{-\lambda_2}(-\mathbf{p})\;. \tag{2.2}$$

Here $\Lambda^{(\rho)}_{\lambda}(\mathbf{p})$ is a projection operator with $\rho=\pm$ denoting sign of energy and λ denoting helicity eigenvalue.[12]

$$\Lambda^{(\rho)}_{\lambda}(\mathbf{p})=\Lambda^{(\rho)}(\mathbf{p})\Sigma_\lambda(\mathbf{p})\;, \tag{2.3}$$

where $\Lambda^{\pm}$ are projection operators to positive and negative energy

$$\Lambda^{(\pm)}(\mathbf{p})=[\pm(E_p\gamma^0-\boldsymbol{\gamma}\cdot\mathbf{p})+m]/(2m) \tag{2.4a}$$

and $\Sigma_{\pm1/2}$ is a helicity projection operator

$$\Sigma_{\pm1/2}(\mathbf{p})=\tfrac{1}{2}(1\pm\gamma^5\not{s}) \tag{2.4b}$$

with $E_p=\sqrt{p^2+m^2}$, $\not{s}=(\gamma^0p-E_p\boldsymbol{\gamma}\cdot\hat{\mathbf{e}})/m$, and $\hat{\mathbf{e}}=\mathbf{p}/p$. Obviously c.m. matrix elements of (2.2) reproduce (2.1). In other frames Dirac spinor matrix elements of (2.2) automatically give rise to Wigner rotation effects. One drawback to (2.2) is that there is no evident connection to the Fermi covariants which have been found useful in previous work. Another is that matrix elements of (2.2) in other than the c.m. frame are not simple.

Proceeding schematically, we eliminate positive energy projection operators by use of

$$\Lambda^{(+)}_{\lambda}=\Sigma_\lambda-\Lambda^{(-)}_{\lambda}\;. \tag{2.5}$$

The term in (2.2) which has four Λ^+ projectors therefore expands to 16 terms. A general form for $\hat{F}$ after all Λ^+ projectors have been eliminated has 16 terms which we choose to divide into four classes each containing four subclasses as follows:

$$\hat{F}=\hat{F}^{1}+\Lambda^{(-)}_{1'}\hat{F}^{2}+\hat{F}^{3}\Lambda^{(-)}_{1}+\Lambda^{(-)}_{1'}\hat{F}^{4}\Lambda^{(-)}_{1}\;, \tag{2.6}$$

$$\hat{F}^{i}=\hat{F}^{i1}+\Lambda^{(-)}_{2'}\hat{F}^{i2}+\hat{F}^{i3}\Lambda^{(-)}_{2}+\Lambda^{(-)}_{2'}\hat{F}^{i4}\Lambda^{(-)}_{2}\;,\quad i=1\text{ to }4\;. \tag{2.7}$$

In this scheme, the first superscript of $\hat{F}^{ij}$ is the class index, $i=1$ to 4, which refers to the associated negative energy projectors for particle 1 and the second superscript of $\hat{F}^{ij}$ is the subclass index, $j=1$ to 4, which refers to the associated negative energy projectors for particle 2. The full expansion is

$$\begin{aligned}\hat{F}=\;&\hat{F}^{11}+\Lambda^{(-)}_{2'}\hat{F}^{12}+\hat{F}^{13}\Lambda^{(-)}_{2}+\Lambda^{(-)}_{2'}\hat{F}^{14}\Lambda^{(-)}_{2}\\&+\Lambda^{(-)}_{1'}[\hat{F}^{21}+\Lambda^{(-)}_{2'}\hat{F}^{22}+\hat{F}^{23}\Lambda^{(-)}_{2}+\Lambda^{(-)}_{2'}\hat{F}^{24}\Lambda^{(-)}_{2}]\\&+[\hat{F}^{31}+\Lambda^{(-)}_{2'}\hat{F}^{32}+\hat{F}^{33}\Lambda^{(-)}_{2}+\Lambda^{(-)}_{2'}\hat{F}^{34}\Lambda^{(-)}_{2}]\Lambda^{(-)}_{1}\\&+\Lambda^{(-)}_{1'}[\hat{F}^{41}+\Lambda^{(-)}_{2'}\hat{F}^{42}+\hat{F}^{43}\Lambda^{(-)}_{2}+\Lambda^{(-)}_{2'}\hat{F}^{44}\Lambda^{(-)}_{2}]\Lambda^{(-)}_{1}\end{aligned} \tag{2.8}$$

Subclass 2, 3, and 4 contributions involve $\Lambda^{(-)}_{2}$ projection operators which tend to be suppressed when it comes to calculating the nucleon-nucleus optical potential. Thus the most important parts of (2.8) are subclass 1, namely $\hat{F}^{11}$, $\hat{F}^{21}$, $\hat{F}^{31}$, and $\hat{F}^{41}$. Contributions which have no $\Lambda^{(-)}$ projectors, namely $\hat{F}^{11}$, are clearly separated from all others. Thus the positive energy matrix elements involve just class 1; subclass 1 and the 15 other classes and subclasses define the remainder which cannot be fixed from knowledge of positive energy scattering data.

Rather than using helicity projectors Σ_λ, each $\hat{F}^{ij}$ in (2.8) is expanded in terms of a set of nine covariants as follows:

$$\hat{F}^{ij}=\sum_{n=1}^{9}F^{ij}_n\mathscr{K}_n\;, \tag{2.9}$$

where

$$\mathscr{K}_n=\{S,V,T,P,A,\gamma_2\cdot Q_1,\gamma_1\cdot Q_2,P\gamma_2\cdot Q_1,P\gamma_1\cdot Q_2\} \tag{2.10}$$

and

$$Q_1^\mu=(p_1+p_1')^\mu/(2m),\quad Q_2^\mu=(p_2+p_2')^\mu/(2m)\ . \tag{2.11}$$

For example, $\mathscr{K}_1=S$ and $\mathscr{K}_9=P\gamma_1\cdot Q_2$. The Fermi covariants used are $S=1$, $V=\gamma_1\cdot\gamma_2$, $T=\sigma_1^{\mu\nu}\sigma_{2\mu\nu}$, $P=\gamma_1^5\gamma_2^5$, and $A=P\cdot V$. The remaining four, $\mathscr{K}_6$ to $\mathscr{K}_9$, have been selected taking into account the equivalence theorems of Scadron and Jones.[16]

A symmetrized expansion is also employed as follows:

$$\hat{F}^{ij}=(2ip)^{-1}\sum_{n=1}^{9} f_n^{ij}k_n\ , \tag{2.12}$$

where

$$k_n=\{S-\tilde{S},\tfrac{1}{2}(T+\tilde{T}),-A+\tilde{A},V+\tilde{V},P-\tilde{P},\gamma_2\cdot Q_1-\gamma_1\cdot Q_2,\gamma_2\cdot Q_1+\gamma_1\cdot Q_2,P(\gamma_2\cdot Q_1-\gamma_1\cdot Q_2),P(\gamma_2\cdot Q_1+\gamma_1\cdot Q_2)\}\ . \tag{2.13}$$

Amplitudes F_n^{ij} are related to f_n^{ij} for $n=1$ to 5 by a Fierz matrix as follows (omitting superscripts ij):

$$\begin{bmatrix}F_1\\F_2\\F_3\\F_4\\F_5\end{bmatrix}=(8ip)^{-1}\begin{bmatrix}3&6&-4&4&1\\-1&0&-2&2&1\\-\frac{1}{2}&1&0&0&-\frac{1}{2}\\-1&6&4&-4&3\\1&0&-6&-2&-1\end{bmatrix}\begin{bmatrix}f_1\\f_2\\f_3\\f_4\\f_5\end{bmatrix}\ . \tag{2.14}$$

See Refs. 7 and 14 for more detailed discussion of the Fierz transformation. Additional relations needed to complete the connection are

$$\begin{aligned}F_6&=(2ip)^{-1}(f_7+f_6)\ ,\\F_7&=(2ip)^{-1}(f_7-f_6)\ ,\\F_8&=(2ip)^{-1}(f_9+f_8)\ ,\\F_9&=(2ip)^{-1}(f_9-f_8)\ .\end{aligned} \tag{2.15}$$

Only eight of the covariants in (2.10) or (2.13) are linearly independent. Therefore (2.9) and (2.12) allow for $16\times 8=128$ independent contributions to $\hat{F}$ consistent with parity invariance. The redundant covariant changes from one class to the next. This complication is easily dealt with by using the overcomplete set of nine covariants supplemented by simple constraint conditions.

Table I shows the constraint conditions used to eliminate the redundant covariant for each class and subclass. As shown by Scadron and Jones,[16] $k_7=\gamma_2\cdot Q_1+\gamma_1\cdot Q_2$ is redundant for $(++,++)$ matrix elements where only positive energy states enter. For matrix elements such as $(-+,++)$ or $(+-,++)$ where one negative energy state is involved, we find that the set formed by omitting k_7 is again linearly independent. Thus in these cases the constraint is $f_7^{ij}=0$, or from (2.15), $F_6^{ij}=-F_7^{ij}$. The remaining cases in Table I can be deduced from their relation to the two just discussed. Using $u^{(-)}=\gamma^5u^{(+)}$ to transform negative energy states to positive energy ones, observe that in the fourth row of Table I, the transformation yields

$$(-+|k_n|-+)=-(++|\gamma_1^5k_n\gamma_1^5|++)\ .$$

Since the redundant covariant in $(++,++)$ states is k_7, we need to find which covariant is transformed into $\pm k_7$. Inspection of (2.10) shows that $\gamma_1^5k_6\gamma_1^5=k_7$ and thus k_6 is the redundant covariant for $(-+,-+)$ matrix elements and we choose $f_6^{41}=0$ as the appropriate constraint. Similar reasoning produces the rest of Table I.

Linear independence of the resulting sets of eight covariants has been verified explicitly by computing a nonzero determinant for the matrix which relates the eight linearly independent f_n^{ij} to the corresponding set of eight parity conserving helicity amplitudes.

TABLE I. Redundant covariants and amplitudes.

