Symmetry-conserving higher-order interaction terms in the interacting boson model

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Symmetry-conserving higher-order terms for the interacting boson model are constructed. Their eigenvalues are explicitly derived within the SU(3) limit. The influence of these interaction terms on the theoretical energy spectrum of several nuclei is discussed. The classical limit of these operators is derived and interpreted.

I. INTRODUCTION

The original interacting boson model (IBM), initially introduced by Arima and Iachello,[1-3] has been rather successful in describing collective properties of several medium- and heavy-mass nuclei. In the first instance only $L=0$ (s) bosons and $L=2$ (d) bosons are considered. This specific version of the model has an inherent U(6) group structure associated with it. In the framework of this boson model, a dynamical symmetry arises whenever the Hamiltonian H can be written in terms of invariants only of maximal subgroups $\mathrm{G}\subset\mathrm{U}(6)$. Such H gives rise to splitting but not to mixing of the various G representations. The subgroups $\mathrm{G}\equiv\mathrm{U}(5)$ (Ref. 1), $\mathrm{G}=\mathrm{SU}(3)$ (Ref. 2), and $\mathrm{G}\equiv\mathrm{O}(6)$ (Ref. 3) can be considered in the IBM. Arima and Iachello made the choice to withhold for each of these cases only up to two-boson interaction terms in H.

Several extensions of the original IBM have been introduced during the past years. Negative-parity states are described by coupling f bosons to the s and d ones.[4] The addition of an extra g,[5] s', and d' boson[6] was necessary for the explanation of certain experimentally observed bands. Heyde *et al.*[7] added to the IBM Hamiltonian cubic terms or three-boson interaction terms, in order to describe nuclei with triaxial features. They studied the influence of such terms on the energy spectrum in each of the three dynamical symmetries. By the introduction of specific higher-order interaction terms, various G representations are mixed. Moreover, the dynamical symmetries of the IBM are broken up, so that the eigenvalue problem of H cannot be solved analytically.

In this paper we shall show that higher-order terms, which conserve the dynamical symmetry, can be added to the Hamiltonian in a natural way. Such operators should belong to the integrity basis of O(3) scalar operators in the G [G=U(5), SU(3), or O(6)] enveloping algebra. The extension of H with higher-order terms in the SU(3) limit will be considered in detail, because the number of O(3) scalars in the integrity bases is limited and the eigenvalue derivation for these operators is straightforward. To obtain a more intuitive insight, the classical limit of the added terms will be calculated. Some comments will be given concerning the third- and fourth-order symmetry-conserving operators in the U(5) and O(6) limits.

II. THIRD-ORDER OPERATORS IN THE SU(3) LIMIT

In the SU(3) limit, the IBM Hamiltonian is written in terms of two O(3) scalars,[2] i.e., the angular momentum operator L^2 and the second-order SU(3) Casimir operator I_2:

$$H=\alpha L^2+\beta I_2\ . \tag{2.1}$$

Both operators can be expressed in terms of the SU(3) generators, which, conforming with the notation used in previous papers,[8-10] will be denoted by $l_\pm$, l_0, and q_μ ($\mu=-2,-1,\ldots,+2$). So L^2 and I_2 read

$$L^2=l_+l_-+l_0(l_0-1) \tag{2.2}$$

and

$$I_2=\frac{1}{36}\left[3L^2+\sum_\mu(-1)^\mu q_\mu q_{-\mu}\right]\ . \tag{2.3}$$

Both L^2 and I_2 are diagonal into the Elliott basis[11] $|(\lambda,\mu),K,l,m\rangle$, and their eigenvalues are given by

$$\langle L^2\rangle=l(l+1)\ , \tag{2.4}$$

$$\langle I_2\rangle=[(\lambda+\mu+3)(\lambda+\mu)-\lambda\mu]/9\equiv C(\lambda,\mu)/9\ . \tag{2.5}$$

Adding third-order terms in the SU(3) generators to H, (2.1), under the restriction of SU(3) symmetry conservation, yields in first instance the construction of the operator integrity basis for O(3) scalars in the SU(3) enveloping algebra. The authors[12] have proved, with the help of the shift operator technique, that only two independent third-order operators can be constructed, i.e., the third-order Casimir operator I_3 and the so-called O(3) scalar shift operator O_l^0, which will be denoted in this paper by Ω. These operators read[10]

$$I_3 = \frac{1}{2\times 36^2}[4(q_0^2 - 3q_{+1}q_{-1} - 6q_{+2}q_{-2})q_0 + 6\sqrt{6}(q_{+2}q_{-1}^2 + q_{-2}q_{+1}^2) - 9\sqrt{6}(q_{-2}l_+^2 + q_{+2}l_-^2)$$
$$-18\sqrt{6}(l_0 - 1)q_{-1}l_+ + 18\sqrt{6}(l_0+1)q_{+1}l_- + 18(L^2 - 3l_0^2 + 3l_0 + 10)q_0] \quad (2.6)$$

and

$$\Omega = \sqrt{6}l(l+1)q_0 - 3(q_{-1}l_+ + q_{+1}l_-) - 3(q_{-2}l_+^2 + q_{+2}l_-^2) , \quad (2.7)$$

where the given form of Ω may be applied to $m=0$ states only. Since the Ω eigenvalues are m independent,[8–10] this restriction does not affect further discussions. In the Elliott basis I_3 is diagonal and its eigenvalue is[10]

$$\langle I_3 \rangle = \tfrac{1}{162}(\lambda-\mu)(2\lambda+\mu+3)(\lambda+2\mu+3) . \quad (2.8)$$

