Exact boson mappings for the nuclear neutron (proton) p shell with the symmetry $SO(7)\supset SU(3)$

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A special closed set of commutation relations satisfied by the fermion pair and multipole operators for a major shell in *LST* coupling is found. The case of the neutron (proton) p shell with the structure $SO(7)\supset SU(3)$ is studied in detail. An exact Dyson boson mapping is constructed, which is subsequently hermitized. The method is directly applicable to higher shells and through the use of the experimentally plausible pseudo-SU(3) symmetry may lead to a theoretical foundation for the interacting boson model near the SU(3) limit.

I. INTRODUCTION

Since the introduction of the phenomenological interacting boson model (IBM),[1,2] which gives a good description of the collective low-lying states in medium and heavy even-even nuclei in terms of monopole (s) and quadrupole (d) bosons, many attempts have been made to provide a theoretical justification. A crucial step is the transition from the fermion space to a boson space, i.e., the boson mapping, through which an expansion of the fermion operators in terms of the boson operators is determined. As it is well known, IBM contains three limiting symmetries, namely the SO(5) (vibrational) limit, suitable as a starting point for spherical nuclei, the SU(3) (rotational) limit, suitable as a starting point for describing deformed nuclei, and the O(6) limit, which applies to γ-unstable nuclei. Most of the number conserving theoretical methods developed so far deal with seniority scheme boson mappings (Ref. 3 and references therein), studying small perturbations around the seniority limit. Thus they offer a way towards the theoretical justification (i.e., connection with the shell model) of the IBM in the vibrational limit. They are not expected to work, however, for deformed nuclei (i.e., in the rotational limit). It appears to us that a new kind of mapping is needed in the deformed region, involving a perturbative expansion around the SU(3) limit. A major difficulty about mappings in the deformed region has been so far the identification of suitable small parameters, since the small parameters used in the seniority limit, which are proportional to the number of d ($J=2$) bosons, g ($J=4$) bosons, etc., are not, in general, small for deformed nuclei, where there is no reason to believe that s ($J=0$) bosons have a predominant presence. It is the purpose of this paper to make the first steps towards the direction of finding a special mapping applicable to the deformed case. We are then looking for an algebra which has an SU(3) subalgebra. [The single-j shell, which has the symmetry $SO(2(2j+1))$ and includes an $SU(2j+1)$ subalgebra, lacks an SU(3) subalgebra, and thus it does not seem particularly suitable for our purposes.]

In this paper we first discover a special closed set of commutation relations satisfied by the fermion pair and multipole operators in a many nondegenerate l level system. Then we find an exact boson mapping for the special case of a single neutron (or proton) p shell, which has the symmetry SO(7) that includes an SU(3) subalgebra. Another important new feature is that although the fermions in our model explicitly have spin and isospin in addition to orbital angular momentum, the (s and d) bosons used for this mapping only have one kind of (total) angular momentum, in contrast to the bosons used by Elliot *et al.* in IBM-3 and IBM-4.[4–7] Once the commutation relations are established, one can either try to directly find a Hermitian boson mapping (as in Ref. 3), or try to first find a (non-Hermitian) Dyson mapping and then hermitize it (as in Ref. 8). Dyson boson mappings can be easily obtained, since the boson expansions of the operators of the pair algebra are finite and include only two-boson terms in the case of the multipole operators, or one-boson and three-boson terms in the case of pair operators, in contrast to Hermitian boson mappings, where the boson expansions of the operators of the pair algebra can be infinite and include all possible terms of any even (odd) number of bosons in the case of the multipole (pair) operators. It turns out that a Dyson boson mapping is particularly useful as an intermediary for a class of mappings where all the pair creation operators (and also the annihilation operators) of the pair algebra belong to the same irreducible representation (irrep) of the corresponding multipole subalgebra. Then one can allow each of the operators belonging to one of these two sets to be mapped simply onto the corresponding boson operator, the operators belonging to the other set being mapped as cubic polynomials in the bosons, still belonging to the same irrep of the multipole subalgebra. It turns out that this is the case for the mapping studied in connection with the present SO(7) model [as it was for the so-called BZM (Belyaev-Zelevinsky-Marshalek) mapping of the SO(5) algebra, which has an SU(2) subalgebra, encountered in Ref. 8]. Thus in this paper the Dyson boson mapping is found first. It is subsequently hermitized. We find the Dyson mapping by a straightforward algebraic approach, in which we directly satisfy the commutation relations. It turns out that the number of equations obtained is larger than the number of unknown coefficients, the surplus equations then providing internal self-consistency checks. Then the hermitization procedure is carried out, in a way parallel to the work by Klein *et al.*[8] for the BZM mapping of the SO(5) algebra. It must be emphasized that if the pair creation operators of the pair algebra do not group themselves into one irrep of the multipole subalgebra, the Dyson mapping turns out to be rather complicated and not particularly useful as a technical device. This was the case for the two seniority mappings given in Ref. 8.

A few clarifying remarks are appropriate at this point:

(i) Since a p shell of identical nucleons can hold at most six particles, using particle-hole conjugation it is easy to see that one can have at most one fermion pair, since four particles are equivalent to a pair of holes and the six-particle state is the conjugate of the 0-particle state. Thus it may seem, at first thought, very strange to attempt a boson mapping for such a small fermion system. There are two reasons for doing so, though. First, the techniques developed in this paper for the p shell are to be applied to higher shells, where a larger number of fermion pairs is allowed. Work is already in progress for the s-d neutron (proton) shell, which has the symmetry $SO(13)\supset U(6)\supset SU(3)$; further remarks concerning this case are made in Sec. VIII. Second, one can always increase the number of particles in the p shell by replacing it with a "toy" model of a nucleon with $l=1$ and a very high spin quantum number, or, as in Ref. 9, by a series of spins differing by three units at least (so that a given angular momentum j does not occur more than once), such as $\frac{1}{2}, \frac{7}{2}, \frac{13}{2}, \ldots$, or $\frac{3}{2}, \frac{9}{2}, \frac{15}{2}, \ldots$. But, in doing so, one will confront the "fatal flaw" mentioned by Ginocchio in Ref. 9 for his $Sp(6)\supset U(3)$ model, namely, the number of states allowed in this model is smaller than the number of states allowed in the IBM, and, most importantly, the missing states happen to be the states belonging to the leading irreducible representations of the SU(3) subalgebra, which are expected to lie lower in energy. Because the Lie algebras Sp(6) and SO(7) are locally isomorphic, this conclusion holds for the SO(7) model, too.

(ii) Since protons and neutrons occupy the same major shell in the cases of the p shell or the s-d shell, our SO(7) model for the p shell [or equivalently the SO(13) model for the s-d shell] cannot provide a theoretical justification for the IBM in these cases, since proton-neutron pairs have been ignored so far. We choose to do so, because our long-term goal is to open a road to the theoretical justification of the IBM in heavy nuclei, where protons and neutrons occupy different major shells, and proton-neutron pairs are not expected to contribute. In order to get some idea about how to apply the IBM in light (e.g., s-d) shell nuclei, one has to generalize the present techniques in order to allow proton-neutron pairs to be present.

Our results immediately invite comparison with those quoted by Ginocchio *et al.* in Refs. 9 and 10, for the so-called Ginocchio model with the symmetry SO(8), which has also been studied in a unified boson expansion framework by Dobaczewski.[11] This model describes a single $j=\frac{3}{2}$ level and has an SU(4) subalgebra but not an SU(3) subalgebra. We refer to our concluding remarks for this comparison.

In Sec. II of this paper the closed set of commutation relations for the many nondegenerate l level system will be found and the special case of a single l shell will be deduced from it in Sec. III. In Sec. IV the exact Dyson mapping for the $l=1$ single shell (the p shell) will be found through use of the commutation relations. The results will be put in a more elegant form, clarifying their group structure in Sec. V, and will be subsequently hermitized in Sec. VI. A slightly different solution will be given in Sec. VII, while Sec. VIII will contain a discussion of the results and plans for future work. Details of the calculations of Secs. IV and V are left for Appendix A and Appendix B, respectively.

II. COMMUTATION RELATIONS FOR A MANY NONDEGENERATE l LEVEL SYSTEM

Let us consider several nondegenerate levels characterized by orbital angular momenta l_i. The associated pair algebra is generated by the pair and multipole operators,

$$\begin{aligned}
F^{\dagger}(LMSsTt) &= \sqrt{1/2}\sum_{m1\,m2}\sum_{s1\,s2}\sum_{t1\,t2}(l1\ m1\ l2\ m2\,|\,L\ M) \\
&\qquad\times(\tfrac{1}{2}\ s1\ \tfrac{1}{2}\ s2\,|\,S\ s)(\tfrac{1}{2}\ t1\ \tfrac{1}{2}\ t2\,|\,T\ t)a^{\dagger}_{l1\ m1\ 1/2\ s1\ 1/2\ t1}a^{\dagger}_{l2\ m2\ 1/2\ s2\ 1/2\ t2} \\
&= \sqrt{1/2}\sum_{m1\,m2}\sum_{s1\,s2}\sum_{t1\,t2}(-1)^{l1+l2-L+3-S-T} \\
&\qquad\times(l2\ m2\ l1\ m1\,|\,L\ M)(\tfrac{1}{2}\ s2\ \tfrac{1}{2}\ s1\,|\,S\ s)(\tfrac{1}{2}\ t2\ \tfrac{1}{2}\ t1\,|\,T\ t) \\
&\qquad\times a^{\dagger}_{l2\ m2\ 1/2\ s2\ 1/2\ t2}a^{\dagger}_{l1\ m1\ 1/2\ s1\ 1/2\ t1} \\
&= \sqrt{1/2}[a^{\dagger}\times a^{\dagger}]_{(LMSsTt)}\ ,
\end{aligned} \tag{2.1}$$

$$G(LMSsTt)=\sum_{m1\,m2}\sum_{s1\,s2}\sum_{t1\,t2}(l1\;m1\;l2\;-m2\,|\,L\;M)$$
$$\times(\tfrac{1}{2}\;s1\;\tfrac{1}{2}\;-s2\,|\,S\;s)(\tfrac{1}{2}\;t1\;\tfrac{1}{2}\;-t2\,|\,T\;t)(-1)^{l2-m2}(-1)^{(1/2)-s2}(-1)^{(1/2)-t2}$$
$$\times a^{\dagger}_{l1\;m1\;1/2\;s1\;1/2\;t1}a_{l2\;m2\;1/2\;s2\;1/2\;t2}$$
$$=[a^{\dagger}\times\tilde{a}]_{(LMSsTt)} \qquad (2.2)$$

with

$$F(LMSsTt)=[F^{\dagger}(LMSsTt)]^{\dagger}\,, \qquad (2.3)$$
$$G^{\dagger}(LMSsTt)=[G(LMSsTt)]^{\dagger}\,. \qquad (2.4)$$

