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Hybrid mass relations

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The properties of several hybrid mass relations are examined through the study of (i) statistical tests, (ii) systematic effects, and (iii) the structure of the solutions of partial difference equations. The usefulness of the relations is discussed. Mass relations based on the effective neutron-proton interaction and on derivatives thereof have particularly small relation inaccuracies.

I. INTRODUCTION

Mass relations are difference equations for the binding energies or masses of atomic nuclei. Hybrid mass relations explicitly include features inherent to mass equations. Mass relations can be used to determine unknown masses of neutron-rich and proton-rich nuclei if only one mass in the relation is unknown; they can usually be applied recursively. When viewed as partial difference equations, (hybrid) mass relations may have unique solutions which represent mass equations. Hybrid mass relations are particularly suited to investigating special aspects of nuclear binding energies, including nucleon interaction energies, curvature terms, the isospin dependence, and pairing effects.

The present study was motivated by the observation that the *repeated* application of the well-known Garvey-Kelson relation[1] (or the equivalent use of the solution of the partial difference equation) leads to difficulties with long-range extrapolations. Introducing an inhomogeneous source term, hence a hybrid mass relation, becomes necessary.[2] Also, it was considered desirable to further examine a new hybrid mass relation[3] derived recently by employing single-particle properties of a Fermi gas.

II. HYBRID BINDING ENERGY RELATIONS

A binding energy equation $B_{\text{eq}}(N,Z)$ is expected to have the property

$$B_{\text{exp}}(N,Z)\approx B_{\text{eq}}(N,Z) \tag{1}$$

for all experimentally known energies $B_{\text{exp}}(N,Z)$. Given an analytical expression for $B_{\text{eq}}(N,Z)$, the parameters contained in the equation are usually obtained by best satisfying Eq. (1) for all known energies $B_{\text{exp}}(N,Z)$. Introducing a difference operator D, it follows that

$$DB_{\text{exp}}(N,Z)\approx DB_{\text{eq}}(N,Z)\ . \tag{2}$$

Equation (1) is contained in Eq. (2) in the limit $D\equiv 1$. The right-hand side of the hybrid binding energy relation (2) is a function $f(N,Z)$ which can easily be obtained in analytical form from any $B_{\text{eq}}(N,Z)$. Depending on D, the function $f(N,Z)$ reflects only upon certain properties of $B_{\text{eq}}(N,Z)$. The function $f(N,Z)$ can also be introduced without explicitly invoking an equation $B_{\text{eq}}(N,Z)$. A special situation arises if $DB_{\text{eq}}(N,Z)\approx 0$. This is the case for the transverse and longitudinal Garvey-Kelson relations,[1] at least on a local level. In this limit, the description of the experimental binding energies $B_{\text{exp}}(N,Z)$ has shifted completely from the mass equation $B_{\text{eq}}(N,Z)$ with its inherent parameters, Eq. (1), to the difference equation $DB_{\text{exp}}(N,Z)=0$.

Difference operators D are easy to construct, but only few are of practical or physical interest. Binding energy relations serve two strongly connected objectives. They make it possible to predict unknown masses of neutron-rich and proton-rich nuclei. This can be achieved by single or by repeated applications of suitable binding energy relations, or by employing a binding energy equation, if it exists, which is the solution of a binding energy relation (partial difference equation). Conversely, the comparison between predicted (extrapolated) and measured binding energies provides a sensitive test for special aspects of nuclear binding energies. For example, the influence of an inhomogeneous source term in the Garvey-Kelson relations[1] has been studied,[2,4] and it was concluded that the effective neutron-proton interaction and the symmetry energy are shell dependent and must contain higher-order terms in isospin. Other difference operators[5] emphasize discontinuous and periodic aspects of the experimental binding energies, such as shell effects or pairing energies. The study of these operators led to a better understanding of pairing and clustering phenomena in nuclear ground states.