Class i	Subclass j	$\rho_1'\ \rho_2'\ \rho_1\ \rho_2$	Redundant covariant	Constraint in Eq. (2.12)	Constraint in Eq. (2.9)
1	1	$+\ +\ +\ +$	k_7	$f_7^{11}=0$	$F_6^{11}=-F_7^{11}$
2	1	$-\ +\ +\ +$	k_7	$f_7^{21}=0$	$F_6^{21}=-F_7^{21}$
3	1	$+\ +\ -\ +$	k_7	$f_7^{31}=0$	$F_6^{31}=-F_7^{31}$
4	1	$-\ +\ -\ +$	k_6	$f_6^{41}=0$	$F_6^{41}=-F_7^{41}$
1	2	$+\ -\ +\ +$	k_7	$f_7^{12}=0$	$F_6^{12}=-F_7^{12}$
2	2	$-\ -\ +\ +$	k_9	$f_9^{22}=0$	$F_8^{22}=-F_9^{22}$
3	2	$+\ -\ -\ +$	k_8	$f_8^{32}=0$	$F_8^{32}=F_9^{32}$
4	2	$-\ -\ -\ +$	k_6	$f_6^{42}=0$	$F_6^{42}=F_7^{42}$
1	3	$+\ +\ +\ -$	k_7	$f_7^{13}=0$	$F_6^{13}=-F_7^{13}$
2	3	$-\ +\ +\ -$	k_8	$f_8^{23}=0$	$F_8^{23}=F_9^{23}$
3	3	$+\ +\ -\ -$	k_9	$f_9^{33}=0$	$F_8^{33}=-F_9^{33}$
4	3	$-\ +\ -\ -$	k_9	$f_6^{43}=0$	$F_6^{43}=F_7^{43}$
1	4	$+\ -\ +\ -$	k_6	$f_6^{14}=0$	$F_6^{14}=F_7^{14}$
2	4	$-\ -\ +\ -$	k_6	$f_6^{24}=0$	$F_6^{24}=F_7^{24}$
3	4	$+\ -\ -\ -$	k_6	$f_6^{34}=0$	$F_6^{34}=F_7^{34}$
4	4	$-\ -\ -\ -$	k_7	$f_7^{44}=0$	$F_6^{44}=-F_7^{44}$

III. SYMMETRY CONSIDERATIONS

Symmetries other than parity invariance exist in the NN system and these significantly reduce the number of independent amplitudes. In this section, we assume that the nucleon-nucleon dynamics is governed by a charge symmetric, time-reversal invariant Lagrangian. Consider first the amplitude for scattering of identical particles, e.g., pp scattering or nn scattering. Figure 1(a) shows the basic process for pp scattering. When momenta and spins of particles 1 and 2 are simultaneously interchanged, the diagram must have the same value since it is exactly the same process. Due to charge symmetry, the same statement holds for pn and np scattering.[17] Thus

$$\langle \mathrm{pn} | \hat{F} | \mathrm{pn} \rangle = \langle \mathrm{np} | \hat{F} | \mathrm{np} \rangle$$

holds if $\hat{F} = e^{-i\pi T_y} \hat{F} e^{i\pi T_y}$, which is the condition of charge symmetry. This means that Figs. 1(b) and (c) are governed by matrix elements of the same $\hat{F}$ (which, of course, is different from the $\hat{F}$ which governs pp and nn collisions). Simultaneously interchanging spins and momenta of particles 1 and 2 in Fig. 1(c) produces exactly the same process as is described by Fig. 1(b). Therefore it is clear that under such an interchange, the nucleon-nucleon amplitude is invariant, and this statement holds for fully-off-shell particles since it derives from the underlying Lagrangian. Note that when Coulomb interactions are included, pp scattering still has the stated symmetry.

For the most general case, the invariant amplitudes, F_n^{ij} or f_n^{ij}, can depend on all the available Lorentz scalars. These are

$$s_1 = (p_1 + p_2)^2 = (p_1' + p_2')^2 ,$$

$$s_2 = (p_1 - p_1')^2 = (p_2' - p_2)^2 ,$$

$$s_3 = (p_1 - p_2')^2 = (p_1' - p_2)^2 ,$$

$$s_4 = \tfrac{1}{2}(p_1^2 + p_2^2) ,$$

$$s_5 = \tfrac{1}{2}(p_1'^2 + p_2'^2) ,$$

$$s_6 = \tfrac{1}{2}(p_1^2 - p_2^2) ,$$

and

$$s_7 = \tfrac{1}{2}(p_1'^2 - p_2'^2) .$$

Under the interchange of momenta of particles 1 and 2, s_1 to s_5 are even but s_6 and s_7 are odd. The meaning of this interchange is somewhat clarified if the momenta are written as $p_1 = \frac{1}{2}P + p$, $p_2 = \frac{1}{2}P - p$, $p_1' = \frac{1}{2}P + p'$, and $p_2' = \frac{1}{2}P - p'$. Then $s_6 = p \cdot P$ and $s_7 = p' \cdot P$. In the c.m. frame where $P^\mu = (\sqrt{s_1}, \mathbf{0})$, it follows that s_6 and s_7 are directly proportional to the time components of relative momenta, i.e., p^0 and $p^{0\prime}$. In general, charge symmetry involves relations between invariant amplitudes with reversed signs of p and p'. For kinematic conditions such that $s_6 = s_7 = 0$, i.e., when $p \cdot P = p' \cdot P = 0$, charge symmetry results in direct equalities between invariant amplitudes with all arguments equal. This includes the interesting case when all particles are on mass shell.

The five Fermi covariants are unchanged if the spin labels and momenta of particles 1 and 2 are interchanged

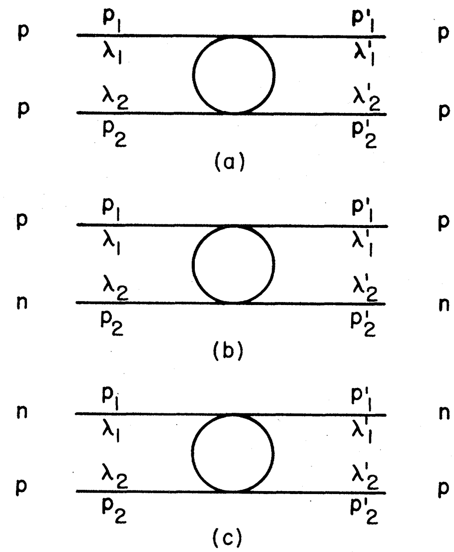

FIG. 1. Diagrams for proton-proton, proton-neutron, and neutron-proton scattering.

simultaneously. Covariants $\mathscr{K}_6$ to $\mathscr{K}_9$ and negative energy projection operators become interchanged:

$$\mathscr{K}_6 \leftrightarrow \mathscr{K}_7 ,$$

$$\mathscr{K}_8 \leftrightarrow \mathscr{K}_9 ,$$

$$\Lambda_1^{(-)}(\mathbf{p}_1) \leftrightarrow \Lambda_2^{(-)}(\mathbf{p}_2) ,$$

and

$$\Lambda_1^{(-)}(\mathbf{p}_1') \leftrightarrow \Lambda_2^{(-)}(\mathbf{p}_2') .$$

Consequently, charge symmetry requires the symmetry with respect to interchange of class and subclass superscripts as follows:

$$\begin{aligned} &F_n^{ij} = F_n^{ji} \quad \text{or} \quad f_n^{ij} = f_n^{ji}, \quad n = 1 \text{ to } 5 , \\ &\left.\begin{aligned} F_6^{ij} &= F_7^{ji} \\ F_7^{ij} &= F_6^{ji} \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} f_6^{ij} &= -f_6^{ji} \\ f_7^{ij} &= f_7^{ji} \end{aligned}\right. \\ &\left.\begin{aligned} F_8^{ij} &= F_9^{ji} \\ F_9^{ij} &= F_8^{ji} \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} f_8^{ij} &= -f_8^{ji} \\ f_9^{ij} &= f_9^{ji} \end{aligned}\right. \end{aligned} \tag{3.1}$$

where i and j refer to any class and subclass, respectively. As noted above, these equalities are to be interpreted as relations between amplitudes with reversed signs of p and p', or as direct equalities if $p \cdot P = p' \cdot P = 0$. Charge symmetry reduces by half the 96 independent amplitudes for the 12 cases where $i \neq j$. For $i = j$, Eqs. (3.1) show that f_1^{ii} to f_5^{ii}, f_7^{ii}, and f_9^{ii} are even under simultaneous reversal of p and p', while f_6^{ii} and f_8^{ii} are odd under the same transformation. In the general case, 48 new relations are thereby found leaving 80 amplitudes independent, for a given isospin, due to parity invariance and charge symmetry. Moreover when $p \cdot P = p' \cdot P = 0$, an additional eight relations hold as follows:

$$f_6^{11} = f_8^{11} = f_6^{22} = f_8^{22} = f_6^{33} = f_8^{33} = f_6^{44} = f_8^{44} = 0 . \tag{3.2}$$

Consequently the number of independent amplitudes for a given isospin is 72 when $p\cdot P=p'\cdot P=0$.

In what follows, we consider the representation of Eqs. (2.12) and (2.13) since it is more convenient for discussion of symmetries. Time-reversal invariance requires that the amplitude for scattering from initial momenta $\mathbf{p}_1,\mathbf{p}_2$ to final momenta $\mathbf{p}'_1,\mathbf{p}'_2$ be equal to the amplitude for scattering from initial momenta $-\mathbf{p}'_1,-\mathbf{p}'_2$ to final momenta $-\mathbf{p}_1,-\mathbf{p}_2$. Time components of initial and final momenta are left unchanged. Upon time reversal, the Lorentz scalars s_1 to s_3 are unchanged, s_4 and s_5 are interchanged, and s_6 and s_7 are interchanged. Stated in terms of the relative momenta, time reversal interchanges p and p'. For the c.m. frame, the special kinematic condition $p\cdot P=p'\cdot P=0$ and $p^2=p'^2$ results in $s_6=s_7$ and $s_4=s_5$. In this case, time-reversal invariance leads to direct equalities between invariant amplitudes at the same kinematic point. This includes the interesting case when all particles are on mass shell. Explicit dependence on the momenta occurs in $\hat{F}$ due to $\Lambda^{(-)}$ projectors and due to covariants $\mathscr{K}_6$ to $\mathscr{K}_9$ (or k_6 to k_9). A formal statement of time-reversal invariance is

$$\bar{\psi}(p'_1p'_2)\hat{F}(p_1p_2\to p'_1p'_2)\psi(p_1p_2)$$
$$=\bar{\psi}(\tilde{p}_1,\tilde{p}_2)\hat{F}(\tilde{p}'_1,\tilde{p}'_2\to\tilde{p}_1,\tilde{p}_2)\psi(\tilde{p}'_1,\tilde{p}'_2)\,, \quad (3.3)$$

where $\psi(\tilde{p}_1,\tilde{p}_2)=\mathscr{T}\psi(p_1,p_2)$. Here $\tilde{p}_i=(p_i^0,-\mathbf{p}_i)$ has reversed space component compared with p_i and $\mathscr{T}$ is the antiunitary time-reversal operator. Expressed in terms of Dirac operators, the requirement is

$$\gamma_1^0\gamma_2^0\mathscr{T}\hat{F}^\dagger(\tilde{p}'_1,\tilde{p}'_2\to\tilde{p}_1,\tilde{p}_2)\mathscr{T}^{-1}\gamma_1^0\gamma_2^0$$
$$=\hat{F}(p_1,p_2\to p'_1,p'_2)\,, \quad (3.4)$$

where[12] $\mathscr{T}=TK$ with $T=(i\gamma_1^1\gamma_1^3)(i\gamma_2^1\gamma_2^3)$ and K is the complex conjugation operator. Covariants $\mathscr{K}_1$ to $\mathscr{K}_7$ (or k_1 to k_7) are time reversal invariant in the sense of (3.4). Covariants $\mathscr{K}_8$ and $\mathscr{K}_9$ (or k_8 and k_9) are odd under time reversal. Negative energy projectors are invariant in the sense that