A general property following from these definitions is

$$G^{\dagger}(LMSsTt)=(-1)^{M}(-1)^{l2-l1}(-1)^{s}(-1)^{t}G(L\;-M S\;-s T\;-t)\,. \qquad (2.5)$$

In the above $a^{\dagger}_{lm1/2s1/2t}$ ($a_{lm1/2s1/2t}$) are fermion creation (annihilation) operators and $(l1\;m1\;l2\;m2\,|\,L\;M)$ are the usual Clebsch-Gordan coefficients.

In Eq. (2.1) we remark that if $l1+l2=$even, even values of L are possible for $(S=0,T=1)$ or $(S=1,T=0)$. If $(S=0,T=0)$ or $(S=1,T=1)$, odd values of L will occur. We intend to choose to study a system of neutrons (protons) with $(T,t)=(1,1)$ $[(T,t)=(1,-1)]$ and we are interested in states of even L. Then $T=1$ implies $S=0$ and the following commutation relations hold:

$$[F^{\dagger}(L1\;M1\;0\;0\;1\;1),F^{\dagger}(L2\;M2\;0\;0\;1\;1)]=0\,, \qquad (2.6)$$

$$[F^{\dagger}(L1\;M1\;0\;0\;1\;1),F(L2\;M2\;0\;0\;1\;1)]$$
$$=-\tfrac{1}{2}\delta_{L1\,L2}\delta_{M1\,M2}[1+(-1)^{l1+l1'-L2}]+\tfrac{1}{2}\sqrt{1/2}$$
$$\times\sum_{T''}3(-1)^{T''}(1\;1\;1\;-1\,|\,T''\;0)\begin{Bmatrix}1&1&T''\\ \frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{Bmatrix}$$
$$\times\sum_{L''}(-1)^{L''}(-1)^{M2}[(2L1+1)(2L2+1)]^{1/2}(L1\;M1\;L2\;-M2\,|\,L''\;M1-M2)$$
$$\times\left[(-1)^{L1}\begin{Bmatrix}L1&L2&L''\\ l2&l1'&l1\end{Bmatrix}+(-1)^{L2}(-1)^{l1+l2'}\right.$$
$$\times\begin{Bmatrix}L1&L2&L''\\ l2'&l1&l1'\end{Bmatrix}+(-1)^{l1+l1'}\begin{Bmatrix}L1&L2&L''\\ l2&l1&l1'\end{Bmatrix}+(-1)^{L1+L2}(-1)^{l1+l2'}$$
$$\left.\times\begin{Bmatrix}L1&L2&L''\\ l2'&l1'&l1\end{Bmatrix}\right]G(L''\;M1-M2\;0\;0\;T''\;0)\,, \qquad (2.7)$$

$$[G^{\dagger}(L1\;M1\;0\;0\;T1\;0),F^{\dagger}(L2\;M2\;0\;0\;1\;1)]$$
$$=\sqrt{(1/2)}3(-1)^{T1}(1\;1\;1\;-1\,|\,T1\;0)\begin{Bmatrix}1&1&T1\\ \frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{Bmatrix}$$
$$\times\sum_{L''}(-1)^{L''}(-1)^{M1}[(2L1+1)(2L2+1)]^{1/2}(L1\;-M1\;L2\;M2\,|\,L''\;M2-M1)$$
$$\times\left[(-1)^{L2}\begin{Bmatrix}L1&L2&L''\\ l2&l1'&l1\end{Bmatrix}+(-1)^{l2'+l1}\right.$$
$$\left.\times\begin{Bmatrix}L1&L2&L''\\ l2'&l1'&l1\end{Bmatrix}\right]F^{\dagger}(L''\;M2-M1\;0\;0\;1\;1)\,, \qquad (2.8)$$

$$[G(L1\ M1\ 0\ 0\ T1\ 0),G(L2\ M2\ 0\ 0\ T2\ 0)]$$

$$=\sqrt{(1/2)}\sum_{T''}(-1)^{1-T''}[(2\,T1+1)(2\,T2+1)]^{1/2}\begin{Bmatrix} T1 & T2 & T'' \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{Bmatrix}(T1\ 0\ T2\ 0\,|\,T''\ 0)$$

$$\times\sum_{L''}[(2\,L1+1)(2\,L2+1)]^{1/2}(L1\ M1\ L2\ M2\,|\,L''\ M1+M2)$$

$$\times\left[(-1)^{L''}(-1)^{l1+l2'}\begin{Bmatrix} L1 & L2 & L'' \\ l2' & l1 & l2\end{Bmatrix}-(-1)^{l1'+l2}(-1)^{L1+L2}\right.$$

$$\left.\times\begin{Bmatrix} L1 & L2 & L'' \\ l2 & l1' & l1\end{Bmatrix}\right]G(L''\ M1+M2\ 0\ 0\ T''\ 0)\,, \tag{2.9}$$

where the curly brackets are the usual 6-j symbols.

A few comments are appropriate at this point:

(i) Our long-term aim is to provide a way towards the theoretical justification of the IBM, which describes low-lying collective states in medium and heavy even-even nuclei. Since protons and neutrons occupy different major shells in these nuclei, our choice to study a system of neutrons (protons) only (with $T=1$) is justified, because no proton-neutron pairs are expected to be present in the realistic case.

(ii) Since the collective low-lying states we have in mind are states of even L, we choose $L=$even in our model and then we are forced by (2.1) to consider pairs with $S+T=$odd only, since major shells contain levels of the same parity, thus guaranteeing $l_1+l_2=$even. This result is in agreement with the well-known fact that states with $S+T=$odd lie lower than states with $S+T=$even. Since we already have $T=1$, this implies $S=0$.

III. COMMUTATION RELATIONS FOR A SINGLE-l SHELL

Considering a single-l shell, and remarking that

$$(-1)^T(1\ 1\ 1\ -1\,|\,T\ 0)\begin{Bmatrix} 1 & 1 & T \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{Bmatrix}=\tfrac{1}{3}\sqrt{1/2} \tag{3.1}$$

for both $T=0$ and $T=1$, the commutation relations (2.6)—(2.9) take the simplified form

$$[F^\dagger(L1\ M1\ 0\ 0\ 1\ 1),F^\dagger(L2\ M2\ 0\ 0\ 1\ 1)]=0\,, \tag{3.2}$$

$$[F^\dagger(L1\ M1\ 0\ 0\ 1\ 1),F(L2\ M2\ 0\ 0\ 1\ 1)]=-\delta_{L1\,L2}\delta_{M1\,M2}$$
$$+\sum_{L''}(-1)^{L''}(-1)^{M2}[(2\,L1+1)(2\,L2+1)]^{1/2}$$
$$\times(L1\ M1\ L2\ -M2\,|\,L''\ M1-M2)$$
$$\times\begin{Bmatrix} L1 & L2 & L'' \\ l & l & l\end{Bmatrix}[G(L''\ M1-M2\ 0\ 0\ 0\ 0)$$
$$+G(L''\ M1-M2\ 0\ 0\ 1\ 0)]\,, \tag{3.3}$$

$$[G^\dagger(L1\ M1\ 0\ 0\ 0\ 0)+G^\dagger(L1\ M1\ 0\ 0\ 1\ 0),F^\dagger(L2\ M2\ 0\ 0\ 1\ 1)]$$
$$=2\sum_{L''}(-1)^{L''}(-1)^{M1}[(2\,L1+1)(2\,L2+1)]^{1/2}(L1\ -M1\ L2\ M2\,|\,L''\ M2-M1)$$
$$\times\begin{Bmatrix} L1 & L2 & L'' \\ l & l & l\end{Bmatrix}F^\dagger(L''\ M2-M1\ 0\ 0\ 1\ 1)\,, \tag{3.4}$$

$$[G(L1\,M1\,0\,0\,0\,0)+G(L1\,M1\,0\,0\,1\,0),G(L2\,M2\,0\,0\,0\,0)+G(L2\,M2\,0\,0\,1\,0)]$$
$$=\sum_{L''}[(2\,L1+1)(2\,L2+1)]^{1/2}(L1\,M1\,L2\,M2\,|\,L''\,M1\,M2)\begin{Bmatrix}L1 & L2 & L''\\ l & l & l\end{Bmatrix}[(-1)^{L''}-(-1)^{L1+L2}]$$
$$\times[G(L''\,M1+M2\,0\,0\,0\,0)+G(L''\,M1+M2\,0\,0\,1\,0)]\,. \quad (3.5)$$

We see that it is always possible to keep the multipole operators

$$G^{\dagger}(L1\,M1\,0\,0\,0\,0)$$

and

$$G^{\dagger}(L1\,M1\,0\,0\,1\,0)$$

together, mapping them onto a multipole operator $B_{M1}^{(L1)\dagger}$ characterized by one kind of angular momentum only,

$$G^{\dagger}(L1\,M1\,0\,0\,0\,0)+G^{\dagger}(L1\,M1\,0\,0\,1\,0)\rightarrow B_{M1}^{(L1)\dagger}\,. \quad (3.6)$$

Furthermore, since $F^{\dagger}(L1\,M1\,0\,0\,1\,1)$ is the only pair operator appearing in these commutation relations, we can also map this onto a pair operator $A_{M1}^{(L1)\dagger}$ characterized by one kind of angular momentum only,

$$F^{\dagger}(L1\,M1\,0\,0\,1\,1)\rightarrow A_{M1}^{(L1)\dagger}\,. \quad (3.7)$$

Thus an important new result emerges: The pair and multipole operators of a system of neutrons (protons) which have explicit spin and isospin dependence can be mapped onto boson operators which satisfy the same commutation relations but do not have any spin or isospin dependence. Finding this exact boson mapping for the p shell will be the subject of the rest of this paper.