Partial difference operators D can be constructed on the basis of mathematical or physical considerations. They are best introduced by defining the partial difference operators

$$\Delta^{i,j}f(N,Z)=f(N,Z)-f(N-i,Z-j)\ . \tag{3}$$

The simplest first-order difference operators $\Delta^{1,0}$, $\Delta^{0,1}$, $\Delta^{1,1}$, and $\Delta^{1,-1}$ have the characteristics of the partial differential operators $\partial/\partial N$, $\partial/\partial Z$, $2\partial/\partial A$, and $2\partial/\partial E$ with $A=N+Z$ and $E=N-Z$. The most basic quantities derived from the binding energies with the use of these operators are

$$
\begin{aligned}
B_{\mathrm{n}}(N,Z) &\equiv \Delta^{1,0}B(N,Z)=B(N,Z)-B(N-1,Z)\ ,\\
B_{\mathrm{p}}(N,Z) &\equiv \Delta^{0,1}B(N,Z)=B(N,Z)-B(N,Z-1)\ ,\\
B_{2\mathrm{n}}(N,Z) &\equiv \Delta^{2,0}B(N,Z)=B(N,Z)-B(N-2,Z)\ ,\\
B_{2\mathrm{p}}(N,Z) &\equiv \Delta^{0,2}B(N,Z)=B(N,Z)-B(N,Z-2)\ , \qquad (4)\\
I_{\mathrm{nn}}(N,Z) &\equiv \Delta^{1,0}\Delta^{1,0}B(N,Z)\\
&=B(N,Z)-2B(N-1,Z)+B(N-2,Z)\ ,\\
I_{\mathrm{pp}}(N,Z) &\equiv \Delta^{0,1}\Delta^{0,1}B(N,Z)\\
&=B(N,Z)-2B(N,Z-1)+B(N,Z-2)\ ,\\
I_{\mathrm{np}}(N,Z) &\equiv \Delta^{1,0}\Delta^{0,1}B(N,Z)\\
&=B(N,Z)-B(N-1,Z)-B(N,Z-1)\\
&\quad +B(N-1,Z-1)\ .
\end{aligned}
$$

Equations (4) represent the operational definitions for the neutron and proton binding energies, the two-neutron and two-proton binding energies, and the effective neutron-neutron, proton-proton, and neutron-proton interaction energies. Particularly the effective neutron-proton interaction plays an important role in mass relations[2] as it represents a quantity related to the curvatures with respect to A and E of the binding energy surface (for odd A). Many hybrid mass relations follow from the operators introduced in Eqs. (4) and from additional higher-order differences with regard to N, Z, A, and/or E.

Another partial difference operator introduced[3] on physical grounds is

$$
\begin{aligned}
D_{SN}B(N,Z) &\equiv (1-N\Delta^{1,0}-Z\Delta^{0,1})B(N,Z)\\
&=B(N,Z)-NB_{\mathrm{n}}(N,Z)-ZB_{\mathrm{p}}(N,Z) \qquad (5)\\
&=(1-A)B(N,Z)+NB(N-1,Z)\\
&\quad +ZB(N,Z-1)\ .
\end{aligned}
$$

It has the property $D_{SN}B(N,Z)=0$ for infinite nuclear matter.

III. SELECTED RELATIONS

About 30 hybrid binding energy relations have been considered in this work, but results will only be reported for a few characteristic examples. These include all relations based on simple first- and second-order difference operators and several relations reported in the literature.[1,3,6] The following relations will be discussed:

$$
\begin{aligned}
&(\mathrm{A})\ \Delta^{1,0}B(N,Z)\approx f_1(N,Z)\ ,\\
&(\mathrm{B})\ \Delta^{1,0}\Delta^{1,0}B(N,Z)\approx f_2(N,Z)\ ,\\
&(\mathrm{C})\ \Delta^{0,1}B(N,Z)\approx f_3(N,Z)\ ,\\
&(\mathrm{D})\ \Delta^{0,1}\Delta^{0,1}B(N,Z)\approx f_4(N,Z)\ ,\\
&(\mathrm{E})\ (1-N\Delta^{1,0}-Z\Delta^{0,1})B(N,Z)\approx f_5(N,Z)\ , \qquad (6)\\
&(\mathrm{F})\ \Delta^{1,0}\Delta^{0,1}B(N,Z)\approx f_6(N,Z)\ ,\\
&(\mathrm{G})\ \Delta^{1,-1}\Delta^{1,0}\Delta^{0,1}B(N,Z)\approx f_7(N,Z)\ ,\\
&(\mathrm{H})\ \Delta^{1,1}\Delta^{1,0}\Delta^{0,1}B(N,Z)\approx f_8(N,Z)\ ,\\
&(\mathrm{I})\ \Delta^{1,-1}\Delta^{1,1}\Delta^{1,0}\Delta^{0,1}B(N,Z)\approx f_9(N,Z)\ .
\end{aligned}
$$

