# Separable potentials for relativistic three-body calculations of the NNN, NN$\pi$, N$\pi\pi$, and $\pi\pi\pi$ systems

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We construct a set of separable potentials for the nucleon-nucleon, pion-nucleon, and pion-pion subsystem for all partial waves with angular momentum $L \leq 2$ within the framework of the relativistic two-body Kadyshevski equation. These interactions serve as input for the relativistic three-body equations proposed by the authors as well as for the three-body formalism derived by Vinogradov.

## I. INTRODUCTION

The relativistic generalization of the Faddeev equations has been considered by many authors[1-4] within the context of the three-body Bethe-Salpeter equation applying the methods developed by Blankenbecler and Sugar.[2] These methods result in integral equations with the same general structure as those of the nonrelativistic three-body problem but satisfying relativistic three-body unitarity and Lorentz invariance. Although these equations have been successful in the treatment of the various reactions of the pion-deuteron system,[5-8] it has been found recently[9] that they are completely inadequate for calculations in the bound-state region, since they contain spurious bound-state solutions and lead to a singular behavior as the invariant mass squared tends to zero. Moreover, in the case of two-body interactions with a very long range in momentum space, this pathological behavior can appear also in the scattering domain. Thus, it has become necessary to modify these equations in order to avoid such problems. The modified equations that we have proposed[9] are well behaved as a function of the invariant mass of the three-body system $\sqrt{s}$ throughout the bound-state region. The main difference between our approach and the standard one is that the two-body amplitudes are now constructed by solving the two-body integral equation proposed by Kadyshevski[10] rather than by the solution of the two-body Blankenbecler-Sugar equation.[2] Thus, our formalism has the same input as the one proposed by Vinogradov,[11] although it differs in other respects.

In this paper, we present a set of separable two-body interactions for the nucleon-nucleon, pion-nucleon, and pion-pion subsystems, which are solutions of the Kadyshevski equation. These interactions can serve as input for relativistic calculations of three-body systems composed of nucleons and pions whether they are performed within our formalism or within Vinogradov's approach. Among the possible applications of these interactions are the systems composed of (i) three nucleons, (ii) two nucleons and one pion, (iii) one nucleon and two pions, and (iv) three pions. As an example of our formalism and the use of these separable interactions, we have studied recently[12] the possible existence of bound states or resonances of a pion and two nucleons in the three-body channels with isospin 0 and 2. Another obvious application of this theory would be the calculation of $\pi$NN resonances with isospin 1 which correspond to the so-called dibaryon resonances as has been done, e.g., by Ueda and collaborators[13,14] within a nonrelativistic three-body theory. Similarly, these interactions can be used to study the scattering and bound-state problems of three nucleons. In addition, one has now a reliable framework in which to study the possible existence of three-pion resonances[15-17] as well as those of a nucleon and two pions.

In order to keep this paper complete we begin with a short summary of the most important steps of the derivation of our equations in Sec. II. By comparison of our result to other relativistic equations it will turn out that the difference between our as well as Vinogradov's approach and the other ones consists in the loss of the so-called clustering property. Since this is an important feature in the scattering domain, we discuss this clustering property in detail in Sec. III. We also give numerical results in order to show whether the loss of clustering is a serious drawback or not. Finally, in Sec. IV we give the analytic forms and parameters of the separable potentials for the nucleon-nucleon, pion-nucleon, and pion-pion systems for all partial waves $L \leq 2$ in the framework of the Kadyshevski equation.

## II. MODIFIED RELATIVISTIC FADDEEV EQUATIONS

The relativistic generalization of the Faddeev equations for the three-body problem has usually been carried out by summing up all possible sets of Feynman diagrams in which two particles interact while the third particle acts

 

as spectator.[1-4] This procedure leads to integral equations of the same form as in the nonrelativistic three-body problem but they are now dependent on four-dimensional variables. In order to eliminate some of these variables without destroying relativistic invariance as well as two- and three-body unitarity, a Blankenbecler-Sugar reduction is performed.[2] If the two-body amplitudes are separable, the relativistic Faddeev equations reduce after a partial wave decomposition to a set of integral equations in one variable. For example, the relativistic Faddeev equations for the bound-state problem corresponding to a separable interaction

$$V_i(p_i,p_i')=g_i(p_i)\lambda_i g_i(p_i') \tag{1}$$

can be written for three spinless particles as

$$T_i(\mathbf{q}_i;S)=\frac{1}{D_i(s_i)}\sum_{j\neq i}\int d\mathbf{q}_i\,\frac{g_i(p_i)g_j(p_j)}{2\omega_j(q_j)2\omega_k(|\mathbf{q}_i+\mathbf{q}_j|)}\,\frac{2[\omega_i(q_i)+\omega_j(q_j)+\omega_k(|\mathbf{q}_i+\mathbf{q}_j|)]}{S-[\omega_i(q_i)+\omega_j(q_j)+\omega_k(|\mathbf{q}_i+\mathbf{q}_j|)]^2+i\epsilon}\,T_j(\mathbf{q}_j;S)\,, \tag{2}$$

where $\mathbf{q}_i$ is the momentum of particle $i$ in the three-body c.m. frame and

$$\omega_j(p)=(m_j^2+p^2)^{1/2}\,.$$

The denominator $D_i(s_i)$ is given by

$$D_i(s_i)=\frac{1}{\lambda_i}-\int_0^\infty p^2dp\,\frac{\omega_j(p)+\omega_k(p)}{2\omega_j(p)\omega_k(p)}\times\frac{g_i^2(p)}{s_i-[\omega_j(p)+\omega_k(p)]^2+i\epsilon} \tag{3}$$

and the argument $s_i$ of $D_i$ is the two-body invariant mass squared and can be written as

$$s_i=[\sqrt{S}-(m_i^2+q_i^2)^{1/2}]^2-q_i^2\,. \tag{4}$$

The origin of the problem with the integral equations (2) lies in the definition of this two-body invariant mass squared $s_i$ as given by Eq. (4). In Fig. 1 we have plotted $s_i$ as a function of $S$ for two values of the momentum $q_i$. The calculation was done for the particular case of the $N\pi\pi$ system in which $m_i=M$ is the mass of the nucleon and $m_j=m_k=\mu$ is the mass of the pion. As we see from Eq. (4) and Fig. 1, $s_i$ has a minimum as a function of $S$ when $S=m_i^2+q_i^2$. Thus, for values of $S>m_i^2+q_i^2$ the two-body invariant mass squared $s_i$ decreases when $S$ decreases, which is the normal behavior, since the momentum of the spectator particle $q_i$ is kept constant; however, for values of $S<m_i^2+q_i^2$ the two-body invariant mass squared $s_i$ has an abnormal behavior as a function of $S$ since it increases, when $S$ decreases. As we have shown in Ref. 9, this gives rise to the appearance of spurious bound states and to the development of a singularity as $S$ tends to zero.

