Pion-helium potential by inverse scattering method

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The inverse scattering formalism is applied to the case of the pion-helium system to obtain the real and imaginary parts of a local phenomenological potential from π^--^{4}He partial wave phase shifts at a fixed energy. The phase shifts and inelasticities for all partial waves up to angular momentum $l=9$ are extracted from the differential cross-section data by parametrizing the nuclear amplitude. The method has been tried for a number of incident pion energies between 51 and 110 MeV. The real and imaginary parts of the potential are seen to compare well with the commonly used pion-nucleus Laplacian potential.

I. INTRODUCTION

In the last decade the study of the interaction of pions with nuclei has been of much theoretical and experimental interest. With the availability of high intensity pion beams from the new generation of meson factories, accurate and extensive experiments on the elastic scattering of low to medium energy pions by nuclei have been performed. The data have been analyzed using pion-nucleus optical potentials, like the Kisslinger potential[1] or some variants of it.[2] Such optical potentials are usually expressed in terms of the densities of protons and neutrons in the target nuclei. They contain several free parameters which are optimized to give the best fit to the scattering, as well as to the pionic atom data. Such model potentials may not reveal the true information content of the data, as sometimes several sets of the free parameters can reproduce good agreement with experiment. An attempt to construct a model-independent local potential has been reported by Friedman.[3] He analyzes the data on the elastic scattering of pions by nuclei in the low to intermediate energy range with a complex local potential which is parametrized by a Fourier-Bessel series. In this paper we pursue the same goal, i.e., to obtain a local potential directly from the scattering data without the explicit use of a model for the interaction. We rely on the use of local potentials. They reproduce the gross features of the experimental data quite well.

Coming to the problem of obtaining a potential directly from the scattering data, several alternative procedures[4] have been suggested. We follow the inverse scattering formalism suggested by Newton[5] to calculate the potential from the partial wave phase shifts at a fixed energy. We have tested a modified form of Newton's method[6] in the case of simple square well potentials which are real. In this paper we generalize the formalism to include potentials and phase shifts which are complex and apply this to a simple physical problem, i.e., to determine a local interaction for the π^--^{4}He system using the experimental phase shifts at a fixed energy.

Theoretically, only an infinite set of phase shifts can contain the necessary information for reproducing the true interaction. Furthermore, small errors in the phase shifts introduce large errors in the reconstructed potential. So we need to know a large number of partial wave phase shifts with the best accuracy possible in order to be able to reproduce the true interaction to some degree of confidence. The objective of this paper is, therefore, twofold. The first is to extract the largest possible number of significant phase shifts with sufficient accuracy from the scattering cross section data. This is achieved by the use of the conformal mapping technique of Cutkosky and Deo;[7] the procedure is described briefly in Sec. II. The second objective is to construct the π^--^{4}He optical potential using the phase shifts. Section III gives an outline of the inverse scattering formalism generalized to accommodate complex forms for the potential and the phase shift. The modification needed to take into account the presence of the Coulomb force is also discussed. In Sec. IV we present the resulting potentials for several values of incident pion energy. A discussion on the salient features of the resulting potentials is also included, and the results are compared with the standard Laplacian potentials.[2]

II. PHASE SHIFT ANALYSIS

The differential cross section for pion-nucleus elastic scattering is

$$\frac{d\sigma}{d\Omega}(\theta)=|f_c(\theta)+f_N(\theta)|^2 . \tag{1}$$

$f_c(\theta)$ is the pure Coulomb scattering amplitude, given by[8]

$$f_c(\theta)=f_c^B\exp\left\{2i\left[\sigma_0-\xi\ln\sin\left[\frac{\theta}{2}\right]\right]\right\} . \tag{2}$$

Here the effective Coulomb coupling constant

$$\xi=\frac{z_1z_2\alpha[s-(m^2+M^2)]}{[s-(m+M)^2]^{1/2}[s-(m-M)^2]^{1/2}} ; \tag{3}$$

z_1 and z_2 are the charges and m and M are the masses of the projectile and the target, respectively; s is the square of the total energy in the center of mass system; and α is the fine structure constant. The Born term is

$$f_c^B=\frac{2\xi k}{t}F(\theta) . \tag{4}$$

$F(\theta)$ is a form factor containing information about the finite sizes of the pion and the target nucleus. In Eq. (4) k is the c.m. momentum and t is the momentum transfer.