The operator Ω is not diagonal in the Elliott basis. Its eigenvalues have been calculated by several authors either analytically, but most of the time numerically, for many cases of physical interest. In a separate paper[13] two of us will discuss a new method for the Ω eigenvalue determination, based upon basis states $|K,l\rangle\rangle$, which are related to the Elliott states by

$$|K,l\rangle\rangle = c(K,l)\,|(\lambda,\mu),K,l,0\rangle , \quad (2.9)$$

where the $c(K,l)$ are the Elliott coefficients.[11] In this basis one can show that for $\lambda \ge \mu$,

$$\Omega\,|K,l\rangle\rangle = \sum_{K'=K,K\pm 2} \Omega_{K'K}\,|K'l\rangle\rangle , \quad (2.10)$$

with

$$\Omega_{KK} = \sqrt{6}(\mu+2\lambda+3)[l(l+1) - 3K^2] , \quad (2.11)$$

and

$$\Omega_{K\pm 2,K} = -3[3(\mu\mp K)(\mu\pm K+2)(l\pm K+2)(l\pm K+1)(l\mp K)(l\mp K-1)/2]^{1/2} . \quad (2.12)$$

Since

$$|-K,l\rangle\rangle = (-1)^{\lambda+\mu+l}\,|K,l\rangle\rangle , \quad (2.13)$$

the relations (2.10)–(2.12) allow one to derive by a diagonalization procedure the eigenvalues of Ω for every SU(3) representation. For the interesting representations, $(\lambda,0)$ and $(\lambda,2)$, which play an important role in the SU(3) limit of IBM, the following expressions for the Ω eigenvalues are obtained:

(i) for $(\lambda,0)$: $\langle\Omega\rangle = \sqrt{6}l(l+1)(2\lambda+3)$, (2.14)

(ii) for $(\lambda,2), l=0$: $\langle\Omega\rangle = 0$, (2.15)

l odd: $\langle\Omega\rangle = \sqrt{6}[l(l+1) - 12](2\lambda+5)$, (2.16)

l even: $\langle\Omega\rangle = \sqrt{6}\{(l-2)(l+3)(2\lambda+5) \pm 6[l(l+1)(l-1)(l+2) + (2\lambda+5)^2]^{1/2}\}$. (2.17)

It is worthwhile to mention that these results have already been obtained piecemeal by several authors.[9,14,15]

III. RESULTS OF THE EXTENDED IBM

In this section we report on a study of two series of nuclei, where many of the positive-parity bands can be explained in the framework of the IBM, i.e., the Gd ($154 \le A \le 158$) and the Er isotopes ($162 \le A \le 166$). Our aim is to compare for levels belonging to the ground-state, β, and γ bands the results of the original IBM with those of the extended IBM version, where the following Hamiltonian is considered:

$$H^{(3)} = \alpha L^2 + \beta I_2 + \gamma\Omega . \quad (3.1)$$

Since we restrict our analysis to the $(\lambda=2N, \mu=0)$ and $(\lambda=2N-4, \mu=2)$ SU(3) irreps (irreducible representations), the introduction of an interaction term proportional to I_3 in (3.1) is of no practical importance. The operators I_2 as well as I_3 determine the energy difference between the above-mentioned representations. This means that the introduction of the third-order Casimir operator in (3.1) results in a rescaling of the β parameter.

For each of the nuclei considered the relevant parameters are obtained from a least-squares fit to all known excitation energies of the levels belonging to the ground-state, β, and γ bands. The derived values are given in Table I, together with the rms deviation defined by

$$d_{\text{rms}} = [(E_{\text{calc}} - E_{\text{exp}})^2/(N-P)]^{1/2} , \quad (3.2)$$

where N is the number of levels taken into account and P is the number of free parameters. For each nucleus the agreement between theory and experiment is improved by adding the Ω interaction term. There is a smooth variation of the parameter values within a given isotope series, although the fact that in certain nuclei the number N of known experimental levels is extremely low makes detailed comparison rather difficult. In Fig. 1 we present the comparison between experimental and calculated spectra for the Gd isotopes and in Fig. 2 the results for the Er isotopes are shown. Because the eigenvalues (2.4) and (2.5) do not depend on K, states of the β and γ bands with identical even angular momentum ($l=2,4,\ldots$) are in the original IBM degenerate. In our extended version this degeneracy is lifted due to the Ω term and the theoretical results are closer to the experimental situation. One should, however, realize that the wave functions describing these specific states are no longer Vergados wave functions[16] as the case was in the original IBM. The wave functions in our extended version are the eigenvectors of the Ω operator as defined in (2.9) and (2.10) for a $(\lambda,2)$SU(3) irrep. One easily derives that