$$\gamma_i^0\mathscr{T}[\Lambda_i^{(-)}(-\mathbf{p})]^\dagger\mathscr{T}^{-1}\gamma_i^0=\Lambda_i^{(-)}(\mathbf{p}),\ \ (i=1,2)\,, \quad (3.5)$$

however, projectors to the right in $\hat{F}(\tilde{p}'_1,\tilde{p}'_2\to\tilde{p}_1,\tilde{p}_2)$ are moved to the left due to Hermitian conjugation. Inspection of (2.8) shows that time-reversal invariance relates amplitudes which transform into one another when $\Lambda^{(-)}$ projectors on the right are moved to the left and vice versa. Specifically, we find

$$\begin{aligned} f_n^{11}&=\pm f_n^{11}\,,\\ f_n^{14}&=\pm f_n^{14}\,,\\ f_n^{41}&=\pm f_n^{41}\,,\\ f_n^{44}&=\pm f_n^{44}\,,\\ f_n^{21}&=\pm f_n^{31}\,,\\ f_n^{12}&=\pm f_n^{13}\,,\\ f_n^{23}&=\pm f_n^{32}\,,\\ f_n^{22}&=\pm f_n^{33}\,,\\ f_n^{42}&=\pm f_n^{43}\,,\\ f_n^{24}&=\pm f_n^{34}\,, \end{aligned} \quad (3.6)$$

where the sign is + for $n=1$ to 7 and − for $n=8$ and 9 (since k_8 and k_9 are odd). These are equalities between amplitudes whose arguments are related by interchange of initial and final relative momenta, $p\leftrightarrow p'$. The first four conditions, in general, specify symmetry properties. For example, f_n^{11}, f_n^{14}, f_n^{41}, and f_n^{44} are even ($n=1$ to 7) or odd ($n=8$ and 9) under interchange of p and p'. The remaining conditions in (3.6) reduce the number of independent amplitudes to the extent that they are not redundant with charge symmetry conditions. For example, $f_n^{21}=\pm f_n^{31}$ means that eight amplitudes, f_n^{21}, can be determined from knowledge of the f_n^{31} amplitudes, which for sake of illustration, we take to be independent. Equation (3.1) fixes eight f_n^{13} amplitudes also from knowledge of f_n^{31} and moreover fixes eight f_n^{12} amplitudes once the f_n^{21} amplitudes are determined. Therefore the time-reversal relation $f_n^{12}=\pm f_n^{13}$ is redundant when charge symmetry holds. Similarly the time-reversal relation $f_n^{23}=\pm f_n^{32}$ is redundant but does contain symmetry information for $p\leftrightarrow p'$. However, $f_n^{22}=\pm f_n^{33}$ provides eight new relations. From the two conditions $f_n^{42}=\pm f_n^{24}$ and $f_n^{43}=\pm f_n^{34}$, another eight new relations are obtained when charge symmetry holds. In all, we find 24 new relations and these reduce the number of independent amplitudes for the most general case to 56 for each isospin due to parity invariance, charge symmetry, and time-reversal invariance.

Twelve additional relations hold when $p\cdot P=p'\cdot P=0$ and $p^2=p'^2$, i.e., when particles 1 and 2 are symmetrically off mass shell by the same amount in initial and final states, or are all on mass shell. Stated more simply, the additional symmetry relations hold when $p_1^2=p_2^2=p_1'^2=p_2'^2$. Eight of these additional relations are given in Eq. (3.2). Two more are $f_6^{23}=f_6^{32}$ and $f_9^{23}=-f_9^{32}$ which follow from (3.6) after elimination of parts of $f_n^{23}=\pm f_n^{32}$ which are redundant with (3.1). However, when (3.2) holds, two relations from $f_n^{22}=\pm f_n^{33}$, which were previously counted as independent, become redundant. Finally, the cases $i,j=1,4$ from (3.6) lead to eight relations,

$$\begin{aligned} &f_8^{11}=f_9^{11}=f_8^{44}=f_9^{44}=0\,,\\ &f_8^{14}=f_9^{14}=f_8^{41}=f_9^{41}=0\,, \end{aligned} \quad (3.7)$$

but four of these relations are redundant when charge symmetry holds. Consequently when $p_1^2=p_2^2=p_1'^2=p_2'^2$, we have 12 additional relations and these reduce the number of independent amplitudes to 44 for a given isospin. In particular when all particles are on mass shell, there are 44 independent amplitudes.

In class 1, subclass 1 (i.e., $\hat{F}^{11}$), there are just five nonzero amplitudes which are associated with the five Fermi covariants when particles are on mass shell. These are determined by positive energy matrix elements and therefore they are exactly the same as defined in Refs. 1, 6, and 7. The amplitudes of covariants $\mathscr{K}_6$ to $\mathscr{K}_9$ in $\hat{F}^{11}$ vanish due to symmetries which hold when $p_1^2=p_2^2=p_1'^2=p_2'^2$. Nonzero amplitudes corresponding to covariants $\mathscr{K}_6$ to $\mathscr{K}_9$ are needed to describe off-mass-shell or negative energy matrix elements of $\hat{F}$.

The final symmetry property of interest is due to the generalized Pauli principle,[11] which may be expressed as

$$P_H\tilde{S}\hat{F}=(-)^I\hat{F}\ , \tag{3.8}$$

where I is the isospin quantum number, 0 or 1, P_H is the operator for exchange of four-momenta of the two nucleons, and $\tilde{S}$ is the Fierz operator for exchange of Dirac spinor indices. In contrast with analyses of charge symmetry, in this case exchange of particles 1 and 2 takes place on just one side of $\hat{F}$. Therefore only the final state (or equivalently, the initial state) is affected. Exchange operators take $\Lambda_1^{(-)}$ into $\Lambda_2^{(-)}$, and vice versa in (2.8). The overall antisymmetry requires

$$f_n^{ij}(\pi-\theta)=(-1)^{n+I}f_n^{ji}(\theta),\quad n=1 \text{ to } 5\ . \tag{3.9}$$

Since covariants k_6 to k_9 are not purely even or odd under particle exchange in the final state, it is not possible to give a simple exchange symmetry for amplitudes f_n^{ij}, $n=6$ to 9. Note that for $i\neq j$, different amplitudes appear on the left- and right-hand sides of (3.9). However, charge symmetry relations (3.1) together with (3.9) produce simple symmetry properties of individual amplitudes when $p\cdot P=p'\cdot P=0$:

$$f_n^{ij}(\pi-\theta)=(-)^{n+I}f_n^{ij}(\theta),\quad n=1 \text{ to } 5\ . \tag{3.10}$$

This concludes the formal considerations. Our general representation (2.8) turns out to be a convenient Lorentz-invariant representation of the NN amplitude with simple symmetry properties. One-meson exchange contributions take an evident form in this representation and the physical positive energy matrix elements involve only the $\hat{F}^{11}$ term of the representation. The five nonzero components of $\hat{F}^{11}$ on mass shell are identical to the usual representations given in Sec. II of Ref. 7 and shown there to be simply expressed in terms of the c.m. frame helicity amplitudes for positive energy states. In order to determine the other amplitudes, it is necessary to relate them to c.m. frame helicity amplitudes which involve negative energy states.

Many other equivalent representations are obviously possible; for example, there is one similar to (2.8) but with all $\Lambda^{(-)}$ projectors eliminated in favor of $\Lambda^{(+)}$ projectors. Another is obtained by using $\Lambda^{(+)}$ projectors in place of unit operators in (2.8) so that $\hat{F}^{11}$ would have four associated $\Lambda^{(+)}$ projectors, and so on. Invariant amplitudes for all such representations can be formed from linear combinations of the invariant amplitudes as defined in (2.8) and (2.9).

IV. RELATION OF INVARIANT AMPLITUDES TO c.m. FRAME HELICITY AMPLITUDES

Although symmetries and constraints greatly reduce the number of independent amplitudes needed on shell, we consider the full set of 128 amplitudes for a given isospin consistent with parity invariance. The object is to determine invariant amplitudes f_n^{ij} appropriate to Eq. (2.12) from knowledge of parity-conserving helicity amplitudes,

$$\phi_n^{ij}=\phi_{\lambda_1'\lambda_2'\lambda_1\lambda_2}^{(\rho_1'\rho_2'\rho_1\rho_2)}/\alpha_n\ , \tag{4.1}$$

in the c.m. frame. Here α_n denotes a kinematic factor to be divided out of the helicity amplitudes. Table I defines the correspondence between energy labels $\rho_1'\rho_2'\rho_1\rho_2$ and class and subclass superscripts ij of ϕ_n^{ij}. Notice that i and j do not have the same meaning as in Eq. (2.8). For example, ϕ_n^{11} involves only positive energy states while $\hat{F}^{11}$ has matrix elements for both positive and negative energy states. Table II defines the correspondence between helicity labels $\lambda_1'\lambda_2'\lambda_1\lambda_2$ for parity-conserving transitions[11] and the subscript n of ϕ_n^{ij} and defines the kinematic factors α_n which are divided out in definitions of ϕ_n^{ij}. Helicity amplitudes ϕ_n^{ij} are free of kinematic singularities at $\theta=0$ or π.

When Eq. (2.12) for $\hat{F}$ is used in Eq. (2.1), helicity matrix elements of the covariants k_m must be evaluated. The 8×9 matrices so defined for a fixed i and j are

$$A_{nm}^{ij}\equiv\bar{u}_{\lambda_1'}^{(\rho_1')}(\mathbf{p}')\bar{u}_{-\lambda_2'}^{(\rho_2')}(-\mathbf{p}')k_m u_{\lambda_1}^{(\rho_1)}(\mathbf{p})u_{-\lambda_2}^{(\rho_2)}(-\mathbf{p})/\alpha_n\ , \tag{4.2}$$

where Tables I and II relate $ij\leftrightarrow\rho_1'\rho_2'\rho_1\rho_2$ and $n\leftrightarrow\lambda_1'\lambda_2'\lambda_1\lambda_2$ and define kinematic factors to be divided out. The matrices A_{nm}^{ij} are defined in Appendix A.