Moreover, as we see in Fig. 1, the minimum of $s_i$ moves to the right, as $q_i$ increases, so that, for the curve labeled $q_i=M$, the minimum is already above the three-body threshold $M+2\mu$. Therefore, the abnormal behavior of $s_i$ in this case does not only effect the bound-state region but also the scattering domain. Consequently, for two-body interactions having a very long range in momentum space [so that the effective values of $q_i$ in Eq. (4) are large] the results of the relativistic Faddeev equations will be unreliable also in the scattering region.

In order to eliminate the spurious bound-state solutions and the abnormal behavior in the scattering region for the case of large $q_i$, we first write the denominator inside the integral in Eq. (3) using Eq. (4) as

$$\frac{1}{s_i-[\omega_j(p_i)+\omega_k(p_i)]^2+i\epsilon}=\frac{1}{2\{q_i^2+[\omega_j(p_i)+\omega_k(p_i)]^2\}^{1/2}}\times\left[\frac{1}{\sqrt{S}-(m_i^2+q_i^2)^{1/2}-\{q_i^2+[\omega_j(p_i)+\omega_k(p_i)]^2\}^{1/2}+i\epsilon}-\frac{1}{\sqrt{S}-(m_i^2+q_i^2)^{1/2}+\{q_i^2+[\omega_j(p_i)+\omega_k(p_i)]^2\}^{1/2}}\right]. \tag{5}$$

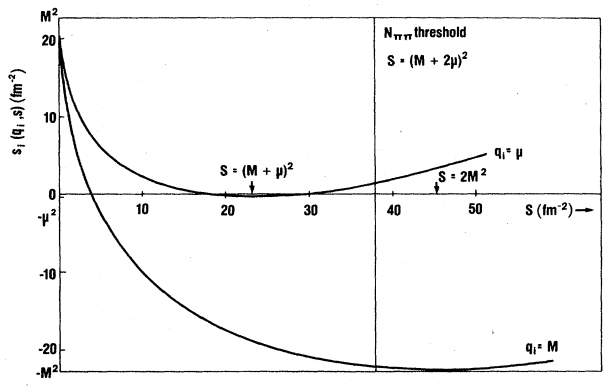


FIG. 1. Two-body invariant mass squared $s_i(q_i,S)$ as a function of the total invariant energy squared $S$ of the $N\pi\pi$ system for two values of the momentum $q_i$.

The conditions that Aaron, Amado, and Young have imposed to construct their theory are to satisfy relativistic invariance and two- and three-body unitarity. With regard to this last point they show that the two- and three-body propagators must obey certain discontinuity relations. In particular, this implies that the function $1/D_i(s_i)$ must have a cut across the positive real axis starting at $s_i=(m_j+m_k)^2$. The conditions of Aaron, Amado, and Young are fulfilled by Eq. (3) with the two-body propagator (5), but they are also fulfilled if one keeps only the first term in the parenthesis. Thus, one is allowed to drop the second term still satisfying the unitarity requirements. That means that from this point of view both theories are equivalent; nevertheless, both models have merits and failures in other respects. The Aaron, Amado, and Young formalism has imposed clus-

tering but gives rise to spurious bound states. These spurious bound states disappear in our formalism but this is necessarily connected with the loss of clustering.

Dropping the second term in Eq. (5), the propagator of the interacting pair, which was given by Eq. (3), now becomes

$$D(s_i)=\frac{1}{\lambda_i}-\int_0^\infty p^2dp\frac{\omega_j(p)+\omega_k(p)}{4\omega_j(p)\omega_k(p)\{q_i^2+[\omega_j(p)+\omega_k(p)]^2\}^{1/2}}\frac{g_i^2(p)}{W_i-\{q_i^2+[\omega_j(p)+\omega_k(p)]^2\}^{1/2}+i\epsilon}\ , \tag{6}$$

where

$$W_i=(s_i+q_i^2)^{1/2}=\sqrt{S}-(m_i^2+q_i^2)^{1/2}\ . \tag{7}$$

It is easy to see that the energy of the interacting pair $W_i$, given by Eq. (7), does not have any abnormal behavior, since it is just a straight line as a function of $\sqrt{S}$. If $q_i=0$, particle $i$ is at rest and the c.m. frame of the interacting pair coincides with the three-body c.m. frame and Eq. (6) becomes

$$D_i(s_i)=\frac{1}{\lambda_i}-\int_0^\infty p^2dp\frac{1}{4\omega_j(p)\omega_k(p)}\frac{g_i^2(p)}{\sqrt{s_i}-\omega_j(p)-\omega_k(p)+i\epsilon}\ , \tag{8}$$

which corresponds to the solution of the Kadyshevski equation[10]

$$t_i(p,p';\sqrt{s_i})=V_i(p,p')+\int_0^\infty\frac{p''^2dp''}{4\omega_j(p'')\omega_k(p'')}V_i(p,p'')\frac{1}{\sqrt{s_i}-\omega_j(p'')-\omega_k(p'')+i\epsilon}t_i(p'',p';\sqrt{s_i}) \tag{9}$$

for the separable interaction of Eq. (1).

The modified relativistic Faddeev equations, given by Eq. (2) with the propagator of the interacting pair $D_i(s_i)$ replaced by Eq. (6), are well behaved as a function of $\sqrt{S}$ throughout the bound state region

$$-\infty\leq\sqrt{S}\leq m_i+m_j+m_k\ .$$

They do not develop any singularity at $\sqrt{S}=0$ nor do they contain any spurious bound-state solutions.

A different formulation of the relativistic three-body problem independent of the Bethe-Salpeter equation has been proposed by Vinogradov.[11] This approach is an extension of the formalism developed by Kadyshevski and co-workers for the relativistic two-body problem to the case of three particles. In this theory, all the particles are kept on their mass shells at every stage, so that no need arises for a Blankenbecler-Sugar reduction. The resulting three-body equations have also as input the propagators of the interacting pair $D_i(s_i)$ calculated from the Kadyshevski equations [Eqs. (6)—(8)]. However, the Vinogradov-Kadyshevski theory differs from our approach in that the three-body propagator

$$2(\omega_i+\omega_j+\omega_k)[S-(\omega_i+\omega_j+\omega_k)^2+i\epsilon]^{-1}$$

of Eq. (2) is replaced by the propagator $(\sqrt{S}-\omega_i-\omega_j-\omega_k+i\epsilon)^{-1}$.

Inherent in both approaches, however, is the feature that the propagator of the interacting pair, given by Eq. (6), does not satisfy the clustering property [as the propagator of the original Eqs. (3) and (4) does]. Since the clustering property is very important for scattering processes, we will concentrate on it in the next section.