$$|(\lambda,2),l^{\text{even}},\pm\rangle=[z_1\,|(\lambda,2),K=0,l^{\text{even}},0\rangle + z_2^{\pm}\,|(\lambda,2),K=2,l^{\text{even}},0\rangle]/z^{\pm}\ , \tag{3.3}$$

where the $\pm$ index corresponds with the two distinct eigenvalues (2.17) and

$$z_1=\{2l^2(l+1)(l-1)[2(\lambda+2)^2-l(l+1)]\}^{1/2}\ , \tag{3.4}$$

$$z_2^{\pm}=\{(2\lambda+5)\mp[(2\lambda+5)^2+l(l+1)(l-1)(l+2)]^{1/2}\}\times[2(\lambda+3)^2-l(l+1)]^{1/2}/2^{1/2}\ , \tag{3.5}$$

and

$$z^{\pm}=[z_1^2+(z_2^{\pm})^2]^{1/2}\ . \tag{3.6}$$

Note that in the limit of large N our solution (3.3) reduces to the Elliott states, i.e.,

$$\lim_{N\to\infty}\ |(\lambda,2),l^{\text{even}},+\rangle\to|(\lambda,2),K=0,l^{\text{even}},0\rangle\ ,$$

$$\lim_{N\to\infty}\ |(\lambda,2),l^{\text{even}},-\rangle\to|(\lambda,2),K=2,l^{\text{even}},0\rangle\ .$$

IV. CLASSICAL LIMIT OF THE Ω OPERATOR

To obtain a more intuitive insight in the geometrical interpretation of the algebraic operator Ω, its classical limit has to be calculated. This can be realized by using the technique of Dieperink *et al.*[22] and Ginocchio and Kirson.[23] For this purpose the operator form (2.7) has to be expressed in terms of s,d boson creation and annihilation operators. One can easily verify that Ω can be written in the following coupled form:

$$\Omega=-6\sqrt{5}(q\,(ll)^2)^0 = -6\sqrt{5}\sum_{\nu,\sigma,\mu}\langle 2\mu 2-\mu\,|\,00\rangle\langle 1\nu 1\sigma\,|\,2-\mu\rangle q_\mu l_\nu l_\sigma\ , \tag{4.1}$$

where l_ν represents the spherical components of the angular momentum operator. The SU(3) generators l_ν,q_μ used here and defined in Refs. 8–10 can be written in quantized form as[2]

$$l_\mu=\sqrt{10}(d^\dagger\tilde{d})^1_\mu\ , \tag{4.2}$$

$$q_\mu=2\sqrt{2}(d^\dagger\tilde{s}+s^\dagger\tilde{d})^2_\mu-\sqrt{14}(d^\dagger\tilde{d})^2_\mu\ . \tag{4.3}$$

The introduction of (4.2) and (4.3) into (4.1) and the reordering of the creation and annihilation operators so that all creation operators stand to the left of the annihilation operators result in the following Ω definition:

$$\begin{aligned}\Omega=-60\sqrt{5}\Bigg[&3\sqrt{70}\sum_{J_1J_2J}(-1)^{J_2}[(2J_1+1)(2J_2+1)]^{1/2}\begin{Bmatrix}2 & J_2 & J\\ J_1 & 2 & 1\end{Bmatrix}\begin{Bmatrix}2 & 2 & 2\\ 2 & 2 & 1\\ J_1 & J_2 & 1\end{Bmatrix}[(d^\dagger d^\dagger)^{J_1}d^\dagger]^J\cdot[(\tilde{d}\,\tilde{d})^{J_2}\tilde{d}]^J+\begin{Bmatrix}2 & 2 & 2\\ 2 & 1 & 1\end{Bmatrix}\\ &\times\Bigg[\sum_J 6\sqrt{10}\begin{Bmatrix}2 & 2 & J\\ 2 & 2 & 1\end{Bmatrix}\{s[(d^\dagger d^\dagger)^2d^\dagger]^J\cdot(\tilde{d}\,\tilde{d})^J+s^\dagger(d^\dagger d^\dagger)^J\cdot[(\tilde{d}\,\tilde{d})^2\tilde{d}]^J-\sqrt{7}(d^\dagger d^\dagger)^J\cdot(\tilde{d}\,\tilde{d})^J\}\\ &-3\sqrt{70}\sum_J\begin{Bmatrix}2 & 2 & J\\ 2 & 2 & 2\end{Bmatrix}(d^\dagger d^\dagger)^J\cdot(\tilde{d}\,\tilde{d})^J+6\sqrt{2}/\sqrt{5}[s(d^\dagger d^\dagger)^2\cdot\tilde{d}+s^\dagger d^\dagger\cdot(\tilde{d}\,\tilde{d})^2]-3\sqrt{14}/\sqrt{5}d^\dagger\cdot\tilde{d}\Bigg]\Bigg]\ .\end{aligned} \tag{4.4}$$