The nuclear amplitude $f_N(\theta)$ of Eq. (1) can be expressed as

$$f_N(\theta)=\sum_{l=0}^{\infty}\frac{2l+1}{2ik}e^{2i\sigma_l}(\eta_l e^{2i\delta_l}-1)p_l(\cos\theta)$$
$$=\sum_{l=0}^{\infty}b_l p_l(\cos\theta)\ . \tag{5}$$

The known Coulomb shift σ_l for the lth partial wave is

$$\sigma_l=\left[\frac{1}{2i}\right]\ln\frac{\Gamma(l+1+i\xi)}{\Gamma(l+1-i\xi)}\ . \tag{6}$$

Usually the phase δ_l and the inelasticities η_l are obtained by varying them as free parameters in order to get the best fit to the cross-section data. However, with increasing energy the higher partial waves gradually become more and more significant. Thus the number of free parameters involved becomes large and the search procedure introduces more uncertainties in their optimized values. To circumvent this difficulty, the nuclear amplitude is parametrized in an optimal polynomial expansion.[7] In the absence of any pole in the pion-^{4}He scattering, the nuclear amplitude is holomorphic in the $\cos\theta=x$ plane except for a t-channel cut extending from $x_+=1+(2m_{\pi^0}^2/k^2)$ up to ∞ and a u-channel cut spreading over $-x_-$ to $-\infty$ where

$$x_-=1+\frac{(m_{^3\mathrm{He}(^3\mathrm{H})}+m_{\mathrm{p(n)}})^2}{2k^2}-\frac{(m_{\pi^\pm}^2-m_{^4\mathrm{He}}^2)^2}{2k^2s}\ .$$

The cuts are symmetrized in a W plane and subsequently mapped onto the boundary of a unifocal ellipse in the z plane. If the real and imaginary parts of the nuclear amplitude $f_N(\theta)$ are expanded in terms of the Tchebyshev polynomial $T_n(z)$ as

$$f_{NR}(\theta)=\sum_{n=0}^{\infty}a_{nR}T_n(z) \tag{7}$$

and

$$f_{NI}(\theta)=\sum_{n=0}^{\infty}a_{nI}T_n(z)\ , \tag{8}$$

the convergence of the series expansion is likely to be maximum. So the number of terms in the series (7) and (8) required for a good fit to the data will be much less than the number of significant terms of the series (5), and hence the optimization procedure with these expansions is easier and analytically more accurate than that with the expansion in terms of the Legendre polynomials.[9] Once the coefficients a_n of series (7) and (8) truncated to any desired degree of accuracy are obtained from the search procedure, δ_l and η_l can be obtained to any order by equating the now truncated series (7) and (8) with the infinite series (5) and then projecting out the various partial waves.

The number of terms in each of the expressions (7) and (8) required for a good fit is three for $E_\pi\leq 180$ MeV. Thus the total number of parameters searched is only six for the range of energies considered. This number is much less compared to the number of significant phase shifts and inelasticity parameters involved.

The number of parameters involved would have been 20 in a conventional phase shift analysis, in order to be able to extract phase shifts and inelasticity parameters up to the partial wave $l=9$. Thus 14 parameters are saved at each energy value, which is a significant achievement.

III. CALCULATION OF THE POTENTIAL FROM THE PHASE SHIFTS

The essential problem is to construct the potential when the corresponding phase shifts for different partial waves are calculated from experiment. It was first solved by Newton[10] within the framework of linear algebraic equations. He considered only real potentials. However, it is a well-known fact that real potentials cannot explain all the properties of nuclear reactions. It is necessary to consider complex potentials and complex phase shifts in order to take into account the possibility of the absorption of particles as well as the influence of closed channels on open channels. It has been shown by Coudray and Coz[11] that the construction of potentials in the real and in the complex cases is performed by solving the same set of equations.