## III. THE CLUSTERING PROPERTY

The loss of the clustering property may be very dangerous for the scattering problem, because it means that the poles of the two-body amplitudes corresponding to bound states are not at the correct position with respect to the logarithmic singularities produced by the breakup channel. The breakup singularities are determined by the equations

$$\sqrt{S}=(m_i^2+q_i^2)^{1/2}+(m_j^2+q_j^2)^{1/2}+[m_k^2+(q_i\pm q_j)^2]^{1/2}\ . \tag{10}$$

Solving this equation for $q_i$ and taking the derivative gives the maximum value of $q_i$ for which the singularities occur (the breakup threshold):

$$q_{i\max}^2=\frac{1}{4S}[S-(m_i+m_{jk})^2][S-(m_i-m_{jk})^2]\ , \tag{11}$$

where

$$m_{jk}=m_j+m_k\ . \tag{12}$$

Equations (11) and (12) mean that the maximum momentum, for which the breakup singularities occur, corresponds to the pair $jk$ having an invariant mass $m_{jk}=m_j+m_k$.

The pole due to a bound-state of the pair $jk$, on the other hand, is determined by the equation

$$\sqrt{S}=(m_i^2+q_i^2)^{1/2}+(\omega_{jk}+q_i^2)^{1/2}\ , \tag{13}$$

where

$$\omega_{jk}=m_j+m_k-B \tag{14}$$

is the invariant mass of the bound state with binding energy $B$. Solution of Eq. (13) for $q_i^2$ yields

$$q_i^2=\frac{1}{4S}[S-(m_i+\omega_{jk})^2][S-(m_i-\omega_{jk})^2]\ . \tag{15}$$

If we compare Eqs. (11) and (15) we see that the pole coming from the two-body bound state is located at a higher momentum than the logarithmic singularities of the breakup channel. We also see that Eqs. (11) and (15)

differ only in that the invariant mass of the pair is in one case $m_{jk}$ and in the other case $\omega_{jk}$ [Eqs. (12) and (14)] and they are separated precisely by the binding energy of the pair $B$. Thus, if the bound state of the pair $jk$ has a very small binding energy, as, e.g., for the deuteron, the break-up singularities lie very close to the bound-state pole. Therefore, it may lead to very large errors if the bound-state pole is not at its correct position due to the loss of the clustering property.

We have calculated the effective binding energy of the deuteron in our theory by first searching the bound-state pole of the Kadyshevski equation for $q_i \neq 0$

$$t_i(p',p'';\sqrt{S},q_i)=V_i(p',p'')+\int_0^\infty p^2dp\frac{\omega_j(p)+\omega_k(p)}{4\omega_j(p)\omega_k(p)\{q_i^2+[\omega_j(p)+\omega_k(p)]^2\}^{1/2}}V(p',p)$$
$$\times\frac{1}{\sqrt{S}-(m_i^2+q_i^2)^{1/2}-\{q_i^2+[\omega_j(p)+\omega_k(p)]^2\}^{1/2}}t_i(p,p'';\sqrt{S},q_i) \quad (16)$$

and then using Eqs. (13)–(15) to find the effective value of the binding energy $B$ as a function of the kinetic energy of the projectile (which is determined by $\sqrt{S}$) for both pion-deuteron and nucleon-deuteron scattering. These results are shown in Fig. 2 for our model of the $^3S_1$-$^3D_1$ nucleon-nucleon interaction. We see that the changes in the effective binding energy are smaller than 0.1 MeV for kinetic energies of the projectile up to 500 MeV in both pion-deuteron and nucleon-deuteron scattering.

In order to estimate how these small changes in the two-body binding energy will affect the three-body results we calculated nonrelativistically the two-nucleon exchange term for nucleon-deuteron scattering at $T_N=500$ MeV. We assumed that the $^1S_0$ and $^3S_1$ nucleon-nucleon channels were both equal to a Yamaguchi potential with form factor $g(p)=(\alpha^2+p^2)^{-1}$, $\alpha=1.45$ fm$^{-1}$ and the strength adjusted such that the two-body binding energy is 2.225 MeV. We then repeated the calculation with the two-body binding energy changed to 2.325 MeV. We found that the magnitude of the nucleon-deuteron on-shell amplitude changed by 1.6% and the phase by 0.05 deg. Similar changes were observed in the half-shell nucleon-deuteron amplitude. Thus, we conclude that the errors generated by the loss of the clustering property in our theory or in the one of Vinogradov will be negligible for most applications in medium energy physics.

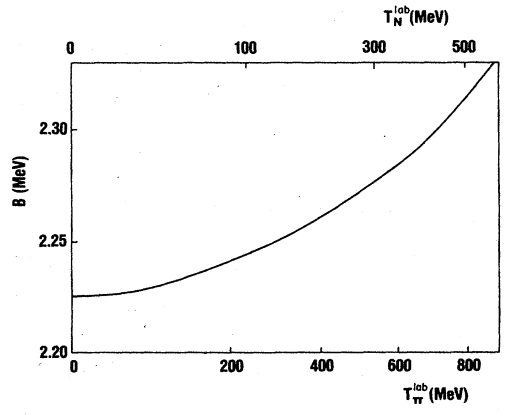


FIG. 2. The effective deuteron binding energy in the Kadyshevski equation as a function of the kinetic laboratory energy of the projectile for $\pi$-d and N-d scattering.

## IV. SEPARABLE POTENTIALS

In this section we present separable potentials fitted to experimental phases in the context of the Kadyshevski equation [Eq. (9)] for the nucleon-nucleon, pion-nucleon, and pion-pion systems for partial waves up to angular momentum $L \leq 2$. To facilitate the use of these potentials in three-body calculations we have kept the rank of the potentials as low as possible and have used the same structure of form factors for all partial waves. For the masses of the particles we have taken $M=938.9$ MeV for the nucleon and $\mu=139.576$ MeV for the pion.

### A. The nucleon-nucleon system

The introduction of relativity (in our case of the Kadyshevski type) to the nonrelativistic equations results in moderate changes of the phases due to the relative large mass of the two nucleons (for the $\pi$-N and especially the $\pi$-$\pi$ system the situation is different). Therefore the separable potentials, constructed in the nonrelativistic Lippmann-Schwinger equation,[18,19] were an excellent starting point for our parameter search. The parameters were adjusted to reproduce the phases of the phase shift analysis of Arndt *et al.*[20] up to a laboratory energy of $E_{lab} \leq 400$ MeV.

#### *1. $^1S_0$ and $^3S_1$ (uncoupled) partial waves*

We find it useful to give a parametrization of the $^3S_1$ wave in an uncoupled formalism in addition to the full $^3S_1$-$^3D_1$ coupled channel treatment because of the following reasons: (i) three-body calculations are facilitated to some extent in taking into account just the uncoupled $^3S_1$ wave; (ii) this approximation is reasonable at low scattering energies since the coupling to the $^3D_1$ state is weak. The most important features of this state at low energies, namely the binding energy and the scattering length, are provided also by the uncoupled parametrization.