IV. EXACT DYSON BOSON MAPPING FOR THE p SHELL

Let us consider a single p shell ($l=1$). The commutation relations for this shell can be easily found from Eqs. (3.2)–(3.5) by putting $l=1$.

The fermion pair and multipole operators appearing in these relations can be mapped onto functions of boson operators as follows (involving at the moment only a change of notation):

$$F^{\dagger}(0\,0\,0\,0\,1\,1)\rightarrow A^{(0)\dagger}\,, \quad (4.1)$$

$$F^{\dagger}(2\,\mu\,0\,0\,1\,1)\rightarrow A_{\mu}^{(2)\dagger}\,, \quad (4.2)$$

$$F(0\,0\,0\,0\,1\,1)\rightarrow A^{(0)}\,, \quad (4.3)$$

$$F(2\,\mu\,0\,0\,1\,1)\rightarrow A_{\mu}^{(2)}\,, \quad (4.4)$$

$$G^{\dagger}(0\,0\,0\,0\,0\,0)+G^{\dagger}(0\,0\,0\,0\,1\,0)\rightarrow B^{(0)\dagger}\,, \quad (4.5)$$

$$G^{\dagger}(1\,\mu\,0\,0\,0\,0)+G^{\dagger}(1\,\mu\,0\,0\,1\,0)\rightarrow B_{\mu}^{(1)\dagger}\,, \quad (4.6)$$

$$G^{\dagger}(2\,\mu\,0\,0\,0\,0)+G^{\dagger}(2\,\mu\,0\,0\,1\,0)\rightarrow B_{\mu}^{(2)\dagger}\,. \quad (4.7)$$

These 21 functions of boson operators are to form the algebra SO(7). The nine multipole operators alone are to form the algebra U(3). If the operator $B^{(0)\dagger}$ (which is essentially the number operator) is removed, the subalgebra SU(3) is obtained.

We seek an exact boson mapping for this set of operators. We will first find an exact mapping for the SU(3) subalgebra, then we will enlarge it into an exact Dyson mapping for the whole SO(7) algebra. Since we are studying the p shell, where only fermion pairs of angular momentum 0 and 2 are present, we expect to find an exact boson mapping employing s ($J=0$) and d ($J=2$) bosons only, satisfying the commutation relations:

$$[a_0,a_0^{\dagger}]=1\,, \quad (4.8)$$

$$[a(2)_{\mu},a(2)_{\nu}^{\dagger}]=\delta_{\mu\nu}\,, \quad (4.9)$$

where $a_0^{\dagger},a(2)_{\mu}^{\dagger}$ $[a_0,a(2)_{\mu}]$ are creation [annihilation] operators for the s and d bosons, respectively. We will also be using

$$\tilde{a}(2)_{\mu}=(-1)^{\mu}a(2)_{-\mu}\,. \quad (4.10)$$

We thus turn to the problem of finding an exact boson mapping for the operators $B_{\mu}^{(1)},B_{\mu}^{(2)}$, forming the SU(3) subalgebra. As discussed in the Introduction, we expect the multipole operators $B_{\mu}^{(1)}$ and $B_{\mu}^{(2)}$ to be of the form

$$B_{\mu}^{(1)}=\alpha[a(2)^{\dagger}\times\tilde{a}(2)]_{\mu}^{(1)}\,, \quad (4.11a)$$

$$B_{\mu}^{(2)}=\beta a(2)_{\mu}^{\dagger}a_0+\beta' a_0^{\dagger}\tilde{a}(2)_{\mu}+\gamma[a(2)^{\dagger}\times\tilde{a}(2)]_{\mu}^{(2)}\,, \quad (4.11b)$$

where α, β, β', and γ are unknown numerical coefficients, to be determined through use of the commutation relations.

The condition

$$B_{\mu}^{(2)\dagger}=(-1)^{\mu}B_{-\mu}^{(2)} \quad (4.12)$$

immediately implies

$$\beta=\beta'\,. \quad (4.13)$$

By means of the commutation relation

$$[B_{\mu}^{(1)},B_{\nu}^{(1)}]=-(1\,\mu\,1\,\nu\,|\,1\,\mu+\nu)B_{\mu+\nu}^{(1)}\,, \quad (4.14)$$

using (4.11a), it is easy to find that

$$[B_\mu^{(1)},B_\nu^{(1)}]=3\alpha^2\sum_J \begin{Bmatrix} 1 & 1 & J \\ 2 & 2 & 2 \end{Bmatrix}[(-1)^J-1](1\;\mu\;1\;\nu\,|\,J\;\mu+\nu)[a(2)^\dagger\times\tilde{a}(2)]^{(J)}_{\mu+\nu}\,. \tag{4.15}$$

Then substituting (4.15) and (4.11a) in (4.14) and equating the coefficients of the tensor $[a(2)^\dagger\times\tilde{a}(2)]^{(1)}$ we obtain

$$\alpha=\sqrt{5}\,. \tag{4.16}$$

We still need to find the coefficients β and γ. We first check the commutation relation

$$[B_\mu^{(1)},B_\nu^{(2)}]=-\sqrt{3}(1\;\mu\;2\;\nu\,|\,2\;\mu+\nu)B^{(2)}_{\mu+\nu}\,. \tag{4.17}$$

Using (4.11a) and (4.11b) it is straightforward to find that

$$[B_\mu^{(1)},B_\nu^{(2)}]=-\alpha\beta\sqrt{3/5}(1\;\mu\;2\;\nu\,|\,2\;\mu+\nu)a(2)^\dagger_{\mu+\nu}a_0-\alpha\beta\sqrt{3/5}(1\;\mu\;2\;\nu\,|\,2\;\mu+\nu)a_0^\dagger\tilde{a}(2)_{\mu+\nu}$$
$$+2\sqrt{15}\begin{Bmatrix} 1 & 2 & 2 \\ 2 & 2 & 2 \end{Bmatrix}(1\;\mu\;2\;\nu\,|\,2\;\mu+\nu)[a(2)^\dagger\times\tilde{a}(2)]^{(2)}_{\mu+\nu}\alpha\gamma\,. \tag{4.18}$$

Substituting (4.18) and (4.11b) in (4.17) and equating the coefficients of

$a(2)^\dagger_{\mu+\nu}a_0$,

$a_0^\dagger\tilde{a}(2)_{\mu+\nu}$,

$[a(2)^\dagger\times\tilde{a}(2)]^{(2)}_{\mu+\nu}$,

in the right- and left-hand sides of the equation, we get three equations which do not provide any additional information. (This was expected because $B_\mu^{(1)}$ is essentially the angular momentum operator and $B_\mu^{(2)}$ already has good angular momentum.)

By use of the commutation relation

$$[B_\mu^{(2)},B_\nu^{(2)}]=\sqrt{5}(2\;\mu\;2\;\nu\,|\,1\;\mu+\nu)B^{(1)}_{\mu+\nu} \tag{4.19}$$

and (4.11b), it is straightforward to find

$$[B_\mu^{(2)},B_\nu^{(2)}]=\beta^2 5\sum_J[(-1)^J-1](2\;\mu\;2\;\nu\,|\,J\;\mu+\nu)\begin{Bmatrix} 2 & 2 & 0 \\ 2 & 2 & J \end{Bmatrix}[a(2)^\dagger\times\tilde{a}(2)]^{(J)}_{\mu+\nu}$$
$$+\gamma^2 5\sum_J[(-1)^J-1](2\;\mu\;2\;\nu\,|\,J\;\mu+\nu)\begin{Bmatrix} 2 & 2 & 2 \\ 2 & 2 & J \end{Bmatrix}[a(2)^\dagger\times\tilde{a}(2)]^{(J)}_{\mu+\nu}\,. \tag{4.20}$$

Substituting (4.20) and (4.11a) in (4.19) and equating the coefficients of $[a(2)^\dagger\times\tilde{a}(2)]^{(1)}$ in both sides we obtain the equation

$$\begin{Bmatrix} 2 & 2 & 0 \\ 2 & 2 & 1 \end{Bmatrix}\beta^2+\begin{Bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \end{Bmatrix}\gamma^2=-\tfrac{1}{2}\,. \tag{4.21}$$

Equating the coefficients of $[a(2)^\dagger\times\tilde{a}(2)]^{(3)}$ we obtain

$$\begin{Bmatrix} 2 & 2 & 0 \\ 2 & 2 & 3 \end{Bmatrix}\beta^2+\begin{Bmatrix} 2 & 2 & 2 \\ 2 & 2 & 3 \end{Bmatrix}\gamma^2=0\,. \tag{4.22}$$

The system of (4.21) and (4.22) provide

$$\beta^2=\tfrac{4}{3} \tag{4.23}$$

and

$$\gamma^2=\tfrac{7}{3}\,. \tag{4.24}$$

Thus β and γ are found, up to some signs which will be fixed later.