Let $V_N(r)$ be the potential associated with the nuclear interaction which is assumed to have the range r_0 and decreases rapidly to zero for $r>r_0$. The interaction of charged particles with the nucleus contains a Coulomb potential $V_C(r)$ which has infinite range in addition to the short range nuclear potential. The radial Schrödinger equation for the pion-nucleus system can be written, in terms of the dimensionless coordinate

$$\rho=kr=\left[\frac{2\mu E}{\hbar^2}\right]^{1/2}$$

as

$$D_\rho\phi_l(\rho)=l(l+1)\phi_l(\rho)\ , \tag{9}$$

where

$$D_\rho=\rho^2\left[\frac{d^2}{d\rho^2}+1-U_N(\rho)-U_C(\rho)\right]\ ,$$
$$U(\rho)=V(r)/E\ ,$$
$$\phi_l(\rho)=rR_l(r)\ .$$

Outside the range of the nuclear interaction, i.e., for $\rho>\rho_0$ ($=kr_0$), the nuclear part of the potential may be neglected and then the wave function $\phi_l(\rho)$ in this region can be expressed as a linear combination of $F_l(\rho)$ and $G_l(\rho)$, the regular and irregular Coulomb wave functions.

$$\phi_l(\rho)=A_l[\cos\delta_l F_l(\rho)+\sin\delta_l G_l(\rho)]=A_l T_l(\rho)\ . \tag{10}$$

The "nuclear" phase shifts δ_l are obtained from experiments and may be complex. The coefficients A_l are the unknown quantities to be determined.

It has been discussed in detail by Newton[10] and by Coudray and Coz[11] that, if one defines a kernel

TABLE I. Phase shifts and inelasticities for π^--^{4}He elastic scattering as computed from analysis.

E_π in (MeV)		$l=0$	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$	$l=6$	$l=7$	$l=8$	$l=9$
51	2δ(deg)	−17.2	17.3	1.90	0.110	0.004	−0.000 39	−0.000 13	−0.000 03	−0.000 004	−0.000 006
	η	0.873	0.985	0.994	0.999	0.999	0.999 76	0.999 95	0.999 99	0.999 99	0.999 99
60	2δ(deg)	−19.5	22.1	2.74	0.186	0.008	−0.000 72	−0.0003	−0.000 07	−0.000 01	−0.000 008
	η	0.889	0.933	0.984	0.998	0.999	0.999 74	0.999 94	0.999 99	0.999 99	0.999 99
68	2δ(deg)	−21.2	25.1	3.21	0.24	0.012	−0.009	−0.0004	−0.0001	−0.000 02	−0.000 01
	η	0.921	0.863	0.972	0.998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
75	2δ(deg)	−23.3	28.9	4.49	0.416	0.026	−0.0012	−0.0009	−0.000 28	−0.000 06	−0.000 02
	η	0.906	0.813	0.953	0.995	0.999	0.999 77	0.999 95	0.999 99	0.999 99	0.999 99
110	2δ(deg)	−30.2	50.42	12.62	2.08	0.397	0.081	0.017	0.0036	0.0008	0.000 18
	η	0.695	0.600	0.811	0.959	0.992	0.998	0.9996	0.9999	0.9999	0.9999

$$K(\rho,\rho')=\sum_l C_l\phi_l(\rho)\phi_l^{(0)}(\rho') , \quad (11)$$

with

$$\phi_l^{(0)}(\rho)=F_l(\rho) ,$$

and the potential $U_N(\rho)$ by

$$U_N(\rho)=-\frac{2}{\rho}\frac{d}{d\rho}[\rho^{-1}K(\rho,\rho)] , \quad (12)$$

the wave functions $\phi_l(\rho)$ satisfy the integral equation

$$\phi_l(\rho)=\phi_l^{(0)}(\rho)-\int_0^\rho K(\rho,\rho')\rho'^{-2}\phi_l^{(0)}(\rho')d\rho' . \quad (13)$$

It turns out that the kernel $K(\rho,\rho')$ is the unique solution of the Gel'fand-Levitan integral equation.