To generate the sign change of both the $^1S_0$ and the $^3S_1$ phase shifts, we are forced to use separable potentials of rank 2

$$V(p,p')=g_1(p)\lambda_1 g_1(p')+g_2(p)\lambda_2 g_2(p') . \quad (17)$$

The form factors consist of the sum of two Yamaguchi-type terms

TABLE I. Potential parameters of the $^1S_0$ and $^3S_1$ (uncoupled) nucleon-nucleon partial waves ($[\lambda_i]=\text{fm}^0$, $[\alpha_{ij}]=\text{fm}^{-2}$, $[\beta_{ij}]=\text{fm}^{-1}$).

| | $^1S_0$ | $^3S_1$ |
|---|---|---|
| $\lambda_1$ | −1 | −1 |
| $\lambda_2$ | 1 | 1 |
| $\alpha_{11}$ | 5.240 525 | 7.767 139 |
| $\alpha_{12}$ | 2.003 268 | −311.961 6 |
| $\alpha_{21}$ | 170.819 9 | 5.225 645 |
| $\alpha_{22}$ | 152.786 3 | −17.074 47 |
| $\beta_{11}$ | 1.077 186 | 1.596 483 |
| $\beta_{12}$ | 1.211 067 | 5.968 069 |
| $\beta_{21}$ | 3.797 065 | 8.015 881 |
| $\beta_{22}$ | 2.262 703 | 1.793 468 |

$$g_i(p)=\frac{\alpha_{i1}(p^{i-1})^2}{(p^2+\beta_{i1}^2)^i}+\frac{\alpha_{i2}(p^i)^2}{(p^2+\beta_{i2}^2)^{i+1}},\quad i=1,2\ . \tag{18}$$

In this case and in most of the following examples (except some $\pi$N states) it has turned out to be more effective to add fractions with different powers of $p$ within one form factor rather than to take the sum of the same expressions. The parameters of the $^1S_0$ and $^3S_1$ potentials are given in Table I and the quality of the fit can be seen in Table II (low-energy parameters) and in Fig. 3 (phase shifts). The $^3S_1$ potential gives rise to a binding energy of 2.225 MeV [experimental value $E_b=2.224\,579(9)$ MeV (Ref. 23)].

### 2. The $^3S_1$-$^3D_1$ channel

This partial wave bears not only the difficulty of the coupling of the $^3S_1$ to the $^3D_1$ state but also of the connection to the only bound state of the two-nucleon system. Therefore in addition to the scattering data (phase shifts and effective range parameters) one has to also reproduce the features of the deuteron like binding energy, quadrupole moment, and so on; furthermore recent experiments on elastic electron-deuteron scattering have given some constraints on the deuteron $s$-state wave function.[24–26] In building up a separable potential, especially for this channel, one has to find a compromise between quality of the

TABLE II. Effective range parameters ($[a]=\text{fm}$, $[r]=\text{fm}$) for the $^1S_0$ and $^3S_1$ (uncoupled and coupled) potentials. The experimental values are taken from Ref. 21 for the $^1S_0$ case and from Ref. 22 for the $^3S_1$ partial wave.

| | $a$ | $a^{\text{exp}}$ | $r$ | $r^{\text{exp}}$ |
|---|---|---|---|---|
| $^1S_0$ | −23.70 | −23.721±0.017 | 2.65 | 2.658±0.062 |
| $^3S_1$ (uncoupled) | 5.41 | 5.423±0.004 | 1.76 | 1.76±0.005 |
| $^3S_1$ (coupled) | 5.41 | | 1.77 | |

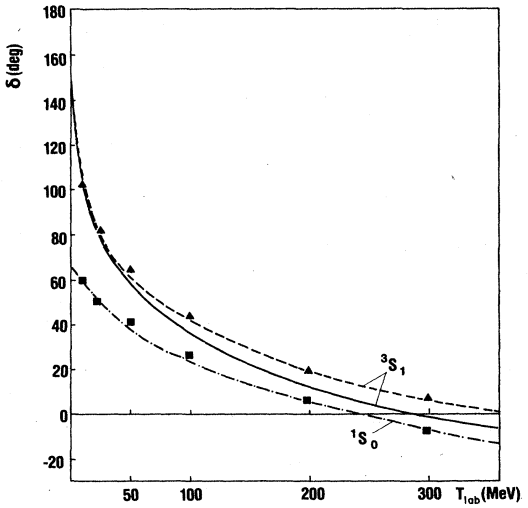


FIG. 3. Nucleon-nucleon phase shift results of the separable potentials for the $^1S_0$ state (—·—·—·) and for the $^3S_1$ state (—— coupled channel calculation, --- uncoupled). The experimental values are taken from the energy-dependent solution of Arndt *et al.* (Ref. 20) ($^3S_1$ state, $^1S_0$ state).

reproduction of the above-mentioned data and complexity of the potential, i.e., rank, structure of the form factors, and number of parameters. Motivated by these considerations we have taken over the form of the separable representation of Ref. 18, namely the rank 3 potential Graz II:

$$V_{LL'}(p,p')=[g_{01}(p),g_{02}(p),g_2(p)]\Delta_L \times\begin{bmatrix}1&\lambda_1&\lambda_2\\ \lambda_1&1&\lambda_3\\ \lambda_2&\lambda_3&1\end{bmatrix}\Delta_{L'}\begin{bmatrix}g_{01}(p')\\ g_{02}(p')\\ g_2(p')\end{bmatrix},\quad L,L'\in\{0,2\} \tag{19}$$

with

$$\Delta_0=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\qquad \Delta_2=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix},$$

TABLE III. Potential parameters of the $^3S_1$-$^3D_1$ nucleon-nucleon potential.

| | |
|---|---|
| $\lambda_1=-0.841\,108\ \text{fm}^0$ | $\gamma_0=1.255\,738\ \text{fm}^2$ |
| $\lambda_2=-2.005\,440\ \text{fm}^0$ | $\gamma_2=2.495\,738\ \text{fm}^2$ |
| $\lambda_3=1.391\,032\ \text{fm}^0$ | $\beta_{01}=1.114\,786\ \text{fm}^{-1}$ |
| $\alpha_{01}=8.651\,126\ \text{fm}^{-4}$ | $\beta_{02}=2.562\,780\ \text{fm}^{-1}$ |
| $\alpha_{02}=60.595\,93\ \text{fm}^{-2}$ | $\beta_{21}=3.983\,242\ \text{fm}^{-1}$ |
| $\alpha_2=27.051\,11\ \text{fm}^{-4}$ | $\beta_{22}=0.877\,584\ \text{fm}^{-1}$ |

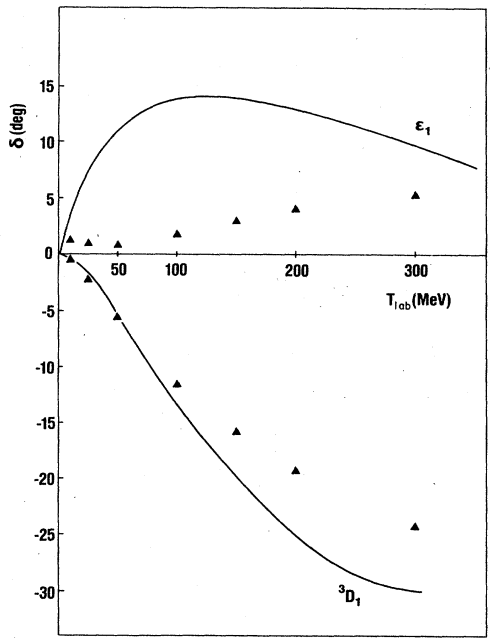


FIG. 4. Mixing parameter $\epsilon_1$ and $^3D_1$ phase shift compared with the values taken from the energy-dependent nucleon-nucleon phase-shift analysis of Arndt *et al.* (Ref. 20).