From Eq. (2.1) we see that the c.m. frame helicity amplitudes can be expressed in terms of invariant amplitudes

TABLE II. Parity conserving helicity transitions.

n	1	2	3	4
$\lambda_1'\lambda_2',\lambda_1\lambda_2$	$\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}$	$\frac{1}{2}\frac{1}{2},-\frac{1}{2}-\frac{1}{2}$	$\frac{1}{2}-\frac{1}{2},\frac{1}{2}-\frac{1}{2}$	$\frac{1}{2}-\frac{1}{2},-\frac{1}{2}\frac{1}{2}$
α_n	1	1	$\cos^2\frac{\theta}{2}$	$-\sin^2\frac{\theta}{2}$
T_n	1	2	3	4
n	5	6	7	8
$\lambda_1'\lambda_2',\lambda_1\lambda_2$	$\frac{1}{2}\frac{1}{2},\frac{1}{2}-\frac{1}{2}$	$\frac{1}{2}\frac{1}{2},-\frac{1}{2}\frac{1}{2}$	$\frac{1}{2}-\frac{1}{2},\frac{1}{2}\frac{1}{2}$	$-\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}$
α_n	$-\frac{1}{2}\sin\theta$	$\frac{1}{2}\sin\theta$	$\frac{1}{2}\sin\theta$	$-\frac{1}{2}\sin\theta$
T_n	8	7	6	5

as follows (summation over $m=1$ to 9 is implied):

$$\phi_n^{11}=A_{nm}^{11}f_m^{11} , \tag{4.3}$$

$$\phi_n^{21}=A_{nm}^{21}(f_m^{21}+f_m^{11}) , \tag{4.4}$$

$$\phi_n^{31}=A_{nm}^{31}(f_m^{31}+f_m^{11}) , \tag{4.5}$$

$$\phi_n^{12}=A_{nm}^{12}(f_m^{12}+f_m^{11}) , \tag{4.6}$$

$$\phi_n^{13}=A_{nm}^{13}(f_m^{13}+f_m^{11}) , \tag{4.7}$$

$$\phi_n^{41}=A_{nm}^{41}(f_m^{41}+f_m^{31}+f_m^{21}+f_m^{11}) , \tag{4.8}$$

$$\phi_n^{14}=A_{nm}^{14}(f_m^{14}+f_m^{13}+f_m^{12}+f_m^{11}) , \tag{4.9}$$

$$\phi_n^{22}=A_{nm}^{22}(f_m^{22}+f_m^{21}+f_m^{12}+f_m^{11}) , \tag{4.10}$$

$$\phi_n^{33}=A_{nm}^{33}(f_m^{33}+f_m^{31}+f_m^{13}+f_m^{11}) , \tag{4.11}$$

$$\phi_n^{23}=A_{nm}^{23}(f_m^{23}+f_m^{21}+f_m^{13}+f_m^{11}) , \tag{4.12}$$

$$\phi_n^{32}=A_{nm}^{32}(f_m^{32}+f_m^{31}+f_m^{12}+f_m^{11}) , \tag{4.13}$$

$$\phi_n^{34}=A_{nm}^{34}(f_m^{34}+f_m^{14}+f_m^{32}+f_m^{33}+f_m^{31}+f_m^{12}+f_m^{13}+f_m^{11}) , \tag{4.14}$$

$$\phi_n^{42}=A_{nm}^{42}(f_m^{42}+f_m^{32}+f_m^{22}+f_m^{41}+f_m^{31}+f_m^{21}+f_m^{12}+f_m^{11}) , \tag{4.15}$$

$$\phi_n^{24}=A_{nm}^{24}(f_m^{24}+f_m^{14}+f_m^{23}+f_m^{22}+f_m^{21}+f_m^{13}+f_m^{12}+f_m^{11}) , \tag{4.16}$$

$$\phi_n^{43}=A_{nm}^{43}(f_m^{43}+f_m^{23}+f_m^{33}+f_m^{41}+f_m^{31}+f_m^{21}+f_m^{13}+f_m^{11}) , \tag{4.17}$$

$$\phi_n^{44}=A_{nm}^{44}\sum_{i=1}^{4}\sum_{j=1}^{4}f_m^{ij} . \tag{4.18}$$

Omitting the constrained amplitude $f_7^{11}=0$ (see Table I), (4.3) defines a system of eight linear equations which may be solved for the eight unconstrained f_m^{11} amplitudes. Once the f_m^{11} are fixed, Eqs. (4.4) to (4.7) together with appropriate constraints from Table I can be solved to determine f_m^{21}, f_m^{31}, f_m^{12}, and f_m^{13} amplitudes. Given these results, Eqs. (4.8) to (4.13) and Table I determine f_m^{41}, f_m^{14}, f_m^{22}, f_m^{33}, f_m^{23}, and f_m^{32} amplitudes, and so on until all amplitudes are determined. Finally when the f_m^{ij} are available, the F_m^{ij} appropriate to Eq. (2.9) can be determined using (2.14) and (2.15).

For each i and j, an 8×8 matrix is inverted to determine eight unconstrained f_n^{ij} from eight parity conserving helicity amplitudes ϕ_n^{ij} and previously determined f_n^{ij}. We have verified that the A matrices are nonsingular, and that for on-shell physical states the f_m^{11} determined by matrix inversion agree with the analytic formulas given in Ref. 7. The matrix inversion method is also applicable off shell and therefore is the more general method. When one-boson exchange helicity amplitudes are used as input, the correct invariant amplitudes are obtained as output.

Symmetries discussed in Sec. III are present in the input helicity amplitudes and therefore they must be automatically obtained from solving the equations given above. We have verified that the expected symmetries for f_n^{ij} are obtained and this provides a very useful check of the analysis. Time-reversal symmetry for helicity amplitudes ϕ_n^{ij} can be analyzed in a manner similar to the discussion above for f_n^{ij}. Due to Hermitian conjugation, class and subclass superscripts are interchanged under time reversal and also initial and final helicities are interchanged. Since parity invariance guarantees the helicity amplitudes are unchanged by sign reversal of all helicities, the net outcome of time-reversal invariance can be expressed as

$$\phi_n^{ij}=\phi_{Tn}^{ji} ,$$

where n and Tn subscripts are as shown in Table II. For example, in $i=j$ cases, $\phi_5=\phi_7$ and $\phi_6=\phi_8$.

V. HELICITY AMPLITUDE ANALYSIS

Relativistic calculations for NN scattering have been performed recently in Refs. 8 and 9. A partial wave expansion for helicity amplitudes is employed and coupled integral equations of the Bethe-Salpeter—type are solved numerically to obtain t-matrix elements. By inclusion of inelastic couplings $NN\leftrightarrow N\Delta$ and $NN\leftrightarrow\Delta\Delta$, the one-boson exchange model employed in these analyses has been found to yield reasonable phase shifts and inelasticity parameters up to 1 GeV. The model assumes pseudovector πN coupling.

In the above calculations, negative energy spinor states were neglected. These states are not very influential for the physical amplitudes when pseudovector πN coupling is used.[18] In the present work, the model of Ref. 8 has been extended to also include negative energy intermediate states in the NN channel. A quasipotential approach is used which gives results for the phase shifts comparable to those of the Bethe-Salpeter equations. By summation of the calculated partial wave t-matrix elements, we are able to determine a complete set of helicity amplitudes for positive and negative energy spinor states with all amplitudes derived from one consistent set of integral equations and meson-baryon couplings. Thus the NN dynamics provides predictions for the helicity amplitudes involving negative energy states within a theoretical model which successfully describes NN phase shifts and inelasticities. Using the methods of Sec. IV, the results can be directly transformed into invariant amplitudes.

We now review the helicity amplitude formalism to the extent necessary to describe our analysis. In this section, spinors defined by Kubis[11] are employed. They differ from the conventions of Bjorken and Drell[12] in the choice of the negative energy spinors. Helicity amplitudes ϕ_n^{ij} defined by Eq. (4.1) possess partial wave expansions (omitting ij superscripts),

$$\phi_n=\sum_J(2J+1)\phi_n^J d_{\lambda\lambda'}^J(\theta)/\alpha_n , \tag{5.1}$$

where $\lambda=\lambda_1-\lambda_2$, $\lambda'=\lambda_1'-\lambda_2'$, $d_{\lambda\lambda'}$ are the usual rotation matrices and α_n is the divisor which eliminates kinematical singularities of the rotation matrices. The helicity amplitudes are calculated using conventions of Kubis for negative energy spinors. Once they are obtained, a simple transformation, as given in Appendix B, is used to obtain the helicity amplitudes using the conventions of Bjorken and Drell for negative energy spinors. It is the latter set which can be used to determine invariant amplitudes using the methods of Sec. IV.

To exhibit the symmetries of the generalized Pauli principle for off-shell states,[11] it is convenient to use symmetrized states in ρ spin made up of products of single-particle states as follows:

$$\begin{aligned}\psi^{+}_{\lambda_1\lambda_2}&=u_1^{(+)}u_2^{(+)}\ ,\\ \psi^{-}_{\lambda_1\lambda_2}&=u_1^{(-)}u_2^{(-)}\ ,\\ \psi^{e}_{\lambda_1\lambda_2}&=2^{-1/2}(u_1^{(+)}u_2^{(-)}+u_1^{(-)}u_2^{(+)})\ ,\\ \psi^{o}_{\lambda_1\lambda_2}&=2^{-1/2}(u_1^{(+)}u_2^{(-)}-u_1^{(-)}u_2^{(+)})\ .\end{aligned} \tag{5.2}$$

Here $u_1^{(\rho_1)}\equiv u_{\lambda_1}^{(\rho_2)}(\mathbf{p})$ and $u_2^{(\rho_2)}\equiv u_{-\lambda_2}^{(\rho_2)}(-\mathbf{p})$ are single-particle states. Under interchange $\rho_1\leftrightarrow\rho_2$, the o state is odd while $+$, $-$, and e states are even. These states comprise a singlet and triplet with respect to total ρ spin.

Since negative energy Dirac states have odd intrinsic parity, ψ^e and ψ^o each have intrinsic parity opposite to that of ψ^+ and ψ^-. Consequently there are two kinds of matrix elements which conserve overall parity. The first group, denoted by T, involves ρ-spin transitions in which intrinsic and spatial parity are conserved, namely, $(\rho',\rho)=(+,+)$, $(-,-)$, $(+,-)$, $(-,+)$, (e,e), (e,o), (o,e), and (o,o). The second group, denoted by U, involves transitions in which intrinsic parity and spatial parity change, namely, $(\rho',\rho)=(+,e)$, $(+,o)$, $(-,e)$, $(-,o)$, $(e,+)$, $(o,+)$, $(e,-)$, and $(o,-)$.