Substituting (11) into (12) we get a coupled system of linear algebraic equations, namely

$$\phi_l(\rho)=\phi_l^{(0)}(\rho)-\sum_{l'=0}^{\infty} C_l L_{ll'}(\rho)\phi_{l'}(\rho) , \quad (14)$$

where the matrix $L_{ll'}$ is

$$L_{ll'}(\rho)=\int_0^\rho \phi_l^{(0)}(\rho')\phi_{l'}^{(0)}(\rho')\rho'^{-2}d\rho' .$$

Equivalently (14) can be written as

$$\sum_{l=0}^{\infty}[\delta_{ll'}T_{l'}(\rho)A_{l'}+L_{ll'}(\rho)T_{l'}(\rho)b_{l'}]=F_l(\rho) , \quad (15)$$

where we have abbreviated $b_l=c_lA_l$. The unknown coefficients A_l and b_l which may, in general, be complex, are determined by solving (15) at two radial distances $\rho=\rho_1,\rho_2$ ($>\rho_0$); the kernel $K(\rho,\rho)$ and the nuclear part of the interaction $U_N(\rho)$ are then obtained from (11) and (12), respectively. The computational details are given in Ref. 6, so we will not reproduce them here.

The Coulomb wave functions F_l,G_l and their derivatives are calculated by using the series expansion method of Froberg.[12] The conditions for the series expansion to be valid are $\eta\rho<50$ and $\rho<10$. For $\rho>10$ we have used the asymptotic expansion method.[13]

IV. RESULTS AND DISCUSSION

The differential cross-section data for negative pions scattered from ^{4}He nuclei have been analyzed at five incident pion energies between 51 and 110 MeV. The resulting phase shifts and inelasticities for the first ten partial waves (up to $l=9$) are given in Table I. Partial waves of angular momentum higher than this contribute insignificantly to the amplitude in the energy region considered in this work. We have not searched for the error limits in these parameters, but they have been shown in Refs. 9 and 14 to be small.

Potentials are calculated at these five incident pion energies by the inverse scattering method using the formulae noted above. The resulting potential curves at different energies have been shown in Figs. 1(a)–(e) along with the commonly used Laplacian potential.[2]