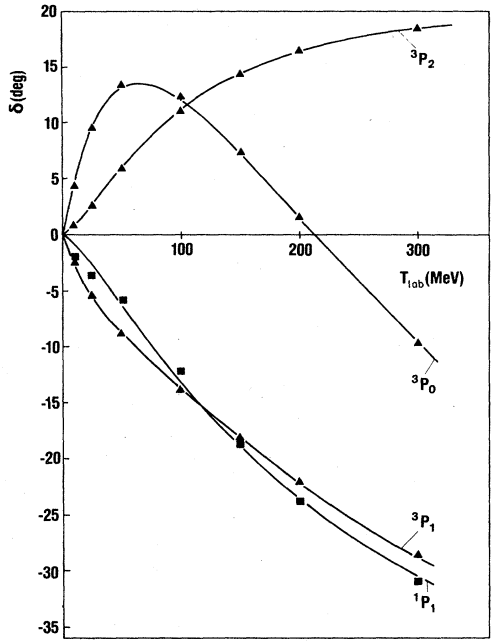


FIG. 5. Nucleon-nucleon $P$-state phase shifts compared with the results of the energy-dependent analysis of Arndt *et al.* (Ref. 20) (the result for the $^1P_1$ state is shown in squares).

and

$$g_{01}(p)=\frac{\alpha_{01}(1+\gamma_0 p^2)}{(p^2+\beta_{01}^2)},$$
$$g_{02}(p)=\frac{\alpha_{02}p^2}{(p^2+\beta_{02}^2)^2}, \qquad (20)$$
$$g_2(p)=\frac{\alpha_2^2 p^2(1+\gamma_2 p^2)}{(p^2+\beta_{21}^2)(p^2+\beta_{22}^2)^2}.$$

The parameters of this potential are given in Table III and the results are very similar to the nonrelativistic case, i.e., bear the same merits and shortcomings: The $^3S_1$ phase (see Fig. 3) and the effective range parameters (Table II) are in sufficient agreement with the experimental data. Figure 4 shows that the same holds for the $^3D_1$ phase shift. The mixing parameter $\epsilon_1$, however, is much too high even at low energies (although the experimental situation of $\epsilon_1$ is not satisfying[21]). The $s$-state wave function fulfills the constraints of elastic e-d scattering, i.e., has a node in momentum space at $p\approx 2.5$ fm$^{-1}$. Further deuteron quantities are in fair agreement with experimental data, namely the binding energy $E_b=2.224$ MeV [experiment, 2.224 579(9) MeV (Ref. 23)], quadrupole moment $Q_d=0.294$ fm$^2$ [experiment, 0.285 90(30) fm$^2$ (Ref. 27)], the $D$ state probability $p_D=5.84\%$ (experiment, 4—7 %), and $D/S$ asymptotic ratio $\eta_D=0.0252$ [experiment, 0.0271(4) (Ref. 28)].

### 3. *P and D waves*

We have found that rank-1 potentials [Eq. (1)] are sufficient for a good reproduction of the $^1P_1$, $^3P_1$, $^3P_2$, $^1D_2$, $^3D_2$, and $^3D_3$ phase shifts. The general structure of the form factors can be written as

$$g_L(p)=p^L\left[\frac{\alpha_1}{(p^2+\beta_1^2)^{L+1}}+\frac{\alpha_2 p^2}{(p^2+\beta_2^2)^{L+2}}\right]. \qquad (21)$$

The $^3P_2$ phase, which is part of the coupled $^3P_2$-$^3F_2$ channel, is fitted in the uncoupled formalism; the arguments which we have given for the uncoupled treatment of the $^3S_1$ state hold much more for this case since the coupling in the $^3P_2$-$^3F_2$ state is even weaker. The parameters for these potentials are given in Table IV and the phases in Figs. 5 and 6.

TABLE IV. Potential parameters of the nucleon-nucleon potentials in the $^1P_1$, $^3P_1$, $^3P_2$, $^1D_2$, $^3D_2$, and $^3D_3$ states. ($[\lambda]=\text{fm}^0$, $[\alpha_i]=\text{fm}^{-L-2}$, $[\beta_i]=\text{fm}^{-1}$.)

| | $^1P_1$ | $^3P_1$ | $^3P_2$ | $^1D_2$ | $^3D_2$ | $^3D_3$ |
|---|---|---|---|---|---|---|
| $\lambda$ | 1 | 1 | −1 | −1 | −1 | −1 |
| $\alpha_1$ | 104.810 1 | 4.393 864 | 15.142 28 | 674.982 8 | 513.690 7 | 357.476 6 |
| $\alpha_2$ | 96.685 22 | 139.975 5 | 158.854 3 | −179.267 5 | −156.741 7 | −111.478 5 |
| $\beta_1$ | 2.485 827 | 0.936 789 8 | 1.707 357 | 2.524 152 | 2.108 456 | 2.645 580 |
| $\beta_2$ | 2.956 462 | 2.089 377 | 2.857 207 | 1.655 343 | 1.438 310 | 2.065 807 |

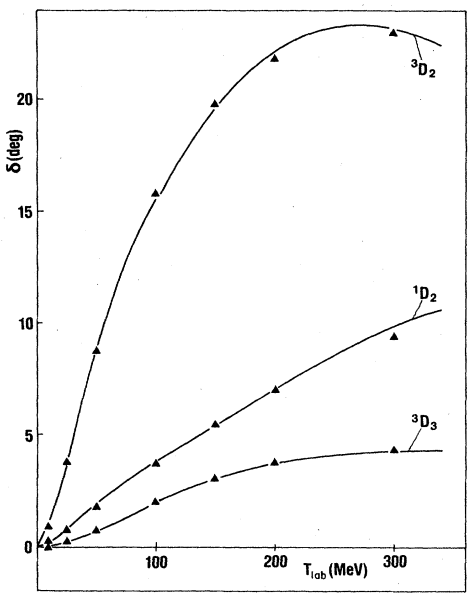


FIG. 6. Nucleon-nucleon $D$-state phase shifts compared with the results of the energy-dependent analysis of Arndt *et al.* (Ref. 20).