The difference operators are schematically displayed in Fig. 1. The functions $f_i(N,Z)=D_iB(N,Z)$ represent inhomogeneous source terms. They may be obtained from any binding energy equation. The relations (A), (B), (C), and (D) are simple first- and second-order differences with regard to the number of neutrons and protons. The functions $f_1(N,Z)$ to $f_4(N,Z)$ represent theoretical expressions for the quantities $B_{\mathrm{n}}(N,Z)$, $I_{\mathrm{nn}}(N,Z)$, $B_{\mathrm{p}}(N,Z)$, and $I_{\mathrm{pp}}(N,Z)$, respectively. The equation lettered (E) is that of Satpathy and Nayak.[3] It has been derived employing single-particle properties of a Fermi gas with two kinds of particles. The difference operator accounts for the nucleon binding energies in infinite nuclear matter. The function $f_5(N,Z)$ describes the influence of the finite nuclear size. An analytical expression for $f_5(N,Z)$ has been derived by the authors[3] assuming $N=Z$. When used as a recursion relation for one of the three nuclei [see Eq. (5) and Fig. 1], the equation must be divided by $(1-A)$, N, or Z, respectively, depending on whether the binding energy of the nucleon-rich, proton-rich, or neutron-rich nucleus in the relation is to be calculated. Results will be given for all three cases. The second-order difference equation lettered (F) represents the effective neutron-proton interaction I_{np}.[2,4] It is a hyperbolic partial difference equation. An analytical expression for $f_6(N,Z)=I_{\mathrm{np}}(N,Z)$ has also been reported.[4] The third- and fourth-order difference equations lettered (G), (H), and (I) represent inhomogeneous extensions of the transverse and longitudinal Garvey-Kelson relations[1] and of the relation of Monahan and Serduke.[6] The former two relations are obtained by considering the dependence on neutron excess and on nucleon number of the effective

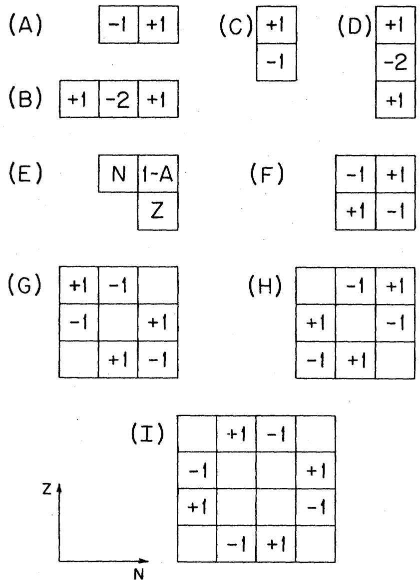

FIG. 1. Difference operators for the hybrid binding energy relations of Eq. (6).

neutron-proton interaction I_{np}, and the latter is obtained by considering their combined dependence. The homogeneous relations (G), (H), and (I) with $f_i(N,Z)=0$ are essentially equivalent to the inhomogeneous relations. Only long-range extrapolations from the repeated application of these relations seem to require certain inhomogeneous source terms.[2]

The hybrid binding energy relations of Eqs. (6) are not independent of each other. Relation (E) is a linear combination of the first-order relations (A) and (C). The second-order relations (B), (D), and (F) are obtained by operating with the difference operators $\Delta^{1,0}$ or $\Delta^{0,1}$ on the first-order relations (A) and (C). The third- and fourth-order relations (G), (H), and (I) are obtained by operating with the difference operators $\Delta^{1,-1}$ or $\Delta^{1,1}$ or both on the second-order relation (F). Possible implications of these connections will be discussed below.

IV. STATISTICAL CONSIDERATIONS

The usefulness of a difference operator D in conjunction with an associated binding energy equation $B_{\mathrm{eq}}(N,Z)$ as a predictive tool for unknown binding energies can be tested by determining the means and standard deviations of the residuals

$$R(N,Z)=D[B_{\mathrm{exp}}(N,Z)-B_{\mathrm{eq}}(N,Z)] \tag{7}$$

for all experimentally known binding energies. Useful relations are characterized by small means and standard deviations. A simple liquid-drop-model binding energy equation (BW1 of Ref. 7) has been used in the present work. The results based on recent mass data[8] are listed in Table I in the second to fifth columns. The mean values are smaller than the standard deviations, reaching about 30% only for relations (E) and (F). The standard deviations with regard to zero are therefore taken as a measure for the relation inaccuracies. They are displayed in the table. Results are given for different ranges of mass numbers A to establish general trends. The number of residuals used for each case is given in parentheses. As mentioned before, results for relation (E) are listed in three variations. The standard deviations range from about 200 keV for relations (F), (G), and (H) to about 1000 keV for relation (E-3), and they generally decrease with increasing A. The standard deviations listed in the last column will be discussed below.