Our next task is to find an exact Dyson boson mapping for the whole SO(7) algebra. Letting

$$A^{(0)}=a_0\,, \tag{4.25}$$

$$A^{(2)}_\mu=a(2)_\mu\,, \tag{4.26}$$

we expect

$$A^{(0)\dagger}=\gamma_0 a_0^\dagger+\gamma_1 a_0^\dagger a_0^\dagger a_0+\gamma_2[a(2)^\dagger\times a(2)^\dagger]^{(0)}a_0+\gamma_3 a_0^\dagger[a(2)^\dagger\times\tilde{a}(2)]^{(0)}+\gamma_4[[a(2)^\dagger\times a(2)^\dagger]^{(2)}\times\tilde{a}(2)]^{(0)}, \quad (4.27)$$

$$A_\mu^{(2)\dagger}=\delta_0 a(2)_\mu^\dagger+\delta_1 a(2)_\mu^\dagger a_0^\dagger a_0+\delta_2 a_0^\dagger a_0^\dagger\tilde{a}(2)_\mu+\delta_3 a_0^\dagger[a(2)^\dagger\times\tilde{a}(2)]_\mu^{(2)}+\delta_4[a(2)^\dagger\times a(2)^\dagger]_\mu^{(2)}a_0$$
$$+\sum_{L=0,2,4}\delta_{5L}[[a(2)^\dagger\times a(2)^\dagger]^{(L)}\times\tilde{a}(2)]_\mu^{(2)}+\sum_{L=0,1,2,3,4}\delta_{6L}[a(2)^\dagger\times[a(2)^\dagger\times\tilde{a}(2)]^{(L)}]_\mu^{(2)}. \quad (4.28)$$

In (4.27) and (4.28), as discussed in the Introduction, all possible one-boson and three-boson terms were included. The three-d-boson terms deserve some discussion though. Since

$$[a(2)^\dagger\times[a(2)^\dagger\times\tilde{a}(2)]^{(2)}]^{(0)}=[[a(2)^\dagger\times a(2)^\dagger]^{(2)}\times\tilde{a}(2)]^{(0)}, \quad (4.29)$$

no complication arises in the case of $A^{(0)\dagger}$. But in $A_\mu^{(2)\dagger}$, the terms

$$P'^{(L)}=[[a(2)^\dagger\times a(2)^\dagger]^{(L)}\times\tilde{a}(2)]^{(2)},\quad L=0,2,4$$

and

$$P^{(L)}=[a(2)^\dagger\times[a(2)^\dagger\times\tilde{a}(2)]^{(L)}]^{(2)},\quad L=0,1,2,3,4$$

are not all linearly independent. In fact, only three of them are linearly independent, since the vanishing of $P'^{(1)}$ and $P'^{(3)}$ (because of the boson principle) implies two conditions among the $P^{(L)}$'s. Either the $P'^{(L)}$'s ($L=0,2,4$) or any three of the $P^{(L)}$'s can be used as the three independent tensors. In order to avoid confusion and have a unique expression for the solution we choose to use the three $P'^{(L)}$'s ($L=0,2,4$) as the linearly independent tensors, although any of the form of the solution is equally adequate. Special care is required during the calculations, though, because only coefficients of linearly independent tensors can be used to provide equations for the unknown numerical coefficients.

The calculation of the unknown numerical coefficients in (4.27) and (4.28) is rather lengthy but otherwise straightforward. We choose to present here only a small part, serving to fix the unknown signs of β and γ, leaving the rest of the details for Appendix A and presenting here only the final results.

Let us first demonstrate how the signs of β and γ are fixed. We know that

$$[A^{(0)},B_\mu^{(2)\dagger}]=\frac{2}{\sqrt{3}}A_\mu^{(2)}. \quad (4.30)$$

Using (4.25) and (4.11b) it is very easy to find that

$$[A^{(0)},B_\mu^{(2)\dagger}]=\beta a(2)_\mu. \quad (4.31)$$

Substituting (4.31) and (4.26) in (4.30) we obtain

$$\beta=+\frac{2}{\sqrt{3}}. \quad (4.32)$$

Thus the sign of β has been already fixed. We then employ the commutation relation

$$[A_\mu^{(2)},B_\nu^{(2)}]=10(-1)^\nu\left[(2\,\mu\,2\,-\mu\,|\,0\,0)\begin{Bmatrix}2&2&0\\1&1&1\end{Bmatrix}A^{(0)}+(2\,\mu\,2\,-\nu\,|\,2\,\mu-\nu)\begin{Bmatrix}2&2&2\\1&1&1\end{Bmatrix}A_{\mu-\nu}^{(2)}\right]. \quad (4.33)$$

From (4.26) and (4.11b) it is very easy to find that

$$[A_\mu^{(2)},B_\nu^{(2)}]=\delta_{\mu\nu}\beta a_0+\gamma\tilde{a}(2)_{\nu-\mu}(-1)^\mu(2\,\mu\,2\,-\nu\,|\,2\,\mu-\nu). \quad (4.34)$$

Using (4.34) and (4.26) in (4.33) and equating the coefficients of a_0 in both sides we obtain

$$\beta=\frac{2}{\sqrt{3}} \quad (4.35)$$

which just provides a consistency check. The coefficients of $a(2)_{\mu-\nu}$ yield

$$\gamma=10\begin{Bmatrix}2&2&2\\1&1&1\end{Bmatrix}=+\sqrt{7/3} \quad (4.36)$$

thus fixing the sign of γ as well.

Having shown how the missing signs of β and γ are found, we just give the final result of the calculation, leaving the rest of it for Appendix A. There the values of the numerical coefficients in (4.27) and (4.28) are found to be

$$\gamma_0=+1, \quad (4.37a)$$

$$\gamma_1=-\tfrac{1}{3}, \quad (4.37b)$$

$$\gamma_2=-\frac{\sqrt{5}}{3}, \quad (4.37c)$$

$$\gamma_3=-\frac{2\sqrt{5}}{3}, \quad (4.37d)$$

$$\gamma_4 = -\frac{\sqrt{35}}{6} , \tag{4.37e}$$

$$\delta_0 = +1 , \tag{4.38a}$$

$$\delta_1 = -\tfrac{2}{3} , \tag{4.38b}$$

$$\delta_2 = -\tfrac{1}{3} , \tag{4.38c}$$

$$\gamma_3 = -\frac{\sqrt{7}}{3} , \tag{4.38d}$$

$$\delta_4 = -\frac{\sqrt{7}}{6} , \tag{4.38e}$$

$$\delta_{50} = -\frac{1}{6\sqrt{5}} , \tag{4.39a}$$

$$\delta_{52} = \tfrac{1}{6} , \tag{4.39b}$$

$$\delta_{54} = -\frac{3}{\sqrt{5}} , \tag{4.39c}$$

$$\delta_{60} = \delta_{61} = \delta_{62} = \delta_{63} = \delta_{64} = 0 . \tag{4.40}$$

V. GROUP STRUCTURE OF THE EXACT DYSON MAPPING FOR THE p SHELL

In Sec. IV an exact Dyson mapping for the pair operators $A^{(0)\dagger}$, $A_\mu^{(2)\dagger}$, $A^{(0)}$, and $A_\mu^{(2)}$ was found. We will now try to put these results in a more elegant form, which will clarify their group structure and give a hint for the way they can be hermitized. Led by similar results in the SO(5) (Ref. 8) and the SO(8) (Refs. 9 and 10) cases, we expect our results to be of the form:

$$A^{(0)\dagger} = a_0^\dagger(1-xN) + [C, a_0^\dagger] , \tag{5.1}$$

$$A_\mu^{(2)\dagger} = a(2)_\mu^\dagger(1-xN) + [C, a(2)_\mu^\dagger] , \tag{5.2}$$

where

$$N = a_0^\dagger a_0 + \sqrt{5}[a(2)^\dagger \times \tilde{a}(2)]^{(0)} \tag{5.3}$$

and

$$C = \epsilon_0[B^{(0)} \times B^{(0)}]^{(0)} + \epsilon_1[B^{(1)} \times B^{(1)}]^{(0)} + \epsilon_2[B^{(2)} \times B^{(2)}]^{(0)} . \tag{5.4}$$

In addition we expect

$$\epsilon_1[B^{(1)} \times B^{(1)}]^{(0)} + \epsilon_2[B^{(2)} \times B^{(2)}]^{(0)} = \epsilon C_2 . \tag{5.5}$$

In the above, N is the boson number, C_2 is the quadratic Casimir invariant of SU(3), and x, ϵ_0, ϵ_1, ϵ_2, and ϵ are unknown numerical coefficients to be determined. It turns out that $A^{(0)\dagger}$ and $A_\mu^{(2)\dagger}$ indeed have the form given in Eqs. (5.1) and (5.2), respectively, the numerical coefficients being

$$\epsilon = -\tfrac{1}{8} , \tag{5.6a}$$

$$\epsilon_0 = \tfrac{5}{8} , \tag{5.6b}$$

$$\epsilon_1 = \sqrt{3}/8 , \tag{5.6c}$$

$$\epsilon_2 = -\sqrt{5}/8 , \tag{5.6d}$$

$$x = 2 . \tag{5.6e}$$

We again choose to present only the final result here, keeping the details for Appendix B. Using (5.6) in (5.1) and (5.2) we find the complete form of the final result:

$$A^{(0)\dagger} = a_0^\dagger(1-2N) + \tfrac{1}{8}[5[B^{(0)} \times B^{(0)}]^{(0)} - C_2, a_0^\dagger] , \tag{5.7}$$

$$A_\mu^{(2)\dagger} = a(2)_\mu^\dagger(1-2N) + \tfrac{1}{8}[5[B^{(0)} \times B^{(0)}]^{(0)} - C_2, a(2)_\mu^\dagger] , \tag{5.8}$$

where

$$C_2 = -\sqrt{3}[B^{(1)} \times B^{(1)}]^{(0)} + \sqrt{5}[B^{(2)} \times B^{(2)}]^{(0)} . \tag{5.9}$$