Because of the more complicated structure of the $^3P_0$ phase (sign change at 210 MeV) we have to use a rank 2 potential [Eq. (17)]. The form factor $g_1(p)$ is given by Eq. (21) and

$$g_2(p)=\frac{\alpha_3 p^3}{(p^2+\beta_3^2)^3} \ . \tag{22}$$

The parameters are stated in Table V and the phases can be seen in Fig. 5.

### B. The pion-nucleon system

For the pion-nucleon interaction we have also considered all $S$, $P$, and $D$ waves; the fits are performed up to $E_{\rm lab}\leq 350$ MeV, and the comparison of our results is made with the phase shifts given by Rowe *et al.*,[29] if not stated otherwise. The phases of most importance and interest concern the $P_{33}$ and $P_{11}$ states, wherefore we will discuss these cases separately.

#### 1. $P_{33}$ state

Because of the $\Delta$ resonance present in $P_{33}$, this partial wave is by far the most important one in the pion-nucleon system; a great deal of three- and more-body calculations including pions have been performed just taking into account this channel. This was also the reason why the authors have constructed energy-independent separable parametrizations of the $P_{33}$ wave.[9] Since it was found to be very fruitful to study three-body systems with different parametrizations for the most important $\pi$-N channel, we have given five potentials with the following form factor:

$$g(p)=p\left[\frac{\alpha_1}{p^2+\beta_1^2}+\frac{\alpha_2}{p^2+\beta_2^2}\right] \tag{23}$$

but with varying range of the second term ($\beta_2$). All of these five parametrizations give essentially the same phase shift up to $E_{\rm lab}=300$ MeV. For completeness we repeat the parameters in Table VI; for a more detailed discussion of the features of these potentials we refer to Ref. 9.

TABLE V. Potential parameters of the $^3P_0$ nucleon-nucleon state.

| | |
|---|---|
| $\lambda_1=-1$ fm$^0$ | $\alpha_2=459.2596$ fm$^{-3}$ |
| $\lambda_2=1$ fm$^0$ | $\beta_{11}=0.7931494$ fm$^{-1}$ |
| $\alpha_{11}=2.072365$ fm$^{-3}$ | $\beta_{12}=1.330105$ fm$^{-1}$ |
| $\alpha_{12}=21.28324$ fm$^{-3}$ | $\beta_2=2.769283$ fm$^{-1}$ |

TABLE VI. Potential parameters ($[\lambda]=$fm$^0$, $[\alpha_i]=$fm$^{-1}$, $[\beta_i]=$fm$^{-1}$) for five different models of the pion-nucleon $P_{33}$ state.

| | $\lambda=-1$ | | | |
|---|---|---|---|---|
| | $\alpha_1$ | $\alpha_2$ | $\beta_1$ | $\beta_2$ |
| $A$ | 0.08674243 | 36.80516 | 0.4764064 | 10.13538 |
| $B$ | 0.1930355 | 74.86341 | 0.8428261 | 20.27076 |
| $C$ | 0.2291463 | 133.8293 | 1.015084 | 35.47382 |
| $D$ | 0.232170 | 193.6752 | 1.097141 | 50.67689 |
| $E$ | 0.2098927 | 395.0781 | 1.209289 | 101.35378 |

#### 2. $P_{11}$ state

The $P_{11}$ state is characterized by the nucleon pole at $\sqrt{S}=M$ (the residuum of the scattering matrix at this point should yield the pion-nucleon coupling constant) and by the small negative phase shift below $T_{\rm lab}\approx 170$ MeV. Among others Mizutani *et al.*[30] have attributed this latter feature to an almost cancellation of two parts, namely the so-called pole and nonpole terms, and they have built up a potential along this line.

As with the $P_{33}$ channel most of the separable potentials so far have been constructed with an energy-dependent coupling constant. For the reasons discussed in

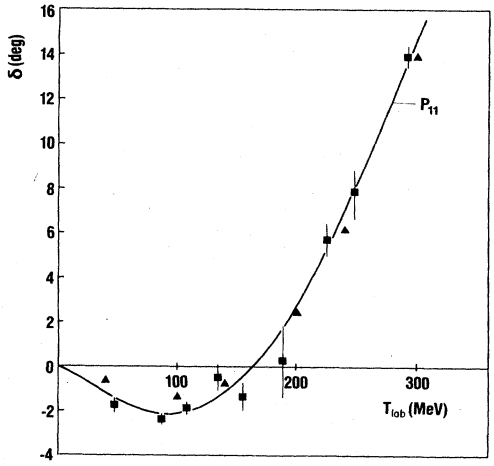


FIG. 7. Pion-nucleon $P_{11}$ phase shift compared with the results of the analyses of Zidell *et al.* (Ref. 33) (squares) and Rowe *et al.* (Ref. 29) (triangles).

TABLE VII. Potential parameters of the pion-nucleon $P_{11}$ state and the resulting scattering volume (in units of the pion mass) compared with experimental results.

| | |
|---|---|
| $\lambda_1=-1$ fm$^0$ | $\alpha_2=245.3575$ fm$^{-3}$ |
| $\lambda_2=-1$ fm$^0$ | $\beta_{11}=8.845090$ fm$^{-1}$ |
| $\alpha_{11}=-2390.1474$ fm$^{-3}$ | $\beta_{12}=7.731272$ fm$^{-1}$ |
| $\alpha_{12}=13592.536$ fm$^{-3}$ | $\beta_2=2.748874$ fm$^{-1}$ |
| $a=-0.114$ | $a^{\text{exp}}=\begin{cases}-0.047\pm0.004 \text{ (Ref. 29)}\\ -0.133\pm0.004 \text{ (Ref. 33)}\\ -0.108\pm0.004 \text{ (Ref. 33)}\end{cases}$ |

Ref. 31 we prefer an energy-independent potential, but because of the complexity of the phase we have to take it of rank 2 [Eq. (17)]. The form factors are the same as for the $^3P_0$ nucleon-nucleon case [Eqs. (21) and (22)].

Similar to Schwarz *et al.*[32] and Mizutani *et al.*[30] it was not possible with our chosen form to fit simultaneously the phase shifts of Rowe *et al.*[29] (including the scattering volume) and the pion-nucleon coupling constant: with phases in agreement with Rowe *et al.* the coupling constant was too low at least by a factor of 2. All of these model calculations favor a larger negative scattering volume and consequently larger negative phase shifts at low energies. This behavior is the outcome of a phase shift analysis of Zidell *et al.*[33] and thus we have used these phases as the basis of our search.