Note that the third-order term in $d^\dagger-\tilde{d}$ has a form comparable to the so-called cubic terms introduced by Heyde *et al.*,[7] i.e.,

$$\sum_L \theta_L(d^\dagger d^\dagger d^\dagger)^L\cdot(\tilde{d}\,\tilde{d}\,\tilde{d})^L\ . \tag{4.5}$$

Such terms however break the IBM dynamical symmetries. In reality they choose all $\theta_L=0$, with the exception of θ_3, which they retain as a parameter. Expression (4.4) shows that a symmetry-conserving operator is composed, besides

TABLE I. Values of the parameters $\alpha,\beta,(\gamma)$ occurring in the Hamiltonians (1.2) and (1.3), as derived in a least-squares fit to the known level energies in a series of Gd and Er nuclei. The rms deviation as defined in (3.2) is indicated for each of the considered Hamiltonians and N denotes the number of considered levels.

	Parameter	^{154}Gd	^{156}Gd	^{158}Gd	^{162}Er	^{164}Er	^{166}Er
N		17	23	11	8	26	10
$H^{(3)}(3.1)$	α	6.532	8.331	6.667	-6.780	-4.140	3.055
	β	-65.466	-73.196	-70.985	-56.154	-54.272	-37.411
	γ	0.071	0.032	0.044	0.148	0.107	0.056
$d_{\rm rms}$ (keV)		112.94	66.54	9.57	13.57	84.27	7.69
$H(2.1)$	α	14.245	12.156	12.447	14.698	12.223	12.580
	β	-59.407	-70.367	-68.416	-48.023	-45.647	-31.664
$d_{\rm rms}$ (keV)		127.56	72.19	32.00	119.57	175.34	14.19

lower order terms, of a specific combination of cubic terms.

Let us for the time being consider the first sum in (4.4). Introducing the short-hand notation

$$d(J_1,J_2,J)=[(d^\dagger d^\dagger)^{J_1}d^\dagger]^J\cdot[(\tilde d\,\tilde d)^{J_2}\tilde d]^J\,, \tag{4.6}$$

this first sum can be written down explicitly, by taking into account that only five linear independent combinations of type (4.6) exist, which can be uniquely characterized by J (0,2,3,4,6), i.e.,

$$\Omega(\text{first sum})=\sqrt{10}/(\sqrt{3}.269\,50)[1617d(2,2,0)+39\,930d(0,0,2)-15\,092d(2,2,3)+13\,622d(2,2,4)-43\,120d(4,4,6)]\,. \tag{4.7}$$

Van Isacker and Chen[24] have shown that the classical limit (CL) of each $d(J_1,J_2,J)$ term is given by

$$d(J_1,J_2,J)_{\rm CL}=N(N-1)(N-2)\frac{\beta^6}{(1+\beta^2)^3}(A+B\cos^2 3\gamma)\,. \tag{4.8}$$

The coefficients A and B for each specific J_1, J_2, and J combination are given in Table II. It is easy to verify that the classical limit of (4.7) disappears completely. In an analogous way one can show that the three terms of the second sum in (4.4) also have a classical limit which vanishes identically. Only the last three terms in (4.4) yield a contribution which is β and γ dependent, i.e.,