Using the basis set of states (5.2), the Bethe-Salpeter equations can be partial wave decomposed to yield coupled-channel integral equations in two continuous variables. When the quasipotential approximation is used, these equations become one-variable integral equations. They are of the form

$$\widetilde{\phi}_i(p^o,p,JLS\rho'\rho)={}_i\langle p^o,p,JL'S'\rho'\,|\,\widetilde{V}\,|\,0\hat{p}JLS\rho\rangle_1+\frac{1}{\pi}\sum_{j=1}^{3}\sum_{\widetilde{L}\widetilde{S}\widetilde{\rho}}\int_0^\infty dq\;{}_i\langle p^o,p,JL'S'\rho'\,|\,\widetilde{V}\,|\,0q,J\widetilde{L}\widetilde{S}\widetilde{\rho}\rangle_j\,Q_j(q,\widetilde{\rho})\widetilde{\phi}_j(0,q,J\widetilde{L}\widetilde{S}\widetilde{\rho}\rho)\ , \tag{5.3}$$

where

$$\widetilde{\phi}_i(p^o,p,JL'S'\rho'\rho)\equiv(2\pi)^{-1}p\hat{p}\;{}_i\langle p^o,p,JL'S'\rho'\,|\,\phi\,|\,0,\hat{p},JLS\rho\rangle\ . \tag{5.4}$$

Here ρ, ρ', and $\widetilde{\rho}$ refer to $+$, $-$, e, or o in the basis set of (5.2). We follow the notation of Refs. 8 and 9 where i and j denote the channel: 1 is NN, 2 is NΔ, and 3 is $\Delta\Delta$. For the NΔ and $\Delta\Delta$ states we have only used $(\rho',\rho)=(+,+)$ and their couplings to the negative energy NN states are neglected. The variable p is the relative momentum in the c.m. frame and $\hat{p}$ is its on-shell value. The time component of relative four-momentum, p^o, is zero in the quasipotential approximation we have used and this eliminates all states of the full Bethe-Salpeter analysis which are odd in p^o. Also with this choice of p^o, the generalized Pauli principle is automatically satisfied by the integral equations.

For a fixed J value, Eq. (5.3) consists of three separate sets of integral equations. Each set can be characterized by the positive energy physical states occurring in the set, i.e., one set contains the physical singlet spin state, another contains the uncoupled triplet state, and a third contains the coupled triplet states. Appendix C presents a summary of the NN states which are coupled and the corresponding t-matrix elements which are obtained by solving (5.3).

The partial wave helicity amplitudes ϕ_n^J from Eq. (5.1) can now be determined from the calculated t-matrix elements based on Eq. (5.3). The analysis is based on helicity states of definite total angular momentum and spatial parity, $P=r(-)^{J-1}$ where $r=\pm1$. These are given by

$$|J,r,\lambda_1,\lambda_2\rangle=2^{-1/2}(\,|J;\lambda_1\lambda_2\rangle+r\,|J;-\lambda_1-\lambda_2\rangle)\ . \tag{5.5}$$

States with $r=-1$ must have orbital angular momentum $L=J$ and states with $r=+1$ must have $L=J\pm1$. Matrix elements in this basis are defined for two cases:

$$\langle J,r',\lambda_1'\lambda_2'\,|\,\phi\,|\,J,r,\lambda_1\lambda_2\rangle=\begin{cases}\langle\lambda_1'\lambda_2'\,|\,T_r^J\,|\,\lambda_1\lambda_2\rangle & \text{if } r'=r\ ,\\ \langle\lambda_1'\lambda_2'\,|\,U^J(r',r)\,|\,\lambda_1\lambda_2\rangle & \text{if } r'\neq r\ ,\end{cases} \tag{5.6}$$

where $r'=r$ when intrinsic parity is conserved and $r'\neq r$ when intrinsic parity changes. There are eight cases of interest in (5.6) when intrinsic parity is conserved as follows (for simplicity we write $|++\rangle$ for $\lambda_1=+\frac{1}{2}$, $\lambda_2=+\frac{1}{2}$, etc.):

$$\begin{aligned}T_1^J&=\langle++\,|\,T_-^J\,|\,++\rangle=\phi_1^J-\phi_2^J\ ,\\ T_3^J&=\langle+-\,|\,T_-^J\,|\,+-\rangle=\phi_3^J-\phi_4^J\ ,\\ T_6^J&=\langle++\,|\,T_-^J\,|\,+-\rangle=\phi_5^J-\phi_6^J\ ,\\ T_8^J&=\langle+-\,|\,T_-^J\,|\,++\rangle=\phi_7^J-\phi_8^J\ ,\\ T_2^J&=\langle++\,|\,T_+^J\,|\,++\rangle=\phi_1^J+\phi_2^J\ ,\\ T_4^J&=\langle+-\,|\,T_+^J\,|\,+-\rangle=\phi_3^J+\phi_4^J\ ,\\ T_5^J&=\langle++\,|\,T_+^J\,|\,+-\rangle=\phi_5^J+\phi_6^J\ ,\\ T_7^J&=\langle+-\,|\,T_+^J\,|\,++\rangle=\phi_7^J+\phi_8^J\ ,\end{aligned} \tag{5.7}$$

where ϕ_n^J are partial wave helicity amplitudes appearing in (5.1),

$$\phi_n^J=\langle J;\lambda_1'\lambda_2'\,|\,\phi\,|\,J;\lambda_1\lambda_2\rangle\ , \tag{5.8}$$

with index n denoting $(\lambda_1'\lambda_2',\lambda_1\lambda_2)$ values as in Table II. Similarly evaluating helicity matrix elements (5.6) for the case when intrinsic parity changes leads to

$$U_1^J=\langle ++\,|\,U^J(+,-)\,|\,++\rangle=\phi_1^J-\phi_2^J\,,$$
$$U_2^J=\langle ++\,|\,U^J(-,+)\,|\,++\rangle=\phi_1^J+\phi_2^J\,,$$
$$U_5^J=\langle ++\,|\,U^J(-,+)\,|\,+-\rangle=\phi_5^J+\phi_6^J\,,$$
$$U_7^J=\langle +-\,|\,U^J(+,-)\,|\,++\rangle=\phi_7^J+\phi_8^J\,,$$
$$U_3^J=\langle +-\,|\,U^J(+,-)\,|\,+-\rangle=\phi_3^J-\phi_4^J\,,$$
$$U_4^J=\langle +-\,|\,U^J(-,+)\,|\,+-\rangle=\phi_3^J+\phi_4^J\,,$$
$$U_6^J=\langle ++\,|\,U^J(+,-)\,|\,+-\rangle=\phi_5^J-\phi_6^J\,,$$
$$U_8^J=\langle +-\,|\,U^J(-,+)\,|\,++\rangle=\phi_7^J-\phi_8^J\,. \tag{5.9}$$

Evidently the ϕ_n^J can be found by inversion of (5.7) or (5.9) as follows:

$$\begin{aligned}
\phi_1^J&=\tfrac{1}{2}(T_1^J+T_2^J)\,,\\
\phi_2^J&=\tfrac{1}{2}(T_2^J-T_1^J)\,,\\
\phi_3^J&=\tfrac{1}{2}(T_3^J+T_4^J)\,,\\
\phi_4^J&=\tfrac{1}{2}(T_4^J-T_3^J)\,,\\
\phi_5^J&=\tfrac{1}{2}(T_5^J+T_6^J)\,,\\
\phi_6^J&=\tfrac{1}{2}(T_6^J-T_5^J)\,,\\
\phi_7^J&=\tfrac{1}{2}(T_7^J+T_8^J)\,,\\
\phi_8^J&=\tfrac{1}{2}(T_8^J-T_7^J)\,.
\end{aligned} \tag{5.10}$$

Equations (5.10) hold when intrinsic parity is conserved; substitution of U_n^J for T_n^J produces the corresponding results for ϕ_n^J when intrinsic parity changes.

Coefficients T_n^J possess a close relation to the t-matrix elements in the LSJ basis which are obtained by solving the quasipotential equation. We find

$$\begin{aligned}
T_1^J&=T_s\,,\\
T_2^J&=c_J^2T_{t,J-1}+c_{J+1}^2T_{t,J+1}\\
&\quad -c_Jc_{J+1}(T_{t,J-1,J+1}+T_{t,J+1,J-1})\,,\\
T_3^J&=T_{t,J}\,,\\
T_4^J&=c_{J+1}^2T_{t,J-1}+c_J^2T_{t,J+1}\\
&\quad +c_Jc_{J+1}(T_{t,J-1,J+1}+T_{t,J+1,J-1})\,,\\
T_5^J&=c_J^2T_{t,J-1,J+1}-c_{J+1}^2T_{t,J+1,J-1}\\
&\quad +c_Jc_{J+1}(T_{t,J-1}-T_{t,J+1})\,,\\
T_6^J&=-T_{st}\,,\\
T_7^J&=-c_{J+1}^2T_{t,J-1,J+1}+c_J^2T_{t,J+1,J-1}\\
&\quad +c_Jc_{J+1}(T_{t,J-1}-T_{t,J+1})\,,\\
T_8^J&=-T_{ts}\,.
\end{aligned} \tag{5.11}$$

Similarly for the cases where spatial parity changes,

$$\begin{aligned}
U_1^J&=c_JU_{ts,J-1}-c_{J+1}U_{ts,J+1}\,,\\
U_2^J&=c_JU_{st,J-1}-c_{J+1}U_{st,J+1}\,,\\
U_3^J&=-c_{J+1}U_{t,J-1,J}-c_JU_{t,J+1,J}\,,\\
U_4^J&=-c_{J+1}U_{t,J,J-1}-c_JU_{t,J,J+1}\,,\\
U_5^J&=c_{J+1}U_{st,J-1}+c_JU_{st,J+1}\,,\\
U_6^J&=-c_JU_{t,J-1,J}+c_{J+1}U_{t,J+1,J}\,,\\
U_7^J&=c_{J+1}U_{ts,J-1}-c_JU_{ts,J+1}\,,\\
U_8^J&=-c_JU_{t,J,J-1}-c_{J+1}U_{t,J,J+1}\,,
\end{aligned} \tag{5.12}$$

where $c_J=[J/(2J+1)]^{1/2}$, $c_{J+1}=[(J+1)/(2J+1)]^{1/2}$ are Clebsch-Gordan coefficients and subscripts refer to singlet (s), triplet (t), singlet-triplet transitions (st), and triplet-singlet transitions (ts).

In order to implement Eq. (5.1) for a given i and j, one must obtain t-matrix elements for the corresponding ρ values as defined in Table I. Appendix C shows how this is done and gives some examples. Given the quasipotential t-matrix elements T and U, Eqs. (5.10) to (5.12) are used to obtain ϕ_n^J partial wave amplitudes and Eq. (5.1) determines the helicity amplitudes in the basis of Kubis spinors. Appendix B shows how to obtain the helicity amplitudes in the basis of Bjorken and Drell spinors. Finally, the invariant amplitudes are determined using the methods of Sec. IV.

VI. CONCLUDING REMARKS

A general Lorentz-invariant representation of nucleon-nucleon scattering amplitudes is derived in this paper. All the necessary linkage is developed to determine the 128 invariant amplitudes of the representation from partial wave t-matrix elements based on solution of a quasipotential equation.[8] Due to charge symmetry and time-reversal invariance, the number of independent amplitudes reduces to 56 for a given isospin and this number further reduces to 44 when $p_1^2=p_2^2=p_1'^2=p_2'^2$. The formalism developed in this work provides the basis for the first calculations of a complete set of Lorentz-invariant NN amplitudes.