The parameters of the final fit are given in Table VII, together with the resulting scattering volume compared to some experimental values. The phase shifts are shown in Fig. 7 together with the phases given by Zidell *et al.*[33] and for comparison also with those given by Rowe *et al.*[29] The value $f^2=0.079$ of the calculated pion-nucleon coupling constant is in agreement with the experimental value ($0.078 \leq f^2 \leq 0.08$).[34]

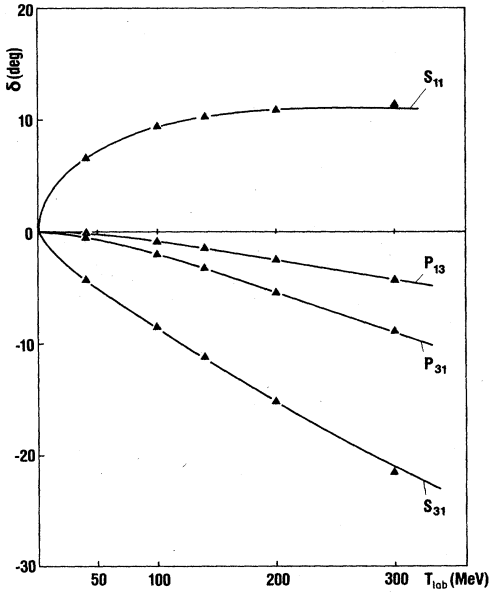


FIG. 8. Pion-nucleon $S$- and $P$-state phase shifts compared with the results given by Rowe *et al.* (Ref. 29).

### 3. $S$, $P_{13}$, $P_{31}$, and $D$ waves

For the $S$ waves it was possible to reproduce the experimental phases by simple Yamaguchi form factors. For the $S_{31}$ wave two terms were needed,

$$g(p)=\frac{\alpha_1}{p^2+\beta_1^2}+\frac{\alpha_2}{p^2+\beta_1^2} \tag{24}$$

whereas for $S_{11}$ one term was sufficient.

For the "small" $P$ waves, $P_{13}$ and $P_{31}$, we have taken the form given by Eq. (23), but for the $P_{31}$ case, again the first term alone already produced satisfactory results.

TABLE VIII. Potential parameters ($[\lambda]=\text{fm}^0$, $[\alpha]=\text{fm}^{-2}$ for $S$ and $D$ waves and $[\alpha]=\text{fm}^{-1}$ for $P$ waves, $[\beta]=\text{fm}^{-1}$) for $S$, $P$, and $D$ waves for pion-nucleon scattering. The resulting low energy parameters (in units of the pion mass) are compared with the results of Rowe *et al.* (Ref. 29) with the exception of the $D_{33}$ and $D_{35}$ waves [Zidell *et al.* (Ref. 33)].

| | $S_{11}$ | $S_{31}$ | $P_{13}$ | $P_{31}$ |
|---|---|---|---|---|
| $\lambda$ | $-1$ | 1 | $-1$ | 1 |
| $\alpha_1$ | 14.645 36 | 95.425 19 | 10.402 29 | 13.079 |
| $\beta_1$ | 3.500 613 | 5.560 206 6 | 3.963 433 | 3.496 |
| $\alpha_2$ | | $-3.137\,41$ | $-2.311\,011$ | |
| $\beta_2$ | | 1.355 239 | 5.583 776 | |
| $a$ | 0.187 | $-0.094$ | $-0.014$ | $-0.032$ |
| $a^{\text{exp}}$ | $0.185\pm0.008$ | $-0.098\pm0.003$ | $-0.13\pm0.002$ | $-0.029\pm0.002$ |
| | $D_{13}$ | $D_{15}$ | $D_{33}$ | $D_{35}$ |
| $\lambda$ | $-1$ | $-1$ | $-1$ | 1 |
| $\alpha$ | 364.057 3 | 10.891 93 | 2.180 775 | 7.525 454 |
| $\beta$ | 7.065 765 | 2.607 608 | 1.790 54 | 2.280 914 |
| $a$ | 0.0014 | 0.0014 | $-0.0011$ | $-0.0019$ |
| $a^{\text{exp}}$ | $0.0013\pm0.0005$ | $0.0012\pm0.0005$ | $-0.0010\pm0.006$ | $-0.011\pm0.002$ |

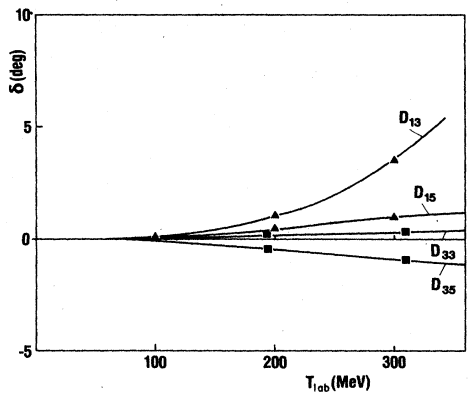


FIG. 9. Pion-nucleon $D$-state phase shifts compared with the results given by Rowe *et al.* (Ref. 29) (triangles) and by Carter *et al.* (Ref. 35) (squares).

All $D$ waves ($D_{13}$, $D_{15}$, $D_{33}$, and $D_{35}$) were fitted with one term

$$g(p)=\frac{\alpha p^2}{(p^2+\beta^2)^2} . \tag{25}$$

Since the $D_{33}$ and $D_{35}$ waves are not given by Rowe *et al.*,[29] we used the phase of Carter *et al.*[35] for these two cases.

The parameters of the potentials are given in Table VIII together with the resulting low-energy parameters and the theoretical phases are compared to the experimental data in Figs. 8 and 9.

### C. The pion-pion system

In contrast to the nucleon-nucleon and to the pion-nucleon system there exist just five partial waves with angular momentum $L \leqq 2$ and the results of the phase shift analyses are connected with larger error bars. Since the phases do not show any complicated structure, we succeeded with separable potentials of rank 1 to reproduce the experimental phases. The expression of the form factors is given by Eq. (21) for $L=0$, Eq. (23) for $L=1$, and Eq. (25) for $L=2$.

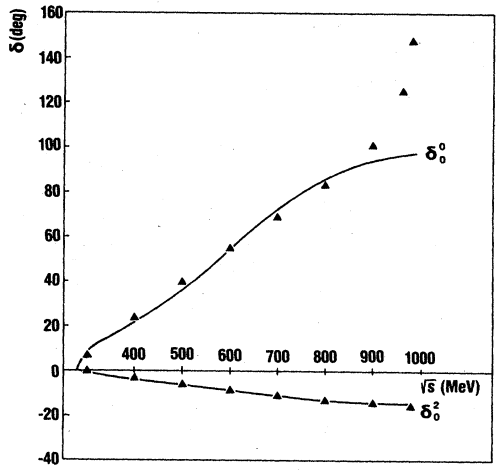


FIG. 11. Pion-pion $S$-state phase shifts compared with the results given by Frogatt *et al.* (Ref. 39).

#### 1. $\delta_1^1$ wave

Like the dominant $P_{33}$ wave with the $\Delta$ resonance in the $\pi$-N system, the $\delta_1^1$ wave in the $\pi$-$\pi$ interaction is governed by the existence of the $\rho$ resonance at $\sqrt{s}=769\pm3$ MeV.[36] Among the various phase shift analyses, which differ just slightly, we have chosen the solution of Protopopescu[37] since it reproduces the mass of the $\rho$ resonance better than the other solutions.