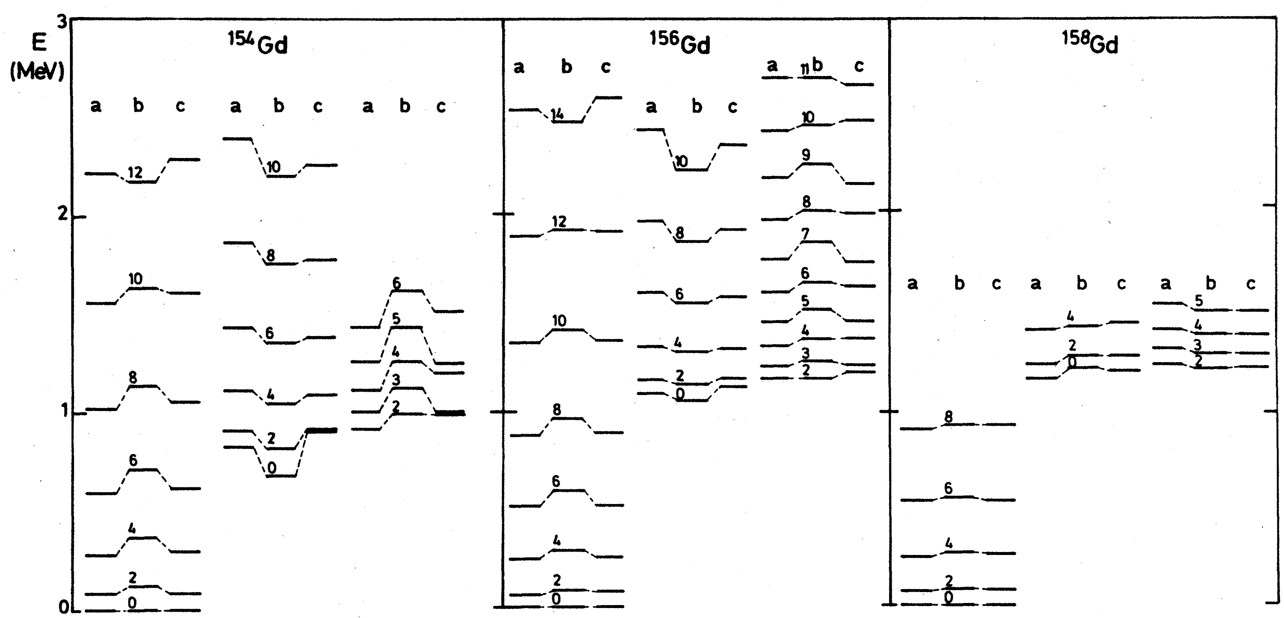

FIG. 1. Comparison between the experimental and calculated spectra for Gd isotopes ($154\le A\le 158$). "a" denotes the results obtained in the original IBM; "b" represents the experimental data; "c" gives the results in the extended IBM with $H^{(3)}$, (3.1). The experimental data for ^{154}Gd are taken from Gono and Sugihara and Sousa *et al.* (Ref. 17), for ^{156}Gd from Konijn *et al.* (Ref. 18), and for ^{158}Gd from Greenwood *et al.* (Ref. 20) and Lederer and Shirley (Ref. 19).

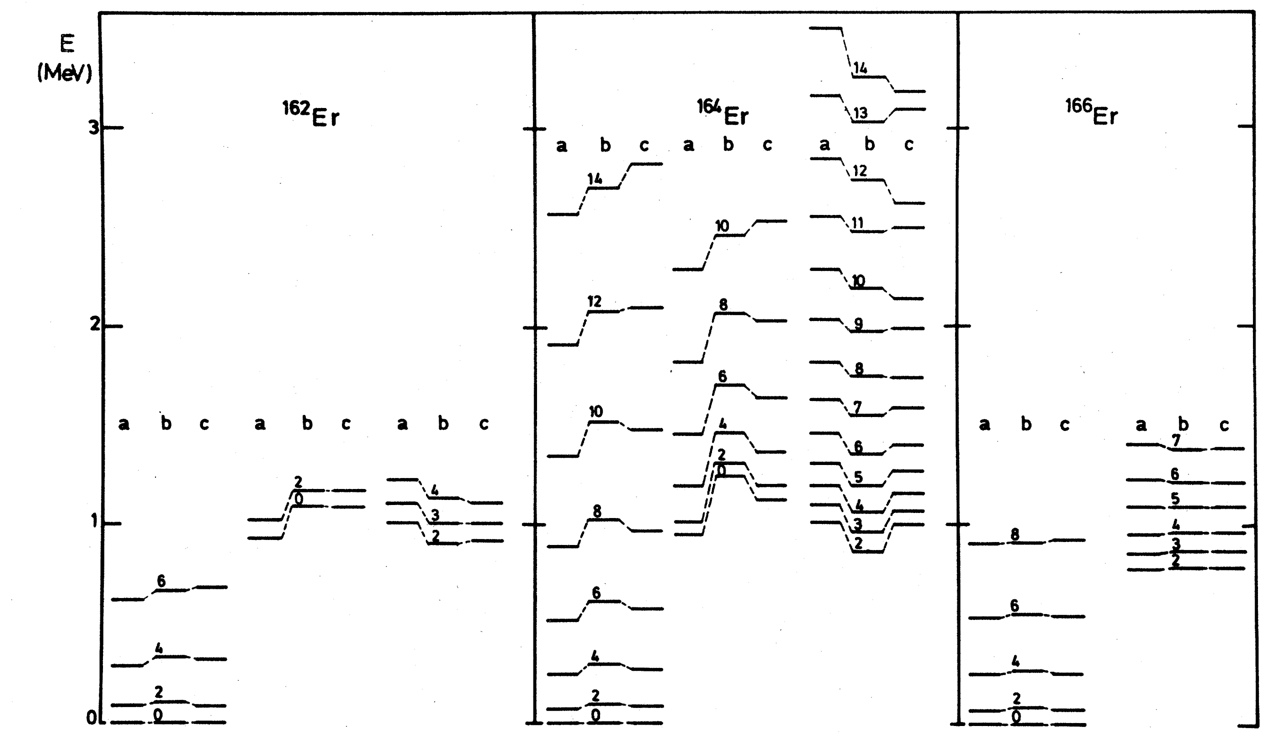

FIG. 2. Comparison between the experimental and calculated spectra for Er isotopes, $(162 \le A \le 166)$. The notations a, b, and c are explained in the caption of Fig. 1. The experimental data for ^{162}Er and ^{166}Er are taken from Lederer and Shirley (Ref. 19), for ^{164}Er from Lederer and Shirley (Ref. 21).