Using

$$B^{(0)} = \frac{2}{\sqrt{3}} N \tag{A31}$$

it is easy to put (5.7) and (5.8) in the following form:

$$A^{(0)\dagger} = a_0^\dagger \left[\frac{11}{6} - \frac{N}{3} \right] - \frac{1}{8}[C_2, a_0^\dagger] , \tag{5.10}$$

$$A_\mu^{(2)\dagger} = a(2)_\mu^\dagger \left[\frac{11}{6} - \frac{N}{3} \right] - \frac{1}{8}[C_2, a(2)_\mu^\dagger] . \tag{5.11}$$

We remark that both $A^{(0)\dagger}$ and $A_\mu^{(2)\dagger}$ have the same structure, namely there is a one-boson term multiplied in each case by the same coefficient, followed by a term involving the commutator of C_2 with the corresponding boson. This result is a very general one, i.e., the exact Dyson boson mapping of the pair operators of an SO(N) algebra having an SU(N') and ($N'<N$) subalgebra involves a one-boson term multiplied by a boson-number dependent coefficient and a term which can be expressed as the commutator of the second-order Casimir invariant of SU(N') with the appropriate boson. More specifically, in the case of a single-j shell with the symmetry SO[2(2j + 1)], which has an SU(2j + 1) subgroup, the second term will involve the commutator of the second-order Casimir invariant of the SU(2j + 1) subalgebra with the appropriate boson. It is obvious that to have an exact mapping we need as many bosons as we have pair operators, i.e., in the case of the single-j shell one has to include bosons up to angular momentum 2j − 1. All the bosons present must belong to the same irrep of SU(N'). Since the three-boson terms in the pair operators must belong to the same SU(N') irrep as the one-boson terms, it turns out that they must consist of the commutator of the second-order Casimir invariant of SU(N') with the corresponding boson, since the Casimir invariant does not change the irrep of SU(N'). This also is a hint about the form of the pair operators of a Hermitian mapping, where higher order terms will be present: these additional terms are expected to be commutators of higher order Casimir invariants of SU(N') with the corresponding bosons, since these higher terms also have to belong to the same irrep of SU(N').

VI. EXACT HERMITIAN MAPPING FOR THE p SHELL

Dyson mappings are elegant and easy to obtain, but there is some difficulty in using them, due to their nonhermiticity. Leaving aside the question of direct use of the results of the previous section, we seek a way to hermitize them. The hermitization procedure we are going to follow is essentially the one given by Klein *et al.* in Ref. 8. We seek a similarity transformation S which will make the pair operators Hermitian, without changing the multipole operators, i.e.,

$$SNS^{-1}=N\ , \tag{6.1}$$

$$SB_{\mu}^{(1)}S^{-1}=B_{\mu}^{(1)}\ , \tag{6.2}$$

$$SB_{\mu}^{(2)}S^{-1}=B_{\mu}^{(2)}\ . \tag{6.3}$$

These conditions imply that S commutes with N, $B_{\mu}^{(1)}$, and $B_{\mu}^{(2)}$. Consequently, it can be a function only of the boson number N and the Casimir invariants C_2,C_3 of SU(3) [or equivalently the symmetrized Casimir invariants I_2,I_3 of SU(3) (Refs. 12 and 13)], since $B_{\mu}^{(1)},B_{\mu}^{(2)}$ are the generators of the SU(3) subalgebra.

Our next task is to find conditions which will provide the form of S. The Dyson pair operators $A^{(0)\dagger}$ and $A_{\mu}^{(2)\dagger}$ given in Eqs. (5.10) and (5.11) have the general form

$$[A^{(J)\dagger}]_D=a(J)^{\dagger}\left[\frac{11}{6}-\frac{N}{3}\right]-\frac{1}{8}[C_2,a(J)^{\dagger}]\ , \tag{6.4}$$

where $J=0,2$, while the Dyson operators $A^{(0)}$ and $A_{\mu}^{(2)}$ have the general form

$$[A^{(J)}]_D=a(J)\ , \tag{6.5}$$

where $J=0,2$. We demand that the similarity transformation S hermitizes them, i.e., we want

$$S[A^{(J)\dagger}]_D S^{-1}=A^{(J)\dagger}\ , \tag{6.6}$$

$$S[A^{(J)}]_D S^{-1}=A^{(J)}\ , \tag{6.7}$$

and

$$A^{(J)\dagger}=[A^{(J)}]^{\dagger}\ , \tag{6.8}$$

where by $[A^{(J)\dagger}]_D,[A^{(J)}]_D$ we mean the Dyson pair operators and by $A^{(J)\dagger},A^{(J)}$ their Hermitian counterparts.

Equations (6.6)–(6.8) imply the condition

$$V^{-1}[[A^{(J)}]_D]^{\dagger}V=[A^{(J)\dagger}]_D\ , \tag{6.9}$$

where

$$V=S^{\dagger}S\ . \tag{6.10}$$

Substituting (6.4) and (6.5) in (6.9) we get the condition

$$V^{-1}a(J)^{\dagger}V=a(J)^{\dagger}\left[\frac{11}{6}-\frac{N}{3}\right]-\frac{1}{8}[C_2,a(J)^{\dagger}]\ . \tag{6.11}$$

We want to study the condition (6.11) by taking all nonvanishing matrix elements in the basis

$$|N(\lambda,\mu)\,\chi\; l\; m\rangle\ . \tag{6.12}$$

Here λ,μ are the quantum numbers introduced by Elliot[14,15]

$$\lambda=f_1-f_2\ , \tag{6.13}$$

$$\mu=f_2\ , \tag{6.14}$$

where f_1 (f_2) is the number of boxes in the first (second) line of the Young tableau corresponding to the SU(3) irrep in question, and χ is the additional quantum number needed to distinguish states with the same angular momentum l belonging to the same SU(3) irrep (λ,μ). [This can be the Vergados quantum number χ,[16] which corresponds to an orthogonal basis, since the Elliot quantum number K (Refs. 14 and 15) corresponds to a nonorthogonal basis.] Since our $a_0^{\dagger}$ and $a(2)^{\dagger}$ bosons belong to the (2,0) irrep of SU(3), there are in general three nonvanishing matrix elements of $a(J)^{\dagger}$ $(J=0,2)$ between the states (6.12), as shown below in two different notations:

$$(\lambda,\mu)\times(2,0)=(\lambda+2,\mu)+(\lambda-2,\mu+2)+(\lambda,\mu-2)\ ,$$

$$[f_1,f_2]\times[2,0]=[f_1+2,f_2]+[f_1,f_2+2]+[f_1-2,f_2-2]\ . \tag{6.15}$$

In the last term of (6.15) the SU(3) property

$$[f_1,f_2,f_3]=[f_1-f_3,f_2-f_3,0] \tag{6.16}$$

has been used. We do not need to calculate the nonvanishing matrix elements of $a(J)^{\dagger}$, since they cancel when we get the nonvanishing matrix elements of (6.11) (the interested reader can find them in Ref. 17, anyway). Taking the nonvanishing matrix elements of (6.11) and applying the Wigner-Eckart theorem we get the following three conditions:

$$V^{-1}(N+1,\lambda+2,\mu)V(N,\lambda,\mu)=\frac{11}{6}-\frac{N}{3}-\frac{1}{8}[C_2(\lambda+2,\mu)-C_2(\lambda,\mu)]\ , \tag{6.17}$$

$$V^{-1}(N+1,\lambda-2,\mu+2)V(N,\lambda,\mu)=\frac{11}{6}-\frac{N}{3}-\frac{1}{8}[C_2(\lambda-2,\mu+2)-C_2(\lambda,\mu)]\ , \tag{6.18}$$

$$V^{-1}(N+1,\lambda,\mu-2)V(N,\lambda,\mu)=\frac{11}{6}-\frac{N}{3}-\frac{1}{8}[C_2(\lambda,\mu-2)-C_2(\lambda,\mu)]\ . \tag{6.19}$$

Assuming

$$S^{\dagger}=S=\text{real}\ , \tag{6.20}$$

i.e.,

$$V=S^2 , \tag{6.21}$$

using the well-known formula

$$C_2=\tfrac{2}{3}[\lambda^2+\mu^2+3(\lambda+\mu)+\lambda\mu] , \tag{6.22}$$

and taking the square roots of (6.17)—(6.19) we get the conditions

$$S^{-1}(N+1,\lambda+2,\mu)S(N,\lambda,\mu)=\left[\frac{11}{6}-\frac{N}{3}-\frac{1}{12}(4\lambda+2\mu+10)\right]^{1/2}=R_1 , \tag{6.23}$$

$$S^{-1}(N+1,\lambda-2,\mu+2)S(N,\lambda,\mu)=\left[\frac{11}{6}-\frac{N}{3}-\frac{1}{12}(-2\lambda+2\mu+4)\right]^{1/2}=R_2 , \tag{6.24}$$

$$S^{-1}(N+1,\lambda,\mu-2)S(N,\lambda,\mu)=\left[\frac{11}{6}-\frac{N}{3}-\frac{1}{12}(-2\lambda-4\mu-2)\right]^{1/2}=R_3 . \tag{6.25}$$