This formalism has been developed in order to answer some basic questions in relativistic nuclear physics. Use of a Dirac equation optical potential in nuclear scattering intrinsically involves introduction of virtual $N\bar{N}$ pair effects.[3] In order to predict these effects, one needs the fully off-shell NN scattering amplitude, in particular, those components which are not determined in positive energy scattering experiments. Meson theory provides the only existing model of relativistic NN scattering dynamics which (a) describes the observations over a wide energy region, and (b) provides theoretical predictions for the negative energy matrix elements. Therefore it has become a central question whether the Dirac successes are compatible with the underlying meson-baryon dynamics conventionally assumed as the theoretical basis for the nuclear force. References 7 and 10 utilize the methods and results of this paper to test the meson theoretical basis for nuclear scattering of protons.

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APPENDIX A: A MATRIX

The matrix A defined by (4.2) to relate Lorentz covariants to helicity amplitudes is developed in detail. Throughout this appendix, we suppress superscripts ij which specify the energy labels $\rho_1'\rho_2'\rho_1\rho_2$ according to Table I. A device introduced by Goldberger, Grisaru, MacDowell, and Wong[14] (GGMW) is used to simplify the task by eliminating covariants $V+\tilde{V}$ and $T+\tilde{T}$ as follows:

$$V+\tilde{V}=S+\tilde{S}-P-\tilde{P} \,, \tag{A1}$$

$$\tfrac{1}{2}(T+\tilde{T})=S+\tilde{S}+P+\tilde{P} \,. \tag{A2}$$

The factor $\frac{1}{2}$ in (A2) accounts for the difference between our tensor covariants and those of GGMW. Equation (2.12) for $\hat{F}^{ij}$ can be rewritten using these relations to obtain the form,

$$\hat{F}=g_1S+g_2\tilde{S}+g_3P+g_4\tilde{P}+g_5(A-\tilde{A})+g_6\mathscr{K}_6+g_7\mathscr{K}_7+g_8\mathscr{K}_8+g_9\mathscr{K}_9 \,, \tag{A3}$$

where

$$\begin{aligned} g_1&=f_1+f_2+f_4 \,,\\ g_2&=-f_1+f_2+f_4 \,,\\ g_3&=f_5+f_2-f_4 \,,\\ g_4&=-f_5+f_2-f_4 \,,\\ g_5&=-f_3 \,,\\ g_6&=f_7+f_6 \,,\\ g_7&=f_7-f_6 \,,\\ g_8&=f_9+f_8 \,,\\ g_9&=f_9-f_8 \,. \end{aligned} \tag{A4}$$

In the c.m. frame, $p_1=\frac{1}{2}P+p$, $p_1'=\frac{1}{2}P+p'$, $p_2=\frac{1}{2}P-p$, and $p_2'=\frac{1}{2}P-p'$, where P is the total momentum while p and p' are relative momenta:

$$P^\mu=(W,\mathbf{0}) \,,$$

$$p^\mu=(p^o,\mathbf{p}) \,,$$

and

$$p'^\mu=(p'^o,\mathbf{p}') \,.$$

Therefore the covariants $\mathscr{K}_6$ and $\mathscr{K}_7$ take the forms

$$\mathscr{K}_6=\gamma_2\cdot(P+p+p')/(2m) \,, \tag{A5}$$

$$\mathscr{K}_7=\gamma_1\cdot(P-p-p')/(2m) \,. \tag{A6}$$

These may be rewritten in terms of on-mass-shell momenta $p_\pm=(E_p,\pm\mathbf{p})$ and $p'_\pm=(E_{p'},\pm\mathbf{p})$ as follows:

$$\mathscr{K}_6=\gamma_2^0\rho_+-\gamma_2\cdot(p_-+p'_-)/(2m) \,, \tag{A7}$$

$$\mathscr{K}_7=\gamma_1^0\rho_--\gamma_1\cdot(p_++p'_+)/(2m) \,, \tag{A8}$$

where

$$\rho_\pm=\frac{W+E_p+E_{p'}\pm(p^o+p'^o)}{2m} \,. \tag{A9}$$

Matrix elements of the covariants as defined by (4.2) can be simplified by use of the Dirac equations for initial states,

$$\gamma_1\cdot p_+u_{\lambda_1}^{(\rho_1)}(\mathbf{p})=\rho_1 m u_{\lambda_1}^{(\rho_1)}(\mathbf{p}) \,, \tag{A10}$$

$$\gamma_2\cdot p_-u_{-\lambda_2}^{(\rho_2)}(-\mathbf{p})=\rho_2 m u_{-\lambda_2}^{(\rho_2)}(-\mathbf{p}) \,, \tag{A11}$$

where $\rho_1=\pm$ and $\rho_2=\pm$ tell whether positive or negative energy states are involved. Replace $p_\pm$ and $\mathbf{p}$ by $p'_\pm$ and $\mathbf{p}'$ in (A10) and (A11) to obtain the Dirac equations for final states. Consequently, for matrix elements involving definite $\rho_1'\rho_2'\rho_1\rho_2$, simplified forms of the covariants in the c.m. frame are

$$\mathscr{K}_6=\gamma_2^0\rho_+-\bar{\rho}_2 \,, \tag{A12}$$

$$\mathscr{K}_7=\gamma_1^0\rho_--\bar{\rho}_1 \,, \tag{A13}$$

where

$$\bar{\rho}_1=\tfrac{1}{2}(\rho_1+\rho_1') \,, \tag{A14}$$

$$\bar{\rho}_2=\tfrac{1}{2}(\rho_2+\rho_2') \,. \tag{A15}$$

Covariants $\mathscr{K}_8$ and $\mathscr{K}_9$ are reducible by the same procedure. We find

$$\mathscr{K}_8=P(\gamma_2^0\rho_++\delta\rho_2) \,, \tag{A16}$$

$$\mathscr{K}_9=P(\gamma_1^0\rho_-+\delta\rho_1) \,, \tag{A17}$$

where

$$\delta\rho_1=\tfrac{1}{2}(\rho_1'-\rho_1) \,, \tag{A18}$$

$$\delta\rho_2=\tfrac{1}{2}(\rho_2'-\rho_2) \,, \tag{A19}$$

and $P=\gamma_1^5\gamma_2^5$ is the usual pseudoscalar covariant. The minus sign in (A18), for example, occurs because of the anticommutation $\gamma_1^5\gamma_1\cdot p'_+=-\gamma_1\cdot p'_+\gamma_1^5$ which must precede use of the Dirac equation in the final state.

Covariants A and $\tilde{A}=\tilde{S}A$ may also be simplified by writing

$$A=A_0-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2V_0 \,, \tag{A20}$$

where

$$A_0=\gamma_1^5\gamma_1^0\gamma_2^5\gamma_2^0 \,, \tag{A21}$$

$$V_0=\gamma_1^0\gamma_2^0 \,. \tag{A22}$$

Consider now the helicity matrix elements as in Eq. (4.2). Initial state Dirac spinors are given by[12] ($\mathbf{p}$ is parallel to the 3 axis):

$$u_{\lambda}^{(+)}(\mathbf{p})=N_P\begin{bmatrix}1\\2\lambda\tilde{p}\end{bmatrix}\chi_\lambda ,$$
$$u_{\lambda}^{(-)}(\mathbf{p})=N_P\begin{bmatrix}2\lambda\tilde{p}\\1\end{bmatrix}\chi_\lambda ,$$
$$u_{-\lambda}^{(+)}(-\mathbf{p})=N_P\begin{bmatrix}1\\2\lambda\tilde{p}\end{bmatrix}\chi_{-\lambda} ,\quad (A23)$$
$$u_{-\lambda}^{(-)}(-\mathbf{p})=N_P\begin{bmatrix}2\lambda\tilde{p}\\1\end{bmatrix}\chi_{-\lambda} ,$$

where $\tilde{p}=p/(E_p+m)$ and $N_p=[(E_p+m)/(2m)]^{1/2}$. Final state Dirac spinors take the same form except for replacement of p by p' and replacement of the Pauli spinor χ_λ by

$$\chi_\lambda(\theta)=\exp(-i\sigma_2\theta/2)\chi_\lambda . \quad (A24)$$

Overlaps of Dirac spinors involve the following spin matrix elements:

$$m_{\lambda'\lambda}\equiv\chi_{\lambda'}^{\dagger}(\theta)\chi_\lambda=\delta_{\lambda'\lambda}\cos\frac{\theta}{2}+(\lambda'-\lambda)\sin\frac{\theta}{2} , \quad (A25)$$

$$\mathbf{m}_{\lambda'\lambda}\equiv\chi_{\lambda'}^{\dagger}(\theta)\sigma\chi_\lambda = \delta_{\lambda'\lambda}\left[(2\lambda\hat{\mathbf{e}}_1+i\hat{\mathbf{e}}_2)\sin\frac{\theta}{2}+2\lambda\hat{\mathbf{e}}_3\cos\frac{\theta}{2}\right] +\delta_{\lambda'-\lambda}\left[(\hat{\mathbf{e}}_1+2i\lambda\hat{\mathbf{e}}_2)\cos\frac{\theta}{2}-\hat{\mathbf{e}}_3\cos\frac{\theta}{2}\right] . \quad (A26)$$

Helicity amplitudes as defined by Eq. (4.1) involve four types of spin factors as follows:

$$u_n=m_{\lambda_1'\lambda_1}m_{-\lambda_2'-\lambda_2}/\alpha_n ,$$
$$\sigma_n=\mathbf{m}_{\lambda_1'\lambda_1}\cdot\mathbf{m}_{-\lambda_2'-\lambda_2}/\alpha_n ,$$

and the exchange counterparts,

$$\tilde{u}_n=m_{\lambda_1'-\lambda_2}m_{-\lambda_2'\lambda_1}/\alpha_n ,$$
$$\tilde{\sigma}_n=\mathbf{m}_{\lambda_1'-\lambda_2}\cdot\mathbf{m}_{-\lambda_2'\lambda_1}/\alpha_n .$$

Table III lists all relevant spin factors which arise in the helicity matrix elements for ϕ_n^{ij}. Kinematic factors α_n listed in the table have been divided out as in (4.1). Note that all spin factors for $n=3$ to 8 are unity after division by α_n. This is a reflection of the fact that spin matrix elements of the different covariants k_m used in this analysis are all equal to α_n in helicity amplitudes ϕ_n, $n=3$ to 8. Thus division by α_n removes zeros which would otherwise be present in these amplitudes at $\theta=0$ or π and the ϕ_n^{ij} are free of kinematic singularities.