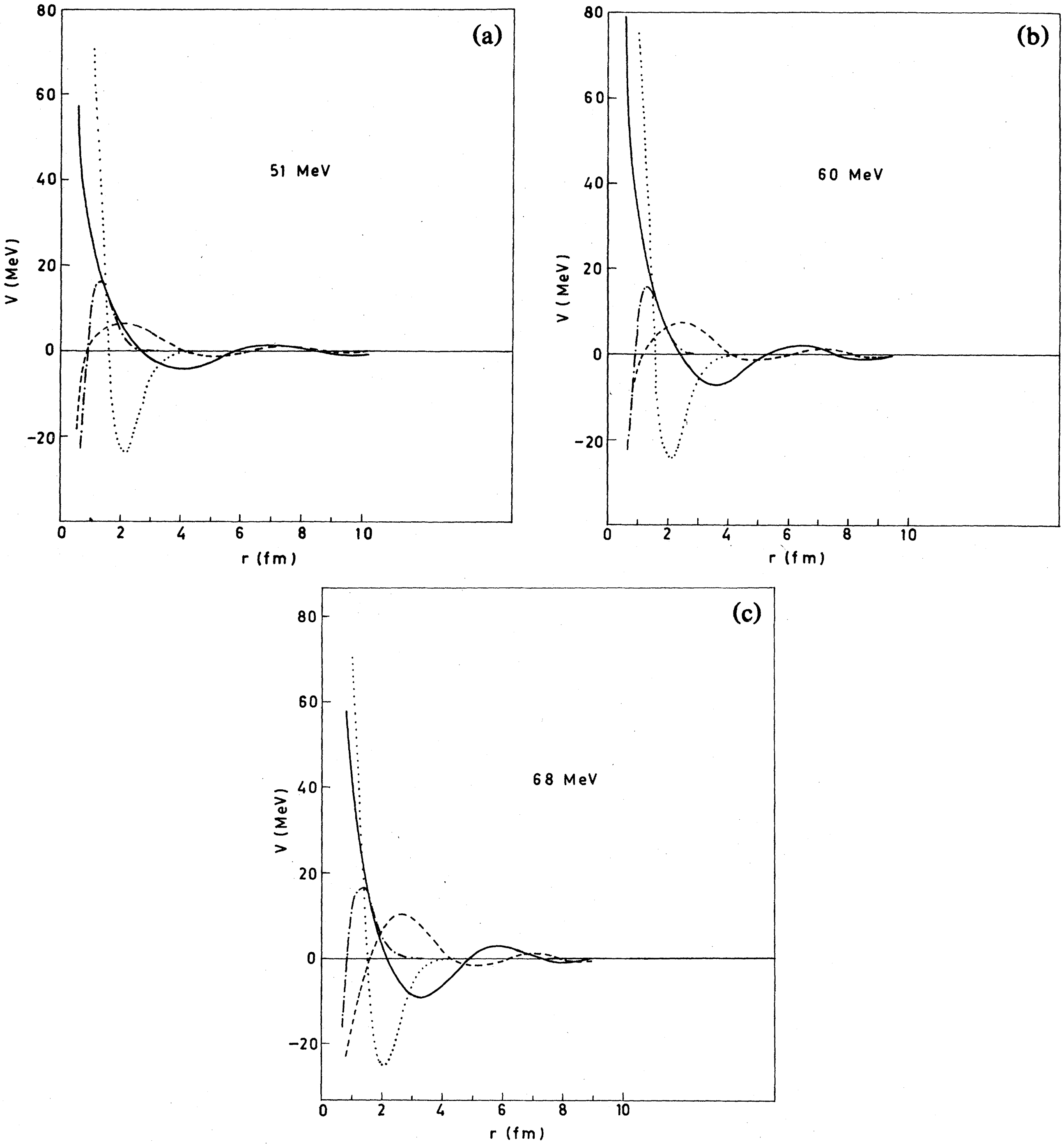

FIG. 1. (a)—(e): π^{-}-^{4}He optical potential. The dotted curve is the real part of the Laplacian potential. The solid curve is the real part of the inversion potential. The dashed-dot curve is twice the negative of the imaginary part of the Laplacian potential. The dashed curve is twice the negative of imaginary of the inversion potential.

Two of the commonly used fundamental potentials which describe the pion-nucleus interaction are the Kisslinger potential and the Laplacian potential. Both explicitly contain terms which originate in the p-wave pion-nucleon interaction. The Laplacian potential, being a local one, may be directly compared with our "inversion" potential. The Laplacian potential is given by

$$V(r)=\frac{1}{2E}[q(r)-k^2\alpha(r)-\tfrac{1}{2}\nabla^2\alpha(r)]\ , \tag{16}$$

where k is the c.m. momentum and E the c.m. total energy of the pion. The term $q(r)$ results from the s-wave part and $\alpha(r)$ results from the p-wave part of the pion nucleon interaction. q and α are expressed in terms of the nuclear densities of the target nucleus.[15]