As we remarked at the end of Sec. V, the hermitized form of the pair operators is expected to contain the commutators of the Casimir invariants (or the symmetrized Casimir invariants) of SU(3) with the corresponding boson operator, since all terms have to belong to the same irrep of SU(3). We choose to use the symmetrized Casimir invariants $I_2=C_2$ (Refs. 12 and 13) and I_3, thus we expect the hermitized pair operators to be of the form

$$A^{(J)\dagger}=a(J)^\dagger f(N,\lambda,\mu)+[C_2,a(J)^\dagger]g(N,\lambda,\mu)+[I_3,a(J)^\dagger]h(N,\lambda,\mu) , \tag{6.26}$$

$$A^{(J)}=f(N,\lambda,\mu)a(J)+g(N,\lambda,\mu)[a(J),C_2]+h(N,\lambda,\mu)[a(J),I_3] , \tag{6.27}$$

where $f(N,\lambda,\mu)$, $g(N,\lambda,\mu)$, and $h(N,\lambda,\mu)$ are unknown functions to be determined. It must be emphasized that the number of terms of Eqs. (6.26) and (6.27) is equal to the number of nonvanishing matrix elements of the boson creation [annihilation] operator $a(J)^\dagger$ [$a(J)$], $J=0,2$, between states of the basis (6.12) and that the quantities λ,μ appearing in the unknown functions $f(N,\lambda,\mu)$, $g(N,\lambda,\mu)$, and $h(N,\lambda,\mu)$ are meant to be operators that can be expressed in terms of the Casimir operators C_2 and I_3 if one solves the system of Eqs. (6.22) and (6.43) (given in the following) for λ and μ.

Equations (6.23)—(6.25) provide a set of conditions to determine the similarity transformation S. Another set of conditions can be found from Eq. (6.7), after substituting (6.5) and (6.27) in it and taking all possible nonvanishing matrix elements of the resulting equation. Using again the Wigner-Eckart theorem and having in mind that the boson annihilation operators $a_0,a(2)_\mu$ belong to the (0,2) irrep of SU(3), we find the following conditions:

$$S(N,\lambda,\mu)S^{-1}(N+1,\lambda+2,\mu)=f(N,\lambda,\mu)+g(N,\lambda,\mu)[C_2(\lambda+2,\mu)-C_2(\lambda,\mu)]+h(N,\lambda,\mu)[I_3(\lambda+2,\mu)-I_3(\lambda,\mu)] , \tag{6.28}$$

$$S(N,\lambda,\mu)S^{-1}(N+1,\lambda-2,\mu+2)=f(N,\lambda,\mu)+g(N,\lambda,\mu)[C_2(\lambda-2,\mu+2)-C_2(\lambda,\mu)]+h(N,\lambda,\mu)[I_3(\lambda-2,\mu+2)-I_3(\lambda,\mu)] , \tag{6.29}$$

$$S(N,\lambda,\mu)S^{-1}(N+1,\lambda,\mu-2)=f(N,\lambda,\mu)+g(N,\lambda,\mu)[C_2(\lambda,\mu-2)-C_2(\lambda,\mu)]+h(N,\lambda,\mu)[I_3(\lambda,\mu-2)-I_3(\lambda,\mu)] . \tag{6.30}$$

We remark that there are again three nonvanishing matrix elements, and the left-hand sides of (6.28)—(6.30) correspond to the known left-hand sides of (6.23)—(6.25), thus providing the equations

$$f(N,\lambda,\mu)+g(N,\lambda,\mu)\Delta C_1+h(N,\lambda,\mu)\Delta I_1=R_1 , \tag{6.31}$$

$$f(N,\lambda,\mu)+g(N,\lambda,\mu)\Delta C_2+h(N,\lambda,\mu)\Delta I_2=R_2 , \tag{6.32}$$

$$f(N,\lambda,\mu)+g(N,\lambda,\mu)\Delta C_3+h(N,\lambda,\mu)\Delta I_3=R_3 , \tag{6.33}$$

where we have used the following definitions:

$$\Delta C_1=C_2(\lambda+2,\mu)-C_2(\lambda,\mu) , \tag{6.34}$$

$$\Delta C_2=C_2(\lambda-2,\mu+2)-C_2(\lambda,\mu) , \tag{6.35}$$

$$\Delta C_3 = C_2(\lambda,\mu-2) - C_2(\lambda,\mu) \,, \tag{6.36}$$

$$\Delta I_1 = I_3(\lambda+2,\mu) - I_3(\lambda,\mu) \,, \tag{6.37}$$

$$\Delta I_2 = I_3(\lambda-2,\mu+2) - I_3(\lambda,\mu) \,, \tag{6.38}$$

$$\Delta I_3 = I_3(\lambda,\mu-2) - I_3(\lambda,\mu) \,. \tag{6.39}$$

Equations (6.31)–(6.33) are a system of three equations for the three unknown functions $f(N,\lambda,\mu)$, $g(N,\lambda,\mu)$, and $h(N,\lambda,\mu)$, which yields the final results

$$f(N,\lambda,\mu) = [(9\lambda^2\mu + 9\lambda\mu^2 + 24\lambda\mu + 9\lambda^2 + 3\mu^2 + 15\lambda + 7\mu + 4)R_1 + (9\lambda^2\mu + 9\lambda\mu^2 + 36\lambda\mu + 3\lambda^2 + 15\mu^2 + 11\lambda + 35\mu + 10)R_2 + (9\lambda^2\mu + 9\lambda\mu^2 + 15\lambda^2 + 9\mu^2 + 48\lambda\mu + 55\lambda + 39\mu + 40)R_3]/[27(\lambda^2\mu + \lambda\mu^2 + \lambda^2 + \mu^2 + 4\lambda\mu + 3\lambda + 3\mu + 2)] \,, \tag{6.40}$$

$$g(N,\lambda,\mu) = [(\mu^2 + 2\lambda\mu + 2\lambda + 2\mu + 1)R_1 + (\lambda^2 - \mu^2 + 4\lambda + 4)R_2 - (\lambda^2 + 2\lambda\mu + 6\lambda + 2\mu + 5)R_3]/[12(\lambda^2\mu + \lambda\mu^2 + \lambda^2 + \mu^2 + 4\lambda\mu + 3\lambda + 3\mu + 2)] \,, \tag{6.41}$$

$$h(N,\lambda,\mu) = [(1+\mu)R_1 - (\lambda+\mu+2)R_2 + (1+\lambda)R_3]/[6(\lambda^2\mu + \lambda\mu^2 + \lambda^2 + \mu^2 + 4\lambda\mu + 3\lambda + 3\mu + 2)] \,. \tag{6.42}$$

In obtaining this result we have used the formula[12,13]

$$I_3 = \tfrac{1}{9}(\lambda-\mu)(2\lambda+\mu+3)(\lambda+2\mu+3) \,. \tag{6.43}$$

VII. A SLIGHTLY DIFFERENT EXACT MAPPING FOR THE p SHELL

In Sec. IV an exact mapping for the SU(3) subalgebra was first found. We remark that the coefficients β (4.23) and γ (4.24) of $B_\mu^{(2)}$ had arbitrary signs, which were fixed later, through use of the commutation relations (4.30) and (4.33), where use of the arbitrary definitions

$$A^{(0)} = a_0 \,, \tag{4.25}$$

$$A_\mu^{(2)} = a(2)_\mu \,, \tag{4.26}$$

was made. With these definitions, β and γ came out having the same (positive) sign [cf. (4.35) and (4.36)]. Since it is well-known that the SU(3) subalgebra can be satisfied equally well with β and γ having opposite signs, we asked if it is possible to obtain an SO(7) mapping in which β and γ would have opposite signs. It turned out that this is in fact the case. Since all calculations are exactly the same, up to some different signs, only the final results will be reported here.

In this second mapping we use the definitions

$$A^{(0)} = -a_0 \,, \tag{7.1}$$

$$A_\mu^{(2)} = a(2)_\mu \,. \tag{7.2}$$

Then the commutation relations (4.30) and (4.33) fix the signs of β and γ as follows:

$$\beta = -\frac{2}{\sqrt{3}} \,, \tag{7.3}$$

$$\gamma = \sqrt{7/3} \,, \tag{7.4}$$

thus β and γ have opposite signs. Then, instead of (4.32)–(4.40), the exact Dyson mapping for the SO(7) algebra contains the coefficients

$$\gamma_0 = -1 \,, \tag{7.5a}$$

$$\gamma_1 = \tfrac{1}{3} \,, \tag{7.5b}$$

$$\gamma_2 = \frac{\sqrt{5}}{3} \,, \tag{7.5c}$$

$$\gamma_3 = \frac{2\sqrt{5}}{3} \,, \tag{7.5d}$$

$$\gamma_4 = -\frac{\sqrt{35}}{6} \,, \tag{7.5e}$$

$$\delta_0 = 1 \,, \tag{7.6a}$$

$$\delta_1 = -\tfrac{2}{3} \,, \tag{7.6b}$$

$$\delta_2 = -\tfrac{1}{3} \,, \tag{7.6c}$$

$$\delta_3 = \frac{\sqrt{7}}{3} \,, \tag{7.6d}$$

$$\delta_4 = \frac{\sqrt{7}}{6} \,, \tag{7.6e}$$

$$\delta_{50} = -\frac{1}{6\sqrt{5}} \,, \tag{7.7a}$$

$$\delta_{52} = \tfrac{1}{6} \,, \tag{7.7b}$$

$$\delta_{54} = -\frac{3}{\sqrt{5}} \,, \tag{7.7c}$$

$$\delta_{60} = \delta_{61} = \delta_{62} = \delta_{63} = \delta_{64} = 0 \,. \tag{7.8}$$