The eight parity conserving helicity amplitudes ϕ_n^{ij}, $n=1$ to 8, are found to be related to nine invariant amplitudes g_m^{ij}, $m=1$ to 9, of Eq. (A3) by a matrix B. Suppressing superscripts ij this relation is

$$\phi_n=\sum_{m=1}^{9}B_{nm}g_m , \quad (A27)$$

where the nine columns of matrix B are given by

$$B_{n1}=u_nS(\rho_1',\rho_1,\lambda_1',\lambda_1)S(\rho_2',\rho_2,\lambda_2',\lambda_2) , \quad (A28)$$
$$B_{n2}=\tilde{u}_nS(\rho_1',\rho_2,\lambda_1',\lambda_2)S(\rho_2',\rho_1,\lambda_2',\lambda_1) , \quad (A29)$$
$$B_{n3}=u_nP(\rho_1',\rho_1,\lambda_1',\lambda_1)P(\rho_2',\rho_2,\lambda_2',\lambda_2) , \quad (A30)$$
$$B_{n4}=\tilde{u}_nP(\rho_1',\rho_2,\lambda_1',\lambda_2)P(\rho_2',\rho_1,\lambda_2',\lambda_1) , \quad (A31)$$
$$B_{n5}=u_nA(\rho_1',\rho_1,\lambda_1',\lambda_1)A(\rho_2',\rho_2,\lambda_2',\lambda_2) -\sigma_nV(\rho_1',\rho_1,\lambda_1',\lambda_1)V(\rho_2',\rho_2,\lambda_2',\lambda_2) -\tilde{u}_nA(\rho_1',\rho_2,\lambda_1',\lambda_2)A(\rho_2',\rho_1,\lambda_2',\lambda_1) +\tilde{\sigma}_nV(\rho_1',\rho_2,\lambda_1',\lambda_2)V(\rho_2',\rho_1,\lambda_2',\lambda_1) , \quad (A32)$$
$$B_{n6}=u_nS(\rho_1',\rho_1,\lambda_1',\lambda_1)V(\rho_2',\rho_2,\lambda_2',\lambda_2)\rho_+-B_{n1}\bar{\rho}_2 , \quad (A33)$$
$$B_{n7}=u_nV(\rho_1',\rho_1,\lambda_1',\lambda_1)S(\rho_2',\rho_2,\lambda_2',\lambda_2)\rho_--B_{n1}\bar{\rho}_1 , \quad (A34)$$
$$B_{n8}=u_nP(\rho_1',\rho_1,\lambda_1',\lambda_1)A(\rho_2',\rho_2,\lambda_2',\lambda_2)\rho_++B_{n3}\delta\rho_2 , \quad (A35)$$
$$B_{n9}=u_nA(\rho_1',\rho_1,\lambda_1',\lambda_1)P(\rho_2',\rho_2,\lambda_2',\lambda_2)\rho_-+B_{n1}\delta\rho_1 . \quad (A36)$$

Energy labels $\rho_1'\rho_2'\rho_1\rho_2$ appearing in these expressions specify the superscripts i and j of the ϕ_n^{ij} and g_m^{ij} as defined in Table I. Functions S, P, V, and A appearing here originate in Dirac matrix elements. They are defined by the following equations:

TABLE III. Kinematic factors associated with helicity matrix elements.

n	1	2	3	4	5	6	7	8
$\lambda_1'\lambda_2',\lambda_1\lambda_2$	$\frac12\frac12,\frac12\frac12$	$\frac12\frac12,-\frac12-\frac12$	$\frac12-\frac12,\frac12-\frac12$	$\frac12-\frac12,-\frac12\frac12$	$\frac12\frac12,\frac12-\frac12$	$\frac12\frac12,-\frac12\frac12$	$\frac12-\frac12,\frac12\frac12$	$-\frac12\frac12,\frac12\frac12$
α_n	1	1	$\cos^2(\frac12\theta)$	$\sin^2(\frac12\theta)$	$-\frac12\sin\theta$	$\frac12\sin\theta$	$\frac12\sin\theta$	$-\frac12\sin\theta$
u_n	$\cos^2(\frac12\theta)$	$-\sin^2(\frac12\theta)$	1	1	1	1	1	1
$\tilde{u}_n$	$-\sin^2(\frac12\theta)$	$\cos^2(\frac12\theta)$	1	1	1	1	1	1
σ_n	$-1-\sin^2(\frac12\theta)$	$1+\cos^2(\frac12\theta)$	1	1	1	1	1	1
$\tilde{\sigma}_n$	$1+\cos^2(\frac12\theta)$	$-1-\sin^2(\frac12\theta)$	1	1	1	1	1	1

TABLE IV. t matrix including physical singlet state.

	$(s,J,+)$	$(s,J,-)$	$(t,J-1,e)$	$(t,J+1,e)$
$(s,J,+)$	$[T_s^{++}]$	T_s^{+-}	$U_{st,J-1}^{+e}$	$U_{st,J+1}^{+e}$
$(s,J,-)$	T_s^{-+}	T_s^{--}	$U_{st,J-1}^{-e}$	$U_{st,J+1}^{-e}$
$(t,J-1,e)$	$U_{ts,J-1}^{e+}$	$U_{ts,J-1}^{e-}$	$T_{t,J-1}^{ee}$	$T_{t,J-1,J+1}^{ee}$
$(t,J+1,e)$	$U_{ts,J+1}^{e+}$	$U_{ts,J+1}^{e-}$	$T_{t,J+1,J-1}^{ee}$	$T_{t,J+1}^{ee}$

TABLE V. t matrix including physical uncoupled triplet state.

	$(t,J,+)$	$(t,J,-)$	$(t,J-1,o)$	$(t,J+1,o)$
$(t,J,+)$	$[T_t^{++}]$	T_t^{+-}	$U_{t,J,J-1}^{+o}$	$U_{t,J,J+1}^{+o}$
$(t,J,-)$	T_t^{-+}	T_t^{--}	$U_{t,J,J-1}^{-o}$	$U_{t,J,J+1}^{-o}$
$(t,J-1,o)$	$U_{t,J-1,J}^{o+}$	$U_{t,J-1,J}^{o-}$	$T_{t,J-1}^{oo}$	$T_{t,J-1,J+1}^{oo}$
$(t,J+1,o)$	$U_{t,J+1,J}^{o+}$	$U_{t,J+1,J}^{o-}$	$T_{t,J+1,J-1}^{oo}$	$T_{t,J+1}^{oo}$

$$\bar{u}_{\lambda'}^{(\rho')}(\mathbf{p}')u_{\lambda}^{(\rho)}(\mathbf{p})=S(\rho',\rho,\lambda',\lambda)m_{\lambda'\lambda}\ , \tag{A37}$$

$$\bar{u}_{\lambda'}^{(\rho')}(\mathbf{p}')\gamma^5 u_{\lambda}^{(\rho)}(\mathbf{p})=P(\rho',\rho,\lambda',\lambda)m_{\lambda'\lambda}\ , \tag{A38}$$

$$\bar{u}_{\lambda'}^{(\rho')}(\mathbf{p}')\gamma^0 u_{\lambda}^{(\rho)}(\mathbf{p})=V(\rho',\rho,\lambda',\lambda)m_{\lambda'\lambda}\ , \tag{A39}$$

$$\bar{u}_{\lambda'}^{(\rho')}(\mathbf{p}')\gamma^5\gamma^0 u_{\lambda}^{(\rho)}(\mathbf{p})=A(\rho',\rho,\lambda',\lambda)m_{\lambda'\lambda}\ , \tag{A40}$$

where $m_{\lambda'\lambda}$ is the Pauli spinor overlap given in (A25). Convenient forms used in our calculations are

$$S(\rho,\rho,\lambda',\lambda)=\begin{cases}\rho N_p N_{p'}(1-4\lambda'\lambda\tilde{p}\tilde{p}'), \\ \text{if } \rho\rho'=+1; \\ N_p N_{p'}(-2\lambda'\rho'\tilde{p}'-2\lambda\rho\tilde{p}), \\ \text{if } \rho\rho'=-1\end{cases} \tag{A41}$$

$$P(\rho,\rho,\lambda',\lambda)=\begin{cases}N_p N_{p'}(-2\lambda'\rho'\tilde{p}'+2\lambda\rho\tilde{p}), \\ \text{if } \rho\rho'=+1; \\ \rho' N_p N_{p'}(1-4\lambda'\lambda\tilde{p}\tilde{p}'), \\ \text{if } \rho\rho'=-1\end{cases} \tag{A42}$$

$$V(\rho,\rho,\lambda',\lambda)=\begin{cases}N_p N_{p'}(1+4\lambda'\lambda\tilde{p}\tilde{p}'), \\ \text{if } \rho\rho'=+1; \\ N_p N_{p'}(2\lambda'\tilde{p}'+2\lambda\tilde{p}), \\ \text{if } \rho\rho'=-1\end{cases} \tag{A43}$$

$$A(\rho,\rho,\lambda',\lambda)=\begin{cases}-N_p N_{p'}(2\lambda'\tilde{p}'+2\lambda\tilde{p}), \\ \text{if } \rho\rho'=+1; \\ -N_p N_{p'}(1+4\lambda'\lambda\tilde{p}\tilde{p}'), \\ \text{if } \rho\rho'=-1\ .\end{cases} \tag{A44}$$

Matrix B defined in (A27) provides helicity matrix elements of the kinematic covariants which appear in (A3). We are seeking a matrix A defined in (4.2) which provides helicity matrix elements of the kinematic covariants k_n defined in (2.13). The matrix A is found by equating two equivalent expressions for helicity amplitude ϕ_n;

$$\phi_n=\sum_{m=1}^{9} A_{nm}f_m=\sum_{m=1}^{9} B_{nm}g_m\ . \tag{A45}$$

Replacing the g_m by Eqs. (A4) and using the linear independence of the f_m, (A45) fixes A_{nm}. We find the following expressions for the nine columns of the A matrix:

$$A_{n1}=B_{n1}-B_{n2}\ , \tag{A46}$$

$$A_{n2}=B_{n1}+B_{n2}+B_{n3}+B_{n4}\ , \tag{A47}$$

$$A_{n3}=-B_{n5}\ , \tag{A48}$$

$$A_{n4}=B_{n1}+B_{n2}-B_{n3}-B_{n4}\ , \tag{A49}$$

$$A_{n5}=B_{n3}-B_{n4}\ , \tag{A50}$$

$$A_{n6}=B_{n6}-B_{n7}\ , \tag{A51}$$

$$A_{n7}=B_{n6}+B_{n7}\ , \tag{A52}$$

$$A_{n8}=B_{n8}-B_{n9}\ , \tag{A53}$$

$$A_{n9}=B_{n8}+B_{n9}\ . \tag{A54}$$

This completes the specification of matrix A used in Eq. (5.2) of this paper.

APPENDIX B: CONVERSION OF KUBIS TO BJORKEN AND DRELL SPINORS

The spinors used in Sec. V are based on the paper of Kubis[11] as follows:

$$u_{K\lambda}^{(+)}(p)=N_K\begin{bmatrix}1\\2\lambda\tilde{p}\end{bmatrix}\chi_\lambda\ ,$$
$$u_{K\lambda}^{(-)}(p)=N_K\begin{bmatrix}-2\lambda\tilde{p}\\1\end{bmatrix}\chi_\lambda\ , \tag{B1}$$

where $N_K=[(E_p+m)/(2E_p)]^{1/2}$, $\tilde{p}=p/(E_p+m)$, and χ_λ represents the Pauli two-component spinor.

In the rest of the paper we use spinors as defined by Bjorken and Drell[12] as follows:

TABLE VI. t matrix including physical coupled triplet states.