V. SYSTEMATIC EFFECTS

The residuals $R(N,Z)$ of Eq. (7) may display systematic effects. The possible presence of such effects is best revealed in a two-dimensional display. Such displays have been obtained for all hybrid relations considered originally. A few of the results are shown in Fig. 2. The display is limited to the region near $A=130$ which includes the shell crossings at $Z=50$ and $N=82$. This region is characteristic for the entire region of nuclei. The various symbols represent residuals, with the small dot assigned to the range from -200 to $+200$ keV. Systematic effects can indeed be recognized. A significant discontinuity at the shell crossing at $Z=50$ can be seen for relations (A) and (E-1), and a similar step at $N=82$ can be seen for relations (B) and (E-1).

VI. SOLUTIONS OF PARTIAL DIFFERENCE EQUATIONS

Additional insight into the usefulness of a binding energy relation may be gained if a solution of the partial difference equation

$$DB_{\mathrm{exp}}(N,Z)=f(N,Z) \tag{8}$$

can be derived. The inhomogeneous equations with $f(N,Z)\neq 0$ will always have $B_{\mathrm{eq}}(N,Z)$ as a special solution, even though certain terms contained in it may suffice (such as the symmetry energy). The most general solutions of the respective homogeneous equations must be added. Neglecting the less important discontinuous pairing contributions, the above (hybrid) binding energy relations have the following solutions:

TABLE I. Standard deviations of the residuals for the hybrid binding energy relations Eqs. (6).

Relation	Standard deviation (keV)				
	$A=40$–260	$A=40$–110	$A=111$–180	$A=181$–260	$Z\leq 82$, $N\geq 126$[c]
(A)	718 (1171)[a]	829 (391)	610 (437)	703 (343)	
(B)	602 (1067)	831 (356)	480 (402)	374 (309)	
(C)	811 (1137)	913 (387)	674 (436)	843 (314)	
(D)	635 (996)	663 (349)	674 (381)	535 (266)	
(E-1)	436 (1015)	441 (352)	393 (386)	485 (277)	627 (151)
(E-2)	749 (1015)	787 (357)	653 (384)	815 (274)	986 (151)
(E-3)	1044 (1006)	984 (348)	930 (384)	1246 (274)	1718 (151)
(F)	196 (980)	262 (346)	157 (377)	128 (257)	255 (151)
(G)[b]	191 (770)	239 (277)	162 (307)	144 (186)	297 (116)
(H)[b]	194 (883)	249 (316)	163 (342)	133 (255)	371 (123)
(I)[b]	273 (673)	338 (247)	231 (276)	203 (150)	548 (90)

[a] The number of residuals used in the calculations is given in parenthesis.

[b] Results for the homogeneous relations are given. The inhomogeneous relations yield practically identical results.

[c] Mostly predicted mass values (Ref. 13) were used as input for these neutron-rich nuclei.