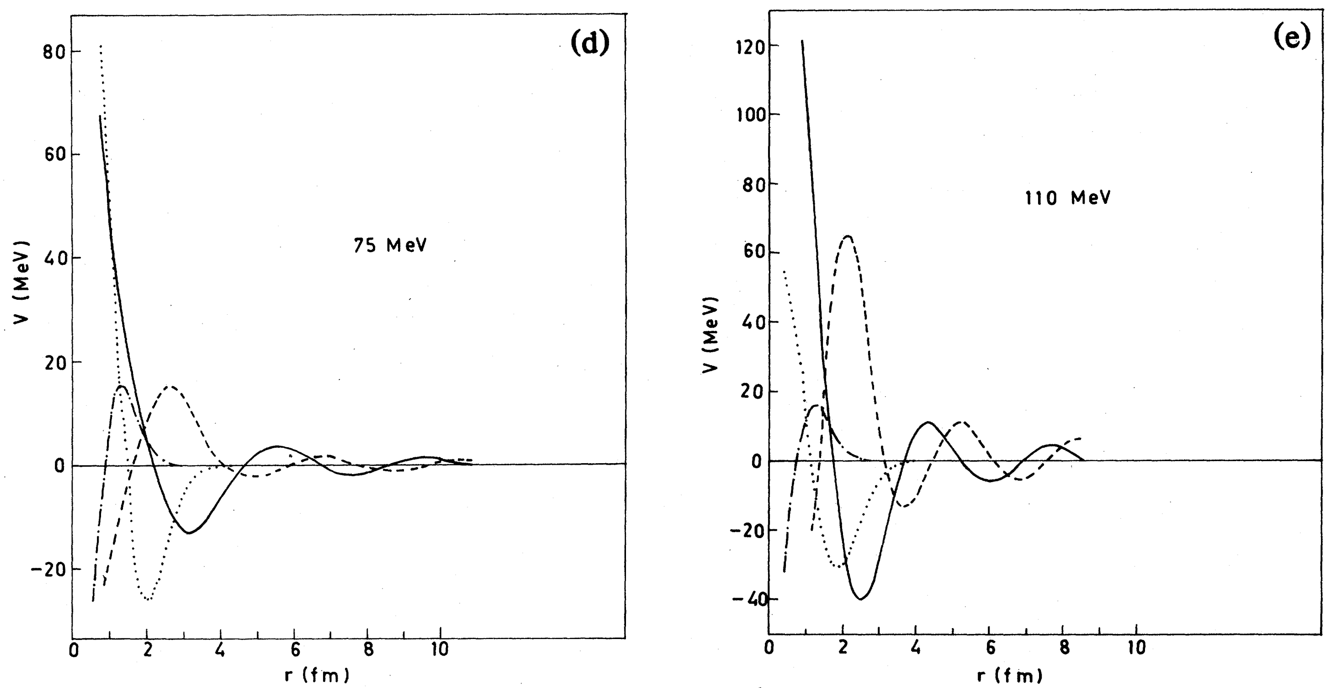

FIG. 1. (*Continued*).

$$q(r)=-4\pi\left\{\left[1+\frac{\mu}{M}\right][b_0(\rho_n+\rho_p)+b_1(\rho_n-\rho_p)]+\left[1+\frac{\mu}{2M}\right]4B_0\rho_n\rho_p\right\} \tag{17}$$

and

$$\alpha(r)=4\pi\left\{\left[1+\frac{\mu}{M}\right]^{-1}[c_0(\rho_n+\rho_p)+c_1(\rho_n-\rho_p)]+\left[1+\frac{\mu}{2M}\right]^{-1}4C_0\rho_n\rho_p\right\}, \tag{18}$$

where M is the nuclear mass; μ is the rest mass of the pion, and ρ_n and ρ_p are the densities of neutron and proton distribution in the nucleus. For the real parameters b_0, b_1, c_0, and c_1 and the complex parameters B_0 and C_0 we have used the values obtained in Ref. 15 from fits to the pionic atom data. They are listed in Table II. For ^{4}He we have used the simple density distribution function ($\rho_n=\rho_p=\rho$)

TABLE II. Parameters of π^--^{4}He optical potentials.

Parameters	Values
$b_0(m_\pi^{-1})$	-0.024 ± 0.005
$b_1(m_\pi^{-1})$	-0.086 ± 0.004
$\mathrm{Re}B_0(m_\pi^{-4})$	-0.017 ± 0.025
$\mathrm{Im}B_0(m_\pi^{-4})$	0.049 ± 0.002
$c_0(m_\pi^{-3})$	0.21
$c_1(m_\pi^{-3})$	0.061 ± 0.017
$\mathrm{Re}C_0(m_\pi^{-6})$	-0.048 ± 0.004
$\mathrm{Im}C_0(m_\pi^{-6})$	0.031 ± 0.004

$$\rho(r)=\frac{2}{\pi^{3/2}a^3}e^{-r^2/a^2},$$

where the constant a is adjusted to fit the rms nuclear radius.

The main features of the inversion potential obtained in the present work can be summarized as follows:

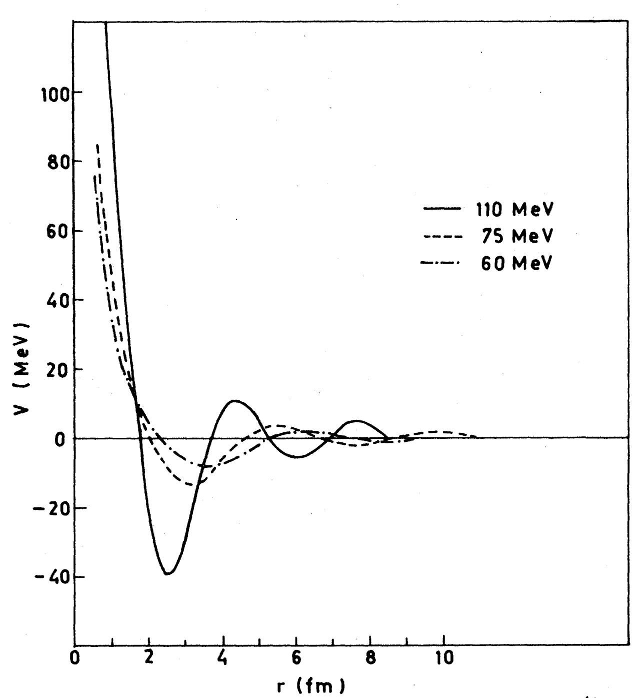

FIG. 2. The real part of the inversion potential for π^--^{4}He at the incident pion energies of 60, 75, and 110 MeV.