$$\Omega_{\mathrm{CL}}=\frac{N(N-1)}{(1+\beta^2)^2}\frac{6\sqrt{6}}{5}(\beta^4+4/\sqrt{2}\beta^3\cos 3\gamma)+\frac{42N\sqrt{3}}{5\sqrt{2}(1+\beta^2)}\beta^2 . \tag{4.9}$$

From this it is clear that $H^{(3)}$, (3.1), cannot give rise to a stable, triaxial shape of the nucleus. The fact confirms the conclusion formulated by Dieperinck *et al.*[22] that the most general Hamiltonian of the IBM model in its original formulation, cannot explain triaxiality.

V. FOURTH-ORDER OPERATORS IN THE SU(3) LIMIT

It is already known for a long time[12] that only one linear independent fourth-order O(3) scalar operator exists in the enveloping algebra of SU(3). In previous papers[8–10] we called it Q_l^0, but here we shall denote this operator as Λ. Its definition in terms of SU(3) generators is given by Van der Jeugt *et al.*[10] (again we restrict our attention to an operator form valid when acting upon $m=0$ states):

$$\Lambda=2l(l+1)q_0^{(2)}-\sqrt{6}(q_{-1}^{(2)}l_+ + q_{+1}^{(2)}l_-)-\sqrt{6}(q_{-2}^{(2)}l_+^2+q_{+2}^{(2)}l_-^2)-81l_+l_- , \tag{5.1}$$

with

$$q_\mu^{(2)}=-\frac{\sqrt{7}}{2}\sum_{\mu_1\mu_2}\langle 2\mu_1 2\mu_2 \,|\, 2\mu\rangle q_{\mu_1}q_{\mu_2} . \tag{5.2}$$

The eigenvalue of this operator has been derived for a limited number of SU(3) irreps and O(3) l values. In Ref. 13 two of us describe a new method for the Λ-eigenvalue derivation in the basis (2.9). Again one can show that for $\lambda \ge \mu$,

$$\Lambda \,|\, K,l\rangle\rangle=\sum_{K'=K,K\pm 2}\Lambda_{K'K}\,|\,K'l\rangle\rangle , \tag{5.3}$$

with

$$\Lambda_{KK}=2[l(l+1)-3K^2](2\lambda+\mu+3)^2-18K^4+6K^2[5l(l+1)-3]-3l(l+1)[4l(l+1)-3]-81l(l+1)+3(\mu-K)(\mu+K+2)[3K^2-l(l+1)]+3(\mu+K)(\mu-K+2)[3K^2-l(l+1)] , \tag{5.4}$$

and

$$\Lambda_{K\pm 2,K}=6[(\mu\mp K)(\mu\pm K+2)(l\pm K+2)(l\pm K+1)(l\mp K)(l\mp K-1)]^{1/2}[(2\lambda+\mu+3)-3(1\pm K)] . \tag{5.5}$$

TABLE II. The coefficients A and B [relation (4.4)] in the expression for the classical limit of $d(J_1,J_2,J)$ defined in (4.6). This table has been copied from Van Isacker and Chen (Ref. 24).

J	0	2	3	4	6
$J_1=J_2$	2	0	2	2	4
A	0	$\frac{1}{5}$	$-\frac{1}{7}$	$\frac{3}{49}$	$\frac{14}{55}$
B	$\frac{2}{35}$	0	$\frac{1}{7}$	$\frac{3}{35}$	$-\frac{8}{385}$

Moreover, we like to stress the fact that Ω and Λ do not commute, i.e.,

$$[\Omega,\Lambda]\neq 0\ . \tag{5.6}$$

Although Λ is the only linear independent fourth-order operator in the SU(3) enveloping algebra, it is evident that other fourth-order operators can be constructed as product forms of lower-order operators, i.e., L^4, I_2L^2, and I_2^2. The most general SU(3) conserving fourth-order Hamiltonian reads

$$H^{(4)}=\alpha L^2+\beta I_2+\gamma\Omega+\delta I_3+\mu\Lambda+\nu L^4+\tau I_2L^2+\rho I_2^2\ . \tag{5.7}$$