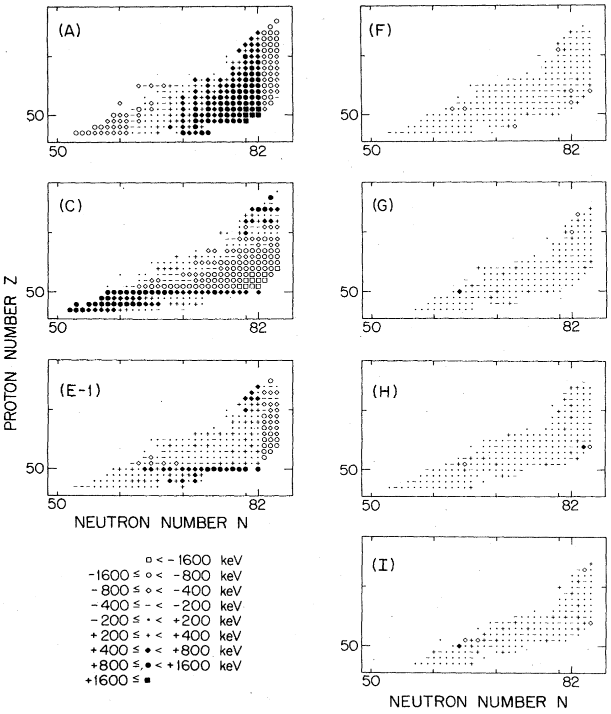

FIG. 2. Residuals for the binding energy relations Eq. (7) for nuclei near $A = 130$. Experimental mass values (Ref. 8) and an associated liquid-drop-model mass equation (Ref. 7) are used. The various symbols denote residuals in the indicated energy ranges.

$$\begin{aligned}
&(\mathrm{A})\ B_{\mathrm{sol}}(N,Z)=B_{\mathrm{eq}}(N,Z)+\alpha(Z)\ ,\\
&(\mathrm{B})\ B_{\mathrm{sol}}(N,Z)=B_{\mathrm{eq}}(N,Z)+\alpha(Z)+\beta(Z)N\ ,\\
&(\mathrm{C})\ B_{\mathrm{sol}}(N,Z)=B_{\mathrm{eq}}(N,Z)+\alpha(N)\ ,\\
&(\mathrm{D})\ B_{\mathrm{sol}}(N,Z)=B_{\mathrm{eq}}(N,Z)+\alpha(N)+\beta(N)Z\ ,\\
&(\mathrm{E})\ B_{\mathrm{sol}}(N,Z)=B_{\mathrm{eq}}(N,Z)+\alpha N+\beta Z\ , \qquad (9)\\
&(\mathrm{F})\ B_{\mathrm{sol}}(N,Z)=B_{\mathrm{eq}}(N,Z)+F_1(N)+F_2(Z)\ ,\\
&(\mathrm{G})\ B_{\mathrm{sol}}(N,Z)=[B_{\mathrm{eq}}(N,Z)]+F_1(N)+F_2(Z)\\
&\qquad\qquad +F_3(N+Z)\ ,\\
&(\mathrm{H})\ B_{\mathrm{sol}}(N,Z)=[B_{\mathrm{eq}}(N,Z)]+F_1(N)+F_2(Z)\\
&\qquad\qquad +F_4(N-Z)\ ,\\
&(\mathrm{I})\ B_{\mathrm{sol}}(N,Z)=[B_{\mathrm{eq}}(N,Z)]+F_1(N)+F_2(Z)\\
&\qquad\qquad +F_3(N+Z)+F_4(N-Z)\ .
\end{aligned}$$

Equations (9) demonstrate how the load for describing the experimental data shifts from the underlying binding energy equation $B_{\mathrm{eq}}(N,Z)$ to the solution $B_{\mathrm{sol}}(N,Z)$ of a homogeneous difference equation.

VII. DISCUSSION

To assess the usefulness and range of applicability of a particular binding energy relation one has to consider the results from the statistical and systematic analyses and the solutions of the partial difference equations.

A. Relations (A), (B), (C), and (D)

These relations are only of limited value as predictive tools. Similar conclusions have been arrived at by Comay and Kelson.[9] The partial differences with regard to N and Z can only be used, if at all, to predict binding energies for sequences of isotopes and isotones, respectively. The standard deviations of Table I are higher than for other relations, and the individual residuals display pronounced shell effects. The first-order differences of Fig. 2 display discontinuities at $N=82$ or $Z=50$. Similarly, the second- and third-order differences generate a "delta function" and a "double layer," respectively. Pairing effects are also visible in Fig. 2 for (A) and (C), but are even more pronounced for the higher-order partial differences. This behavior is caused by the use of simple liquid-drop-model expressions for the binding energies B_{n} and B_{p} and the effective interaction energies I_{nn} and I_{pp}. More sophisticated expressions could be used, but the usefulness of these hybrid relations lies more in the possibility of establishing shell and pairing effects. They are therefore related to the class of partial difference operators introduced by Jensen *et al.*[5] to extract discontinuous and periodic components of nuclear binding energies.