The result of our parameter search can be seen in Fig. 10, and the corresponding low energy parameter together with the potential parameters are listed in Table IX. The resonance is located at 770.6 MeV and the width at $\approx 160$ MeV, which is in accordance with experimental data.[36]

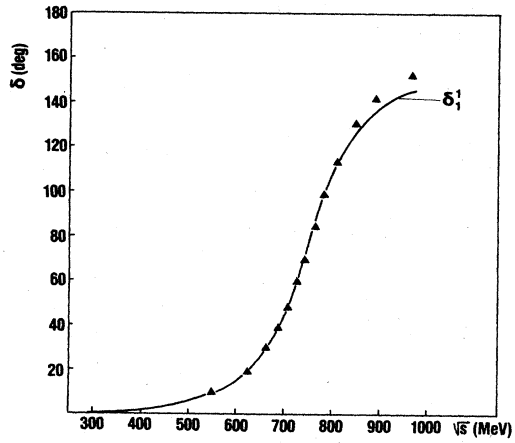


FIG. 10. Pion-pion $P$-state phase shift compared with the result given by Protopopescu (Ref. 37).

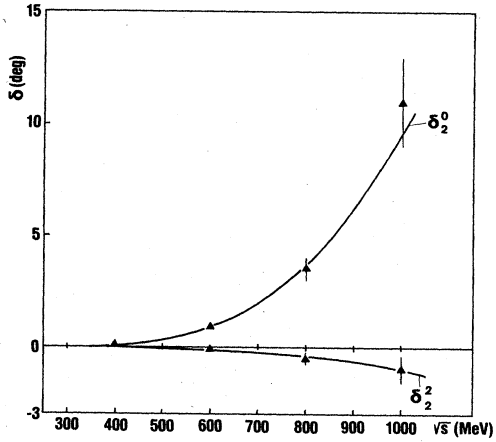


FIG. 12. Pion-pion $D$-state phase shifts compared with the results given by Basdevant *et al.* (Ref. 40).

TABLE IX. Potential parameters ($[\lambda]=\text{fm}^0$, $[\alpha_i]=\text{fm}^{-L-2}$, $[\beta_i]=\text{fm}^{-1}$) of the $S$, $P$, and $D$ states of the pion-pion interaction; the resulting low energy parameters (in units of the pion mass) are compared with the experimental data of Rosselat *et al.* (Ref. 38).

| | $\delta_0^0$ | $\delta_0^2$ | $\delta_1^1$ | $\delta_2^0$ | $\delta_2^2$ |
|---|---|---|---|---|---|
| $\lambda$ | −1 | 1 | −1 | −1 | 1 |
| $\alpha_1$ | 60.263 75 | 9.281 88 | 247.296 62 | 284.862 5 | 489.288 9 |
| $\beta_1$ | 11.708 48 | 5.086 58 | 44.718 831 | 7.322 809 | 10.051 82 |
| $\alpha_2$ | 331.754 9 | 13.959 35 | −31.789 013 | | |
| $\beta_2$ | 7.331 817 | 2.469 03 | 13.017 966 | | |
| $a$ | 0.26 | −0.029 | 0.041 | $10\times10^{-4}$ | $-1.3\times10^{-4}$ |
| $a^{\text{exp}}$ (Ref. 38) | 0.26±0.05 | −0.028±0.012 | 0.038±0.002 | $(17\pm4)\times10^{-4}$ | $(1.3\pm3)\times10^{-4}$ |

### 2. $\delta_0^0$ wave

The behavior of this wave at higher energies (see Fig. 11) is characterized by a steep increase of the phase shift connected with the $K\bar{K}$ threshold. Since we stick to an elastic treatment of the phases throughout this paper, it was not possible to reproduce the behavior above $\sqrt{S}=900$ MeV with our formalism. Below this energy we could get satisfactory agreement with the phases given by Frogatt and Petersen[39] (see Fig. 11), and also the scattering length is in accordance with the experimental value (Table IX). The potential parameters are also given in Table IX.

### 3. $\delta_0^2$, $\delta_2^0$, and $\delta_2^2$ waves

These phase shifts are smaller than the other $\pi$-$\pi$ phases by a factor of 10 at least and do not show any special structure below $\sqrt{S}=900$ MeV. for $\delta_2^0$ and $\delta_2^2$ a rank 1, single-term, separable potential yielded a sufficient reproduction of the experimental data; for $\delta_0^2$, however, a second term was needed. The basis of our parameter search was the results of Frogatt and Petersen[39] for $\delta_0^2$ and of Basdevant *et al.*[40] for $\delta_2^0$ and $\delta_2^2$, and the parameters of the fits are given in Table IX, and the results are shown in Figs. 11 and 12 and in Table IX.

## V. CONCLUSION

Though the relativistic three-body formalism given by Aaron, Amado, and Young was used with great success in the scattering domain, it leads to an unrealistic behavior below threshold and gives rise to spurious bound states. To overcome this disadvantage the authors have proposed modified equations which still satisfy relativistic three-body unitarity and are Lorentz invariant. The spurious bound-states disappeared, but—as in the case of the similar Vinogradov equations—the so-called clustering property has been lost. However, we have shown in this paper that the errors connected with the loss of the clustering property are small in the energy range, where usual three-body calculations are performed.

Contrary to the Aaron, Amado, and Young formalism, where the two-body input is given by the Blankenbecler-Sugar equation, in our case (as well as in Vinogradov's theory) the two-particle subsystems have to be calculated in the framework of the Kadyshevski equation. Therefore we constructed separable potentials for the nucleon-nucleon, pion-nucleon, and pion-pion systems. To facilitate the use of these potentials in three-body calculations we kept the rank of the potentials as low as possible: rank 3 for the coupled $^3S_1$-$^3D_1$ NN channel, rank 2 to permit phase shifts to change sign, and rank 1 elsewhere. The experimental phase shifts and other properties of the specific states are reproduced in a satisfactory way, in general.

As a by-product of the parameter search we can strengthen the finding of Schwarz *et al.* and Mizutani *et al.* that in the $\pi$-N $P_{11}$ channel small values of the phase shift and of the scattering volume as given for example by the solution of Rowe *et al.* are incompatible with the experimental value of the pion-nucleon coupling constant. This statement holds at least for the case of separable interactions.

We emphasize that it is the first time that separable potentials in a specific framework, namely the Kadyshevski equation, are given for the three different systems involving nucleons and pions and for all $S$, $P$, and $D$ partial waves. Therefore it is now possible to perform three-body calculations in the scattering as well as in the bound-state domain in a relativistic manner for all systems consisting of pions and nucleons. In a forthcoming paper[12] we shall discuss the possible existence of bound states or resonances in the $\pi$NN channels, with isospin 0 and 2, but also the $\pi$NN isospin 1 resonances (the so-called dibaryon resonances) should be interesting to discuss in our formalism. Last but not least, a study of possible resonances of systems with more than one pion, namely $\pi\pi$N and $\pi\pi\pi$, also provided by our formalism and the given potentials, could bring up new interesting features.

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