It is interesting to compare the results obtained for levels belonging to the ground-state, β, and γ bands derived in an extended IBM version, where $H^{(4)}$ is introduced as the Hamiltonian with the experimental spectrum of ^{156}Gd which exhibits a rich band structure. Due to (5.6), a least-squares procedure based on $H^{(4)}$ gives rise to a nonlinear set of equations, which we have solved by using the EO4FDF subroutine of the NAG (Numerical Algorithms Group) library.[25] By studying the influence of the various terms of (5.7) on the energy spectrum it became clear that I_3, I_2L^2, and I_2^2 did not change the theoretical results drastically; therefore, we have chosen $\delta=\tau=\rho=0$. The other parameter values following from the least-squares fit are

$$\alpha=12.05\ ,\quad \beta=-71.19\ ,\quad \gamma=0.020\ ,$$

$$\nu=-0.014\ ,\quad \mu=-5.72\times10^{-5}\ ,$$

while the rms derivation for this fit equals 21.67. Comparison of these last results with the ones given in Table I derived in the original IBM or in the third-order extension, shows that the α, β (and γ) values are not much changed by adding the fourth-order terms, but that the rms value is decreased enormously. In Fig. 3 we compare the theoretically obtained spectrum of ^{156}Gd with the experimental one.[18]

It is worthwhile to mention that for the nondegenerate states [i.e., the states belonging to the $(\lambda,0)$ irrep and the one belonging to the $(\lambda,2)$ irrep with odd l] the Λ eigenvalues follow directly from Λ_{KK}, (5.3):

$$\Lambda(\lambda,0)=2l(l+1)(4\lambda^2+12\lambda-6l^2-6l-27)$$
$$=2l(l+1)[4\lambda^2+12\lambda-6l(l+1)-27] \tag{5.8}$$

and

$$\Lambda(\lambda,2)^{l\ \mathrm{odd}}=2[l(l+1)-12](4\lambda^2+20\lambda-12)$$
$$+(-12l^4-24l^3+86l^2+98l-960)$$
$$=2[l(l+1)-12](4\lambda^2+20\lambda-12)$$
$$+[-12l^2(l+1)^2+98l(l+1)-960]\ . \tag{5.9}$$

Note that these eigenvalues are, besides other terms, $l^2(l+1)^2$ dependent. Similarly, the eigenvalues of Ω for this same set of states are all $l(l+1)$ dependent [see (2.14) and (2.16)]. This means that through these eigenvalues and the ones following from L^2 and L^4, $H^{(4)}$ reproduces in a certain way the previously empirically introduced three-parameter energy formula

$$E=E_0+Al(l+1)+Bl^2(l+1)^2$$

often used for the description of rotational-like bands.

VI. HIGHER-ORDER OPERATORS IN THE O(6) AND U(5) LIMITS

It is evident that in a similar way symmetry-conserving higher-order terms can be constructed in the two other dynamical symmetries of the IBM.

The generators of the O(6) algebra can be expressed in terms of d-s creation and annihilation operators:[3]

$$l_\mu=(d^\dagger\tilde{d})^1_\mu\ ,\quad p_\mu=(d^\dagger\tilde{d})^3_\mu\ ,\quad q'_\mu=(d^\dagger\tilde{s}+s^\dagger\tilde{d})^2_\mu\ . \tag{6.1}$$

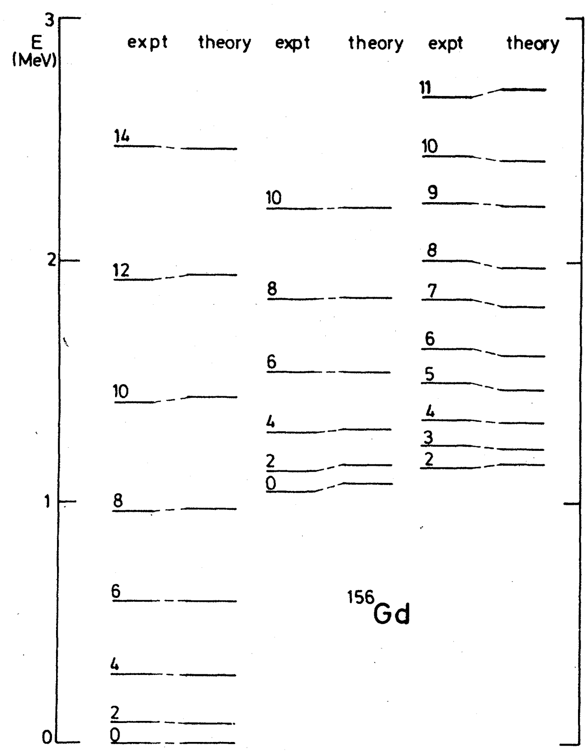

FIG. 3. Comparison between the experimental and calculated spectra for ^{156}Gd. The theoretical results are obtained in the extended IBM with $H^{(4)}$, (5.7). The experimental data are taken from Konijn *et al.* (Ref. 18).

A third-order O(6) symmetry-conserving operator can be constructed in an analogous way as it was done for Ω [see (4.1)], i.e.,

$$\Omega' = a[q'(ll)^2]^0 \quad (a:\ \text{an arbitrary constant}) . \tag{6.2}$$

To our knowledge, no systematic study has been performed to calculate eigenvalues of such an operator. Its classical limit however can be derived quite easily. The operator Ω' can be transformed to a form of the type (4.4). Due to the definition of q'_μ in terms of d-s creation and annihilation operators, it is clear that Ω' can be written as a linear combination of the second, third, and sixth term in (4.4). Only the sixth term has a nonzero classical limit, proportional to $\beta^3\cos 3\gamma$, again showing that an additional third-order symmetry conserving term does not give rise to a stable, triaxial shape of the nucleus.