B. Relation (E)

The standard deviations for the binding energy relation of Satpathy and Nayak[3] are significantly higher than for other relations. This is particularly true for the relations (E-2) and (E-3). The individual residuals displayed for (E-1) in Fig. 2 show discontinuities at neutron *and* proton magic numbers. This is not surprising as the relation represents essentially an average of the relations (A) and (C). The discontinuities of (E-1) are further amplified by factors of essentially A/N and A/Z for the relations (E-2) and (E-3), which must be used for predicting proton-rich and neutron-rich nuclei, respectively. Equation (9) shows that only the volume term of the liquid-drop-model equation (and a term proportional to $N-Z$) are regenerated by the solution of the inhomogeneous partial difference equation. One has to conclude that relation (E) has only limited applicability.

The binding energy relation (E-1) has previously[3] been compared to the Garvey-Kelson relation (G), particularly for certain neutron-rich nuclei. The conclusions are at variance with those from the present work because the authors compare predictions from a *single* application of relation (E-1) to predictions from the Garvey-Kelson equation.[1] The latter are essentially identical to predictions from the *repeated* application of relation (G) with up to 15 steps (cf. Refs. 7 and 10).

C. Relation (F)

While relations (B) and (D) compare experimental and calculated effective interaction energies I_{nn} and I_{pp}, relation (F) compares experimental and calculated effective neutron-proton interaction energies I_{np}. Surprisingly, despite the simple model used to calculate I_{np}, excellent agreement is observed for relation (F) contrary to relations (B) and (D), and the individual residuals of Fig. 2 do not display shell effects. The shell dependence of I_{np} seems to be weaker than that of I_{nn} and I_{pp}. Also, the operational definition of I_{np}, Eq. (4), favors averaging of shell effects at shell crossings. Furthermore, pairing contributions for the valence neutrons and protons in the ground states of odd-odd nuclei are much less than those for $J=0$ coupled neutron-neutron or proton-proton pairs in the ground states of even-even nuclei. The standard deviations for relation (F) are comparable to those of the transverse and longitudinal Garvey-Kelson relations (G) and (H). Relation (F) appears to be the first known hybrid mass relation with this property, and its use as recursion relation is justified. The structure of the binding energy equation which is obtained as solution of the partial difference equation suggests that it could become a useful binding energy equation. This has to be done with caution, though, because small residuals observed for a relation do not necessarily reflect upon the long-range characteristics of the associated binding energy equation. In fact, shell effects for the two components of the effective neutron-proton interaction have been observed[2] on a very low energy level and would probably have to be included in the solutions of the difference equation.

D. Relations (G), (H), and (I)

As noted above, results for the hybrid $[f_i(N,Z)\neq 0]$ and homogeneous $[f_i(N,Z)=0]$ relations are essentially equivalent, and the results for the latter are presented in Table I and Fig. 2. The standard deviations are very small, particularly for (G) and (H). Properties and applications of these relations have been reported in the literature[1,6] and will not be repeated here. The slightly increased standard deviation for (I) is probably due to the fact that the relation contains eight binding energies, instead of only six for (G) and (H). Relation (G) is known to be better than (H) in repeated applications, and the equality of the standard deviations for (G) and (H) shows that relation inaccuracies do not necessarily reflect upon the long-range behavior. Only recently with the availability of new very neutron-rich and proton-rich data have the long-range characteristics of the systematic behavior of (G) been investigated.[2]

E. Other considerations

(1) Several of the hybrid binding energy relations included in Eqs. (6) are derived from each other by operating with a difference operator $\Delta^{i,j}$ on both sides of a lower-order relation (see Sec. III). This leads to certain connections between the relations but does not translate into general statements about the usefulness of a relation and its relation inaccuracies.

"Random" contributions to the relation inaccuracies in the absence of obvious systematic effects are expected to increase approximately as $\sqrt{n}$ where n is the number of nuclei contained in a relation. This increase can be seen, for example, for relation (I). Another increase observed for higher-order relations is in the number of adjustable parameters of the solutions of the partial difference equations. The number of parameters for the solutions (F), (G), (H), and (I) of Eqs. (9), for example, increases significantly with increasing order and may exceed reasonable limits.