	$(t,J-1,+)$	$(t,J+1,+)$	$(t,J-1,-)$	$(t,J+1,-)$	(s,J,e)	(t,J,o)
$(t,J-1,+)$	$[T_{t,J-1}^{++}]$	$[T_{t,J-1,J+1}^{++}]$	$T_{t,J-1}^{+-}$	$T_{t,J-1,J+1}^{+-}$	$U_{ts,J-1}^{+e}$	$U_{t,J-1,J}^{+o}$
$(t,J+1,+)$	$[T_{t,J+1,J-1}^{++}]$	$[T_{t,J+1}^{++}]$	$T_{t,J+1,J-1}^{+-}$	$T_{t,J+1}^{+-}$	$U_{ts,J+1}^{+e}$	$U_{t,J+1,J}^{+o}$
$(t,J-1,-)$	$T_{t,J-1}^{-+}$	$T_{t,J-1,J+1}^{-+}$	$T_{t,J-1}^{--}$	$T_{t,J-1,J+1}^{--}$	$U_{ts,J-1}^{-e}$	$U_{t,J-1,J}^{-o}$
$(t,J+1,-)$	$T_{t,J+1,J-1}^{-+}$	$T_{t,J+1}^{-+}$	$T_{t,J+1,J-1}^{--}$	$T_{t,J+1}^{--}$	$U_{ts,J+1}^{-e}$	$U_{t,J+1,J}^{-o}$
(s,J,e)	$U_{st,J-1}^{e+}$	$U_{st,J+1}^{e+}$	$U_{st,J-1}^{e-}$	$U_{st,J+1}^{e-}$	T_s^{ee}	T_{st}^{eo}
(t,J,o)	$U_{t,J,J-1}^{o+}$	$U_{t,J,J+1}^{o+}$	$U_{t,J,J-1}^{o-}$	$U_{t,J,J+1}^{o-}$	T_{ts}^{oe}	T_t^{oo}

$$u_{B\lambda}^{(+)}(p)=N_B\begin{bmatrix}1\\2\lambda\tilde{p}\end{bmatrix}\chi_\lambda ,\quad u_{B\lambda}^{(-)}(p)=N_B\begin{bmatrix}2\lambda\tilde{p}\\1\end{bmatrix}\chi_\lambda \tag{B2}$$

where $N_B=[(E_p+m)/(2m)]^{1/2}$. It follows that

$$u_{B\lambda}^{(\rho)}=\sum_{\rho'} R^{\rho\rho'}(p,\lambda)u_{K\lambda}^{(\rho')}(p) , \tag{B3}$$

where

$$R^{++}=\left[\frac{E_p}{m}\right]^{1/2} , \tag{B4}$$

$$R^{+-}=0 , \tag{B5}$$

$$R^{-+}=\frac{2\lambda p}{E_p}\left[\frac{E_p}{m}\right]^{1/2} , \tag{B6}$$

$$R^{--}=\frac{m}{E_p}\left[\frac{E_p}{m}\right]^{1/2} . \tag{B7}$$

Notice that particle 2 conventions for helicity amplitudes assign spinors $u_{-\lambda}^{(\rho)}(-\mathbf{p})$. These take the same form as $u_\lambda^{(\rho)}(\mathbf{p})$ except for replacement of χ_λ by $\chi_{-\lambda}$ and thus the R matrix applies to particle 2 spinors without any sign changes. Consequently, helicity amplitudes ϕ defined as in (2.1) using the Bjorken-Drell basis can be calculated as follows from helicity amplitudes ϕ defined using the Kubis basis:

$$\phi_{\lambda_1'\lambda_2'\lambda_1\lambda_2}^{(\rho_1'\rho_2'\rho_1\rho_2)}(\text{Bjorken})=\sum_{\sigma_1\sigma_2\sigma_1'\sigma_2'} R^{\rho_1'\sigma_1'}(p',\lambda_1')R^{\rho_2'\sigma_2'}(p',\lambda_2')R^{\rho_1\sigma_1}(p,\lambda_1)R^{\rho_2\sigma_2}(p,\lambda_2)\phi_{\lambda_1'\lambda_2'\lambda_1\lambda_2}^{(\sigma_1'\sigma_2'\sigma_1\sigma_2)}(\text{Kubis}) . \tag{B8}$$

APPENDIX C: t-MATRIX ELEMENTS BASED ON QUASIPOTENTIAL EQUATION

For a fixed J value, three types of couplings in the NN sector arise in Eq. (5.3). For example, the physical singlet state is coupled to three other states with negative energy content as shown in Table IV. Singlet spin states are labeled (s,L,ρ) and similarly triplet spin states are labeled (t,L,ρ), where L is the orbital angular momentum ($L=J$ or $J\pm1$), and $\rho=+,\ -,\ e$, or 0 is the ρ-spin index. Physical t-matrix elements are enclosed in brackets. Table IV shows the 4×4 t matrix for each J obtained by solving the equations which include the physical singlet state. Symbol T is used for matrix elements in which spatial parity is conserved and symbol U is used for matrix elements in which spatial parity changes. In similar fashion, physical uncoupled triplet states ($L=J$) are coupled to three other states as shown in Table V and the corresponding solutions of (5.3) yield a second 4×4 matrix.

TABLE VII. (ρ',ρ) content of the helicity amplitudes ϕ^{ij}.

ij	$(\rho_1'\rho_2',\rho_1\rho_2)$	$(\rho'\rho)$
11	$(++,++)$	$2(++)$
22	$(--,++)$	$2(-+)$
33	$(++,--)$	$2(+-)$
44	$(--,--)$	$2(--)$
41	$(-+,-+)$	$(ee)-(eo)-(oe)+(oo)$
32	$(+-,-+)$	$(ee)-(eo)+(oe)-(oo)$
23	$(-+,+-)$	$(ee)+(eo)-(oe)-(oo)$
14	$(+-,+-)$	$(ee)+(eo)+(oe)+(oo)$
21	$(-+,++)$	$\sqrt{2}[(e+)-(o+)]$
31	$(++,-+)$	$\sqrt{2}[(+e)-(+o)]$
12	$(+-,++)$	$\sqrt{2}[(e+)+(o+)]$
13	$(++,+-)$	$\sqrt{2}[(+e)+(+o)]$
42	$(--,-+)$	$\sqrt{2}[(-e)-(-o)]$
43	$(-+,--)$	$\sqrt{2}[(e-)-(o-)]$
24	$(--,+-)$	$\sqrt{2}[(-e)+(-o)]$
34	$(+-,--)$	$\sqrt{2}[(e-)+(o-)]$

Physical coupled triplet states ($L=J\pm1$) are within a group of six coupled states which yield the 6×6 matrix shown in Table VI. These three matrices contain all t-matrix elements which are coupled to the positive energy states by the quasipotential equation. They are sufficient to define the helicity amplitudes of interest in this work.

However, it is necessary to link the ρ-spin states (ρ',ρ) which are used in the quasipotential t matrices to the $ij\leftrightarrow(\rho_1'\rho_2'\rho_1\rho_2)$ labels used in helicity states ϕ_n^{ij} as shown in Table VII. This table takes into account the definitions of ρ-spin states given in Eqs. (5.2) and an overall factor 2 which originates in Pauli antisymmetrization (only states consistent with the Pauli principle are to be used in the partial wave expansion). The first eight entries of Table VII involve ρ-spin transitions which conserve intrinsic parity. For these cases, amplitudes ϕ_n^{ij} are expanded in terms of T_n^J coefficients in (ρ',ρ) basis. Table VII shows how to use Tables IV—VI to construct the T_n^J using Eqs. (5.11). The last eight entries are for cases where intrinsic parity changes and then ϕ_n^{ij} is obtained from corresponding U_n^J coefficients. Table VII then shows how to use Tables IV—VI to construct the U_n^J using Eqs. (5.12). For example, consider that ϕ_n^{41}, $n=1$ to 8, are to be determined. The fifth line of Table VII shows that ee and oo superscripted elements of T from Tables IV—VI which are needed in Eqs. (5.11) enter with plus one multiplier while the eo and oe superscripted elements of T which are needed in Eqs. (5.11) enter with a minus one multiplier. Similarly for amplitudes ϕ_n^{21}, $e+$ superscripted elements

TABLE VIII. t matrix for group IV states.

	(s,J,o)	(t,J,e)
(s,J,o)	T_s^{oo}	T_{st}^{oe}
(t,J,e)	T_{ts}^{eo}	T_t^{ee}

of U from Tables IV–VI which are needed in Eqs. (5.12) enter with plus $\sqrt{2}$ multiplier, while $o+$ superscripted elements enter with minus $\sqrt{2}$ multiplier. Proceeding in this fashion, partial wave expansions for all helicity amplitudes ϕ_n^{ij} for any i and j can be obtained. Any t-matrix elements which occur in (5.11) or (5.12) that are not present in the quasipotential results are taken to be zero. This prescription omits coupled states (s,J,e) and (s,J,o) which are completely decoupled from the physical states as shown in Table VIII. Although one-meson exchange contributions exist for these states, they have been neglected in our analysis.

[1]J. A. McNeil, J. R. Shepard, and S. J. Wallace, Phys. Rev. Lett. **50**, 1439 (1983); J. R. Shepard, J. A. McNeil, and S. J. Wallace, *ibid.* **50**, 1443 (1983).
[2]B. C. Clark, S. Hama, R. L. Mercer, L. Ray, and B. D. Serot, Phys. Rev. Lett. **50**, 1644 (1983).
[3]M. V. Hynes, A. Picklesimer, P. C. Tandy, and R. M. Thaler, Phys. Rev. Lett. **52**, 978 (1984).
[4]B. C. Clark, S. Hama, R. L. Mercer, L. Ray, G. W. Hoffmann, and B. D. Serot, Phys. Rev. C **28**, 1421 (1983).
[5]L. Ray and G. W. Hoffmann, Phys. Rev. C **31**, 538 (1985).
[6]J. A. McNeil, L. Ray, and S. J. Wallace, Phys. Rev. C **27**, 2123 (1983).
[7]J. A. Tjon and S. J. Wallace, Phys. Rev. C **32**, 267 (1985).
[8]E. E. van Faassen and J. A. Tjon, Phys. Rev. C **30**, 285 (1984).
[9]E. E. van Faassen and J. A. Tjon, Phys. Rev. C **28**, 2354 (1983).
[10]J. A. Tjon and S. J. Wallace, Phys. Rev. Lett. **54**, 1357 (1985).
[11]J. J. Kubis, Phys. Rev. D **6**, 547 (1972).
[12]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), Vol. I.
[13]M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) **7**, 404 (1959).
[14]M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960).
[15]N. Hoshizaki, Prog. Theor. Phys. Suppl. **42**, 107 (1968).
[16]M. Scadron and H. F. Jones, Phys. Rev. **173**, 1734 (1968).
[17]M. L. Goldberger, Y. Nambu, and R. Oehme, Ann. Phys. (N.Y.) **2**, 226 (1957).
[18]M. Zuilhof and J. A. Tjon, Phys. Rev. C **24**, 736 (1981).