The generators of the O(5) algebra can all be expressed in terms of $d^\dagger$ and $\tilde{d}$ alone, i.e.,

$$l_\mu = (d^\dagger \tilde{d})^1_\mu , \quad q''_\mu = (d^\dagger \tilde{d})^3_\mu . \tag{6.3}$$

A few years ago two of the authors[26] developed a method for constructing operator forms of third- and fourth-order in the generators, which commute with the Casimir operators of the groups appearing in the chain U(5)$\supset$O(5) $\supset$O(3), i.e., operators which conserve the U(5) symmetry. There, it has been proven that at the level of third-order operators there is no room for a new independent operator. The only existing operators at that level are constructed by considering products of the U(5) Casimir operator, n_d; the O(5) one, I_2; and the O(3) one, L^2. These specific operators can be written in terms of $d(J_1,J_2,J)$ [see (4.6)] as follows:[26]

$$n_d^3 = 3n_d^2 - 2n_d + d(2,2,0) + \tfrac{15}{7} d(0,0,2) - \tfrac{7}{5} d(2,2,3) + \tfrac{105}{55} d(2,2,4) + d(4,4,6) , \tag{6.4}$$

$$L^2 n_d = 6n_d^2 - 12n_d + 2L^2 - 3d(2,2,0) - \tfrac{30}{7} d(0,0,2) + \tfrac{7}{5} d(2,2,3) + \tfrac{7}{11} d(2,2,4) + 4d(4,4,6) , \tag{6.5}$$

$$I_2 n_d = -\tfrac{1}{2} n_d^3 - \tfrac{1}{2} n_d^2 + 3n_d + 2I_2 + \tfrac{5}{2} d(0,0,2) . \tag{6.6}$$

It is again interesting to consider for each of the above operators the classical limit of the third-order terms. Each of the $d(J_1,J_2,J)$ terms has, again, a classical limit of the form (4.8). Making use of the typical A and B values of Table II, one can easily check that for n_d^3 the classical limit of the third-order terms reduces to

$$N(N-1)(N-2)\beta^6/(1+\beta^2)^3 ,$$

while for $L^2 n_d$ and $I_2 n_d$ these terms produce a completely vanishing classical limit. Here again no triaxiality can be introduced in this way.

For O(5) two of us[26,27] have constructed symmetry conserving fourth-order operators. They have shown that there exists an infinity of such operators. The simplest is given by

$$S = -\frac{2^3 5^2}{3\times 7\times 11} f(2,3,5) + f(2,4,6) + \frac{5\times 13}{2^4\times 7} f(4,6,8) , \tag{6.7}$$

whereby

$$f(J_1,J_2,J) = \{[(d^\dagger d^\dagger)^{J_1} d^\dagger]^{J_2} d^\dagger\}^J \cdot \{[(\tilde{d}\,\tilde{d})^{J_1} \tilde{d}]^{J_2} \tilde{d}\}^J . \tag{6.8}$$

By the method of Van Isacker and Chen[24] it is easy to verify that the classical limit of the three occurring f operators can be written as

$$f(J_1,J_2,J) = \beta^8/(1+\beta^2)^4 N(N-1)(N-2)(N-3) \times (A + B\cos^2 3\gamma) , \tag{6.9}$$

where for

$f(2,3,5)$: $A = -3/(2\times 5\times 7)$, $B = 3/(2\times 5\times 7)$;

$f(2,4,6)$: $A = 2^3/(7^2\times 11)$, $B = 2^2/(7\times 11)$;

$f(4,6,8)$: $A = 2^3\times 79/(5\times 7\times 11\times 13)$, $B = -2^7/(5\times 7\times 11\times 13)$.

Introducing these numbers into (6.7) results in

$$S_{\mathrm{CL}} = \frac{3^3\times 5}{2\times 7^2\times 11} \beta^8 \frac{N(N-1)(N-2)(N-3)}{(1+\beta^2)^4} ,$$

and again triaxiality does not occur.

VII. CONCLUSIONS

We have shown how one can introduce dynamical symmetry-conserving higher-order terms in the IBM. The influence of these terms on the energy spectrum has been studied in the SU(3) limit. It is important to note that these higher-order terms remove the degeneracies which exist for members of the β and γ bands. The wave functions for these states approximate in the limit of large N, the Elliott basis. Since the SU(3) limit is conserved the conclusions stated in the original IBM for transition rates between members of the ground state band remain valid. In the same way, transitions from members of the β and γ bands to levels of the ground state band remain forbidden. It has been demonstrated, by considering the classical limit, that in general such higher-order terms cannot give rise to a stable triaxial shape of the nucleus. Thus, triaxiality can be introduced only by breaking up the dynamical symmetries of the IBM.

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