Systematic effects and the ensuing relation inaccuracies do not seem to display a predictable pattern, and each relation must be studied on its own merits. Systematic effects may become more pronounced for higher-order differences (derivatives) as observed, for example, for the pairing energy contributions [compare relation (B) with (A) or relation (D) with (C)]. Systematic effects may remain essentially constant when higher-order differences are constructed from a relatively smooth quantity, such as I_{np} [compare relations (G), (H), and (I) with (F)]. Systematic effects may also decrease, as observed unexpectedly for the second-order relation representing the effective interaction I_{np} [compare relation (F) with (A) and (C)]. This behavior is very different from that of the other second-order relations representing I_{nn} and I_{pp} [compare relation (B) with (A) or relation (D) with (C)], presumably because relation (F) connects differences with regard to the number of neutrons *and* protons.

The characteristics of long-range extrapolations from repeated applications of binding energy relations, such as relations (F), (G), (H), and (I), are very difficult to predict and require individual studies. Here again, there does not seem to exist a predictable pattern related to the structure of the difference operators.

(2) Hybrid mass relations are applicable to both neutron-rich and proton-rich nuclei. The reliability for proton-rich nuclei depends on the accuracies of the Coulomb energy term and the symmetry energy term for small T which are used in the associated binding energy equation $B_{eq}(N,Z)$. Homogeneous relations cannot usually be used to cross the $N=Z$ line. However, other procedures, such as the charge-symmetric Kelson-Garvey relation[1] or the isobaric multiplet mass equation (IMME),[12] may be used instead.

(3) Most binding energy equations $B_{sol}(N,Z)$ of Eq. (9) are multiparameter equations with more parameters than the majority of binding energy equations from the literature. The reason for this is that the solutions of (hybrid) binding energy relations make extended use of the known data (and the physics implicitly contained in them), and only special aspects of nuclear binding energies (e.g., the shell dependence of the effective nn, pp, and/or np interactions, higher-order effects in isospin, etc.) must be treated theoretically or by using models. However, the stability of extrapolations for such multiparameter equations requires attention.

The functions $F_1(N)$ and $F_2(Z)$ in the solutions (F), (G), (H), and (I) of Eqs. (9) may be considered as shell-correction terms if $B_{eq}(N,Z)$ is taken as a liquid-drop-model-type equation. They will also account for neutron and for proton pairing effects if these are not already included in $B_{eq}(N,Z)$. Shell correction terms with the above structure are occasionally introduced phenomenologically into mass equations to better account for the nn and pp interactions.

For some of the solutions $B_{sol}(N,Z)$ of Eq. (9) the multiparameter terms have the character of expansion coefficients of a local Taylor expansion of the atomic mass surface as a function of neutron and proton number. This has been studied, for example, for the solutions of the homogeneous relation (G), and it was found[2] that the leading correction term *not* included in the solutions is proportional to $(N-Z)^3$. Long-range extrapolations of the data do indeed support the conclusion that higher-order effects in isospin are important.

The functions $F_1(N)$ and $F_2(Z)$ in the homogeneous solution (G) account for the Coulomb energy difference between isobaric analog states.[1] If, on the other hand, $B_{\text{eq}}(N,Z)$ in the inhomogeneous solution includes a Coulomb energy term, then $F_1(k)$ and $F_2(k)$ must be equal. This provides a test for the goodness of the Coulomb term. Similarly, if $B_{\text{sol}}(N,Z)$ of Eqs. (9) is valid for neutron-rich *and* proton-rich nuclei, charge symmetry of nuclear forces imposes another condition on the functions $F_i(k)$ which must be satisfied and can be used as a test.

(4) Useful information about binding energy systematics may also be obtained if in some of the relations of Eqs. (6) the nucleon binding energies B_n and B_p are replaced by two-nucleon binding energies B_{2n} and B_{2p} as defined in Eq. (4).

(5) A simple liquid-drop-model mass equation has been used in this work as the associated mass equation for the hybrid relations. More sophisticated equations could be used, presumably leading to improved results. However, greatly improved results would reflect the influence of the improved equation and not necessarily that of a more useful hybrid relation.

(6) Comay and Kelson[9,11] have studied extensively the propagation of errors in the repeated application of mass relations. This leads to an additional criterion, not considered in the present work, for the usefulness of a mass relation. The authors[9] introduced a procedure which tests the stability or instability encountered in the repeated application of a relation. An inaccuracy for one nucleus will propagate linearly for (G), but will increase exponentially with fluctuating sign for (E-2) and (E-3). Significantly, an inaccuracy for one nucleus will not increase in the repeated application of (F).