(a) In the range of energies considered, the real part of the π^--^{4}He optical potential has the typical shape of being attractive at large radii and repulsive at short distances. There is a distinctive repulsive core at all energies. The oscillations at the tail end of the potential are a characteristic feature of the inversion procedure. They arise due to the finite size of the matrices used in the computation rather than the infinite dimensional matrices that occur in the theory. For this reason we get nonzero values for the potential in some cases for values of r for which there is essentially no nuclear matter.

(b) The strength of the attractive part of the real potential increases with energy.

(c) The point where the real potential crosses the zero value moves to smaller radii as the energy increases. This means that the radius of the repulsive core decreases with energy. Figure 2 clearly shows the energy dependence of the real potential.

(d) The imaginary potential increases with increasing energy. This agrees with the observation of an increase in the inelastic cross sections at higher energies.

It is easily seen that the Laplacian potential shown for comparison in Figs. 1(a)—(e) does indeed possess the characteristics listed above. Similar features of the pion-nucleus potential have also been noted by Friedman[3] in a model-independent analysis of the pion-nucleus scattering, using an unbiased Fourier-Bessel series for the potential.

It is quite satisfying that the inverse scattering which directly uses the experimental data is able to reproduce the essential features of the pion-nucleus optical potential. However, the inversion potential differs from the standard Laplacian potential in its finer details, and it may be interesting to see how these affect other pionic processes in nucleus. One such interesting feature, as can easily be seen in Figs. 1(a)—(e), is that the inversion potential is much stronger at large distances than in the Laplacian version. This should affect pionic reactions such as inelastic scattering of pions. This is being looked into and will be reported on later.

In this work we have assumed a local pion-nucleus potential. Most of the attempts to study the information content of the experimental results on pion-nucleus scattering by way of large scale fits to the data at present rely on the use of local potentials, mainly for practical reasons. However, if we look at our results, it is obvious that the inversion potential is in strong disagreement which lessens near the Δ resonance. This result agrees with the observation in the isobar-doorway model that the potential is very nonlocal at low energies and becomes almost local at the Δ resonance.[16] This suggests the importance of a nonlocal formulation of the inverse scattering method which should be studied carefully in the future.

[1] L. S. Kisslinger, Phys. Rev. **98**, 761 (1955).
[2] G. Fäldt, Phys. Rev. C **5**, 400 (1972).
[3] E. Friedman, Phys. Rev. C **28**, 1264 (1983).
[4] K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory* (Springer, New York, 1977).
[5] R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966).
[6] B. B. Deo, S. Jena, and S. Swain, J. Phys. A **17**, 2767 (1984).
[7] R. E. Cutkosky and B. B. Deo, Phys. Rev. **174**, 1859 (1968).
[8] A. Messiah, *Quantum Mechanics* (Wiley, New York, 1966), p. 424.
[9] K. M. Das and B. B. Deo, Pramana **23**, 91 (1984).
[10] R. G. Newton, J. Math. Phys. **3**, 75 (1962).
[11] C. Coudray and M. Coz, Ann. Phys. (N.Y.) **61**, 488 (1970).
[12] C. E. Froberg, Rev. Mod. Phys. **27**, 399 (1955).
[13] M. Abramowitz and A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).
[14] K. M. Crowe, A. Fainberg, J. Miller, and A. S. L. Parsons, Phys. Rev. **180**, 1349 (1969); F. Binon, P. Duteil, M. Gouanere, L. Hugon, J. Jansen, P. Lagnaux, H. Palevsky, J. P. Peigneux, M. Spighel, and J. P. Stroot, Nucl. Phys. **A298**, 499 (1978).
[15] C. J. Batty, E. Friedman, and A. Gal, Nucl. Phys. **A402**, 411 (1983).
[16] A. N. Saharia, R. M. Woloshyn, and L. S. Kisslinger, Phys. Rev. C **23**, 2140 (1981).