(7) *Predicted* masses have been used[3] for comparing and testing mass relations. This is not entirely unreasonable. In fact, using the mostly predicted masses for very neutron-rich nuclei of Viola *et al.*[13] with $Z \leq 82$ and $N \geq 126$ (to avoid shell effects), one obtains the standard deviations shown in the last column of Table I. The relative values are in reasonable agreement with other results, giving the largest standard deviations for Eqs. (E-1), (E-2), and (E-3). However, the absolute values are increased. This reflects upon problems with the underlying *predicted* mass values, not with the relations. Individual residuals are displayed in Fig. 3 for two examples. Besides the increased residuals of (E-1) relative to those of (G), systematic variations of α-decay energies and pairing energies are clearly present. Similar effects are observed for most of the other relations. Binding energy relations could actually be used to decrease the "roughness" of a predicted mass surface. Another related problem arises in the comparison[3] between binding energies which are calculated from equations[1,14] and binding energies which are derived from nuclear energy systematics.[13] No conclusion can be drawn about the overall preference of one over the other.

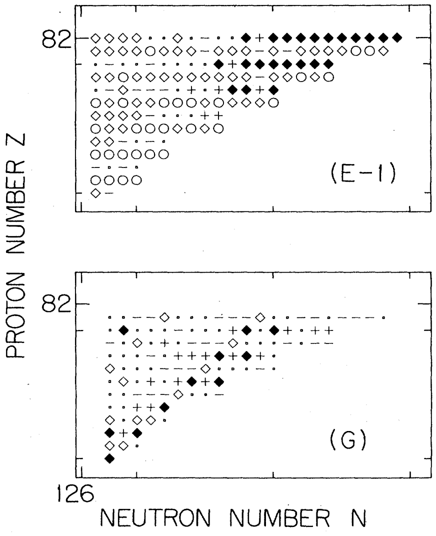

FIG. 3. Residuals for two binding energy relations for nuclei near ^{208}Pb. Predicted mass values (Ref. 13) and an associated liquid-drop-model mass equation (Ref. 7) are used. The symbols of Fig. 2 are used for the energy ranges of the residuals.

VIII. CONCLUSIONS

Statistical properties and systematic effects of several hybrid binding energy relations have been studied, and the respective solutions of the partial difference equations are given. A recently published hybrid relation[3] is found to have significant relation inaccuracies, particularly at major shell crossings. A simple hybrid relation involving the effective neutron-proton interaction is found to have relation inaccuracies as small as the transverse and longitudinal Garvey-Kelson relations,[1] which justifies its use as recursion relation.

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[1] G. T. Garvey, W. C. Gerace, R. L. Jaffe, I. Talmi, and I. Kelson, Rev. Mod. Phys. **41**, S1 (1969); G. T. Garvey, Annu. Rev. Nucl. Sci. **19**, 433 (1969).

[2] J. Jänecke and E. Comay, Nucl. Phys. **A436**, 108 (1984).

[3] L. Satpathy and R. Nayak, Phys. Rev. Lett. **51**, 1243 (1983).

[4] J. Jänecke and H. Behrens, Phys. Rev. C **9**, 1276 (1973).

[5] A. S. Jensen, P. G. Hansen, and B. Jonson, Nucl. Phys. **A431**, 393 (1984).

[6] J. E. Monahan and F. J. D. Serduke, Phys. Rev. C **15**, 1080 (1977); **17**, 1196 (1978).

[7] J. Jänecke and B. P. Eynon, Nucl. Phys. **A243**, 326 (1975); At. Data Nucl. Data Tables **17**, 467 (1978).

[8] A. H. Wapstra, 1981 Mass Evaluation (unpublished).

[9] E. Comay and I. Kelson, Phys. Rev. C **30**, 1017 (1984).

[10] E. Comay and I. Kelson, At. Data Nucl. Data Tables **17**, 463 (1978).

[11] E. Comay and I. Kelson, Z. Phys. A **310**, 107 (1984).

[12] M. S. Antony and A. Pape, Phys. Rev. C **30**, 1286 (1984).

[13] V. E. Viola, J. A. Swant, and J. Graber, At. Data Nucl. Data Tables **13**, 35 (1974).

[14] W. D. Myers and W. J. Swiatecki, Nucl. Phys. **81**, 1 (1966); W. D. Myers, *Droplet Model of Atomic Nuclei* (Plenum, New York, 1977).