Notice that the signs of γ_0, γ_1, γ_2, γ_3, δ_3, and δ_4 have changed, the rest of them remaining unchanged. These changes have as a result that instead of (5.1) and (5.2) we obtain the equations

$$-A^{(0)\dagger}=a_0^\dagger(1-xN)+[C,a_0^\dagger]\ , \qquad (7.9)$$

$$A_\mu^{(2)\dagger}=a(2)_\mu^\dagger(1-xN)+[C,a(2)_\mu^\dagger]\ , \qquad (7.10)$$

where (5.3)—(5.6) remain unchanged. This has as a consequence that the final Hermitian result reads

$$-A^{(0)\dagger}=a_0^\dagger f(N,\lambda,\mu)+[C_2,a_0^\dagger]g(N,\lambda,\mu)+[I_3,a_0^\dagger]h(N,\lambda,\mu)\ , \qquad (7.11)$$

$$A_\mu^{(2)\dagger}=a(2)_\mu^\dagger f(N,\lambda,\mu)+[C_2,a(2)_\mu^\dagger]g(N,\lambda,\mu)+[I_3,a(2)_\mu^\dagger]h(N,\lambda,\mu)\ , \qquad (7.12)$$

where $f(N,\lambda,\mu)$, $g(N,\lambda,\mu)$, and $h(N,\lambda,\mu)$ are still given by Eqs. (6.40)—(6.42), respectively.

VIII. DISCUSSION

An exact boson mapping for the neutron (proton) p shell is found, by first constructing an exact Dyson mapping and then hermitizing it. A test similar to that of Ref. 10 can be easily performed, namely one can choose a schematic shell-model Hamiltonian and do both the exact diagonalization and the diagonalization in the boson space, after mapping the Hamiltonian and consistently keeping terms up to some order. For the latter case, either the $|N(\lambda,\mu)\chi lm\rangle$ basis mentioned above or the already existing[18] seniority basis of the SO(5) limit of the IBM can be used. Of more interest is the fact that the method can provide exact boson mappings for higher shells as well, the relevant commutation relations having already been given in general form in Sec. II. The number of bosons needed in these exact mappings is equal to the number of fermion pairs which can be formed in the shell under discussion. Work on the s-d shell is already in progress. It turns out that in the case of a neutron (proton) s-d shell, which has the symmetry SO(13)$\supset$U(6)$\supset$SU(3), five bosons are present, namely two s bosons, two d bosons, and one g boson, all of them belonging to the (2,0,0,0,0,0) irrep of the U(6) subalgebra, thus allowing an exact Dyson mapping to be found, which involves the second-order Casimir invariant of U(6), and in addition an exact Hermitian mapping to be found according to the procedure discussed in this paper, which involves all Casimir invariants of U(6).

The present method also implies that the SO(8) mappings of Ref. 10 are incomplete, since in this case there are four nonvanishing matrix elements of the boson operators, thus implying that all possible symmetrized Casimir invariants C_2,I_3,I_4 must appear in the hermitized results. The complete form of the SO(8) mappings will be given in a forthcoming publication.

So far we were dealing with exact boson mappings only. It is interesting to see how one can get approximate boson mappings, especially for higher shells, as perturbative expansions around the SU(3) limit. It is well-known[14,15,19–22] that the leading SU(3) irreps are the ones which dominate in reality. By leading SU(3) irreps we mean the irreps with maximum $\lambda+\mu$, and, among those with equal $\lambda+\mu$, the ones with maximum λ. It is obvious that for leading SU(3) irreps μ/λ (and $2N-\lambda/\lambda$) will remain small, being hopeful candidates for the small parameter role in such an expansion. [It is well-known that one of the main obstacles in obtaining approximate mappings in the SU(3) limit has been so far the identification of a small parameter.] Such an approximate mapping will certainly be of physical interest for the s-d shell[14,15] [which has an SU(3) subgroup], and work in this direction is already in progress. It turns out that one can select the five bosons mentioned above in a way that one s boson, one d boson, and the g boson belong to the (4,0) irrep of the SU(3) subalgebra, while the next two bosons (one s boson and one d boson) belong to the (0,2) irrep of the SU(3) subalgebra. It seems plausible that the first of these two sets of bosons will be predominantly present in states belonging to the leading irreps of the SU(3) subalgebra. Because the SU(3) symmetry of the three-dimensional harmonic oscillator is destroyed by strong spin-orbit coupling in higher shells, the method is not directly applicable there, but the presence of a pseudo-SU(3) symmetry[19–22] can provide the intermediate link previously missing. It is at least possible that by pushing in this direction, there is a chance of finding a way to theoretically justify the IBM in the SU(3) limit. It is worth mentioning that by contrast the SO(8) model and any single-j model with the symmetry of SO[2(2j + 1)], in general, cannot be useful in this direction, since they lack an SU(3) subgroup.

It is also of interest to generalize the commutation relations of Sec. II to include proton-neutron pairs, in addition to proton-proton and neutron-neutron pairs. Then one can find exact boson mappings which will be applicable to the realistic p-shell and s-d shell nuclei, where protons and neutrons occupy the same major shell and thus presence of proton-neutron pairs is expected.

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APPENDIX A

In this appendix we apply the commutator method in order to determine the unknown numerical coefficients in Eqs. (4.27) and (4.28). The numerical coefficients in Eqs. (4.11a) and (4.11b) have already been calculated in Sec. IV.

We first make use of the commutator

$$[A^{(0)},A^{(0)\dagger}]=1-\frac{B^{(0)}}{\sqrt{3}}\ . \qquad (A1)$$

Using (4.25) and (4.27) it is very easy to find

$$[A^{(0)},A^{(0)\dagger}]=\gamma_0+2\gamma_1 a_0^\dagger a_0+\gamma_3[a(2)^\dagger\times\tilde{a}(2)]^{(0)}\ . \qquad (A2)$$

Substituting (A2) in (A1) implies

$$\gamma_0=+1 \qquad (A3)$$

and

$$B^{(0)}=-2\sqrt{3}\gamma_1 a_0^\dagger a_0-\sqrt{3}\gamma_3[a(2)^\dagger\times\tilde{a}(2)]^{(0)}\ . \qquad (A4)$$

We know that

$$[B^{(0)},B_\mu^{(1)}]=0\ , \tag{A5}$$

$$[B^{(0)},B_\mu^{(2)}]=0\ . \tag{A6}$$

These imply that $B^{(0)}$ has to be proportional to the boson number

$$N=a_0^\dagger a_0+\sqrt{5}[a(2)^\dagger\times\tilde{a}(2)]^{(0)}\ . \tag{A7}$$

Thus

$$2\gamma_1=\frac{\gamma_3}{\sqrt{5}}\ . \tag{A8}$$

We then use the commutator

$$[A_\mu^{(2)},A^{(0)\dagger}]=-\frac{1}{\sqrt{3}}(-1)^\mu B_{-\mu}^{(2)}\ . \tag{A9}$$

From (4.26) and (4.27) it is easy to find

$$\begin{aligned}[A_\mu^{(2)},A^{(0)\dagger}]=&\frac{2}{\sqrt{5}}(-1)^\mu\gamma_2 a(2)_{-\mu}^\dagger a_0\\&+\frac{1}{\sqrt{5}}(-1)^\mu\gamma_3 a_0^\dagger\tilde{a}(2)_{-\mu}\\&+\frac{2}{\sqrt{5}}(-1)^\mu\gamma_4[a(2)^\dagger\times\tilde{a}(2)]_{-\mu}^{(2)}\ .\end{aligned} \tag{A10}$$

Substituting (A10) and (4.11b) in (A9) and equating coefficients of the same tensor in the left- and right-hand sides we obtain the following conditions:

The coefficient of $a(2)_{-\mu}^\dagger a_0$ gives:

$$\gamma_2=-\frac{\sqrt{5}}{3}\ . \tag{A11}$$

The coefficient of $a_0^\dagger\tilde{a}(2)_{-\mu}$ gives:

$$\gamma_3=-\frac{2\sqrt{5}}{3}\ . \tag{A12}$$

The coefficient of $[a(2)^\dagger\times\tilde{a}(2)]_{-\mu}^{(2)}$ gives:

$$\gamma_4=-\frac{\sqrt{35}}{6}\ . \tag{A13}$$

Then (A8) provides

$$\gamma_1=-\tfrac{1}{3}\ . \tag{A14}$$

Thus all coefficients in (4.27) are already found. Our next task is to find the coefficients in (4.28). We first use the commutator

$$[A^{(0)},A_\mu^{(2)\dagger}]=-\frac{1}{\sqrt{3}}B_\mu^{(2)}\ . \tag{A15}$$

Using (4.25) and (4.28) in (A15) it is easy to find

$$\begin{aligned}[A^{(0)},A_\mu^{(2)\dagger}]=&\delta_1 a(2)_\mu^\dagger a_0+2\delta_2 a_0^\dagger\tilde{a}(2)_\mu\\&+\delta_3[a(2)^\dagger\times\tilde{a}(2)]_\mu^{(2)}\ .\end{aligned} \tag{A16}$$

Substituting (A16) and (4.11b) in (A15) and equating coefficients of the same tensors we obtain the following conditions:

The coefficient of $a(2)_\mu^\dagger a_0$ gives:

$$\delta_1=-\tfrac{2}{3}\ . \tag{A17}$$

The coefficient of $a_0^\dagger\tilde{a}(2)_\mu$ gives:

$$\delta_2=-\tfrac{1}{3}\ . \tag{A18}$$

The coefficient of $[a(2)^\dagger\times\tilde{a}(2)]_\mu^{(2)}$ gives:

$$\delta_3=-\frac{\sqrt{7}}{3}\ . \tag{A19}$$

From the commutator

$$\begin{aligned}[A_\mu^{(2)},A_\nu^{(2)\dagger}]=\delta_{\mu\nu}-(-1)^\mu 5\Bigg[&(2\ -\mu\ 2\ \mu\,|\,0\ 0)\begin{Bmatrix}2&2&0\\1&1&1\end{Bmatrix}B^{(0)}+(2\ -\mu\ 2\ \nu\,|\,2\ \nu-\mu)\begin{Bmatrix}2&2&2\\1&1&1\end{Bmatrix}B_{\nu-\mu}^{(2)}\\&+(2\ -\mu\ 2\ \nu\,|\,1\ \nu-\mu)\begin{Bmatrix}2&2&1\\1&1&1\end{Bmatrix}B_{\nu-\mu}^{(1)}\Bigg]\end{aligned} \tag{A20}$$

it is immediately seen that

$$\delta_0=+1\ . \tag{A21}$$

Finally, we use the commutator

$$[B_\mu^{(2)},A^{(0)\dagger}]=\frac{2}{\sqrt{3}}A_\mu^{(2)\dagger}\ . \tag{A22}$$

From (4.11b) and (4.27) it is straightforward to find

$$\begin{aligned}[B_\mu^{(2)},A^{(0)\dagger}]=&\beta a(2)_\mu^\dagger+2\beta\gamma_1 a(2)_\mu^\dagger a_0^\dagger a_0-\beta\gamma_1 a_0^\dagger a_0^\dagger\tilde{a}(2)_\mu+\beta\gamma_3 a(2)_\mu^\dagger[a(2)^\dagger\times\tilde{a}(2)]^{(0)}\\&-\frac{\beta\gamma_3}{\sqrt{5}}a(2)_\mu^\dagger a_0^\dagger a_0+\frac{2\beta\gamma_2}{\sqrt{5}}a(2)_\mu^\dagger a_0^\dagger a_0-\beta\gamma_2[a(2)^\dagger\times a(2)^\dagger]^{(0)}\tilde{a}(2)_\mu+\frac{\beta\gamma_3}{\sqrt{5}}a_0^\dagger a_0^\dagger\tilde{a}(2)_\mu\\&-\beta\gamma_4[a(2)^\dagger\times a(2)^\dagger]_\mu^{(2)}a_0+\frac{2\beta\gamma_4}{\sqrt{5}}a_0^\dagger[a(2)^\dagger\times\tilde{a}(2)]_\mu^{(2)}+\frac{2}{\sqrt{5}}\gamma\gamma_2[a(2)^\dagger\times a(2)^\dagger]_\mu^{(2)}a_0\\&+\frac{2}{\sqrt{5}}\gamma\gamma_4[a(2)^\dagger\times[a(2)^\dagger\times\tilde{a}(2)]^{(2)}]_\mu^{(2)}-\frac{1}{\sqrt{5}}\gamma\gamma_4[[a(2)^\dagger\times a(2)^\dagger]^{(2)}\times\tilde{a}(2)]_\mu^{(2)}\ .\end{aligned} \tag{A23}$$

Substituting (A23) and (4.28) in (A22) and equating coefficients of the same tensors we obtain the conditions:

The coefficient of $[a(2)^\dagger \times a(2)^\dagger]^{(2)}_\mu a_0$ gives:

$$\delta_4 = -\frac{\sqrt{7}}{6} . \tag{A24}$$

The coefficient of $[a(2)^\dagger \times a(2)^\dagger]^{(0)} \tilde{a}(2)$ gives:

$$\delta_{50} = -\frac{1}{6\sqrt{5}} . \tag{A25}$$

The coefficient of $[[a(2)^\dagger \times a(2)^\dagger]^{(2)} \times \tilde{a}(2)]^{(2)}_\mu$ gives:

$$\delta_{52} = \tfrac{1}{6} . \tag{A26}$$

The coefficient of $[[a(2)^\dagger \times a(2)^\dagger]^{(4)} \times \tilde{a}(2)]^{(2)}_\mu$ gives:

$$\delta_{54} = -\frac{3}{\sqrt{5}} . \tag{A27}$$

The coefficients of $a(2)^\dagger_\mu$, $a(2)^\dagger_\mu a_0^\dagger a_0$, $a_0^\dagger a_0^\dagger \tilde{a}(2)_\mu$, and $a_0^\dagger [a(2)^\dagger \times \tilde{a}(2)]^{(2)}_\mu$ provide equations which are already identities. In obtaining (A25)—(A27) the recoupling formula

$$[a(2)^\dagger \times [a(2)^\dagger \times \tilde{a}(2)]^{(L)}]^{(2)}_\mu = \sum_J [(2L+1)(2J+1)]^{1/2} \begin{Bmatrix} 2 & 2 & L \\ 2 & 2 & J \end{Bmatrix} \times [[a(2)^\dagger \times a(2)^\dagger]^{(J)} \times \tilde{a}(2)]^{(2)}_\mu \tag{A28}$$

has been used. Since we are using the

$$[[a(2)^\dagger \times a(2)^\dagger]^{(L)} \times \tilde{a}(2)]^{(2)}_\mu \quad (L = 0,2,4)$$

as the three linearly independent 3-d boson terms, no

$$[a(2)^\dagger \times [a(2)^\dagger \times \tilde{a}(2)]^{(L)}]^{(2)}_\mu \quad (L = 0,1,2,3,4)$$

terms appear, thus

$$\delta_{60} = \delta_{61} = \delta_{62} = \delta_{63} = \delta_{64} = 0 . \tag{A29}$$

By now all the unknown coefficients have been found. It is straightforward to check that all the remaining commutation relations, e.g., (A20) and

$$[A^{(0)\dagger}, A^{(2)\dagger}_\mu] = 0 , \tag{A30}$$

are satisfied. Also (A4) provides

$$B^{(0)} = \frac{2}{\sqrt{3}} a_0^\dagger a_0 + \frac{2\sqrt{5}}{\sqrt{3}} [a(2)^\dagger \times \tilde{a}(2)]^{(0)} = \frac{2}{\sqrt{3}} N . \tag{A31}$$

APPENDIX B

In this appendix we verify that our results of Sec. IV can be put in the form (5.1) and (5.2) and we calculate the constants ϵ, ϵ_0, ϵ_1, ϵ_2, and x, appearing in Eqs. (5.1), (5.2), (5.4), and (5.5).

If the quantity C_2 of Eq. (5.5) is the second-order Casimir invariant of SU(3), it has to commute with the generators of this algebra, i.e., it must satisfy the conditions

$$[B^{(1)}_\mu, C_2] = 0 , \tag{B1}$$

$$[B^{(2)}_\mu, C_2] = 0 . \tag{B2}$$

The first of these conditions is always satisfied, without imposing any restrictions on the unknown coefficients ϵ_1, ϵ_2, since $B^{(1)}_\mu$ is the angular momentum operator and the terms included in C_2 already have good angular momentum. But using (5.5), (4.14), and (4.19) in (B2) we obtain

$$[B^{(2)}_\mu, C_2] = \left[-\frac{\epsilon_1}{\epsilon} - \frac{\epsilon_2}{\epsilon} \sqrt{3/5} \right] ([B^{(1)} \times B^{(2)}]^{(2)}_\mu - [B^{(2)} \times B^{(1)}]^{(2)}_\mu) . \tag{B3}$$

This vanishes if the condition

$$\epsilon_1 = -\sqrt{(3/5)} \epsilon_2 \tag{B4}$$

holds.

We then check if our result for $A^{(0)\dagger}$ of Sec. IV agrees with (5.1). Using (5.3) and the explicit form of (5.4) [obtained when $B^{(0)}$, $B^{(1)}_\mu$, and $B^{(2)}_\mu$ are substituted into it from Eqs. (A31), (4.11a), and (4.11b), respectively] in (5.1) we obtain

$$\begin{aligned} A^{(0)\dagger} = {} & a_0^\dagger - x a_0^\dagger a_0^\dagger a_0 - x\sqrt{5} a_0^\dagger [a(2)^\dagger \times \tilde{a}(2)]^{(0)} + \tfrac{4}{3} \epsilon_0 (a_0^\dagger + 2 a_0^\dagger a_0^\dagger a_0) + \frac{8\sqrt{5}}{3} \epsilon_0 a_0^\dagger [a(2)^\dagger \times \tilde{a}(2)]^{(0)} \\ & + \tfrac{4}{3} \epsilon_2 2 [a(2)^\dagger \times a(2)^\dagger]^{(0)} a_0 + \tfrac{4}{3} \epsilon_2 a_0^\dagger [a(2)^\dagger \times \tilde{a}(2)]^{(0)} + \tfrac{4}{3} \epsilon_2 a_0^\dagger [\tilde{a}(2) \times a(2)^\dagger]^{(0)} \\ & \times \frac{2\sqrt{7}}{3} \epsilon_2 [a(2)^\dagger \times [a(2)^\dagger \times \tilde{a}(2)]^{(2)}]^{(0)} + \frac{2\sqrt{7}}{3} \epsilon_2 [[a(2)^\dagger \times \tilde{a}(2)]^{(2)} \times a(2)^\dagger]^{(0)} . \end{aligned} \tag{B5}$$

Notice that the above expression is not in normal order. It can be put in such a form using the relations

$$[\tilde{a}(2) \times a(2)^\dagger]^{(0)} = [a(2)^\dagger \times \tilde{a}(2)]^{(0)} + \sqrt{5} , \tag{B6}$$

$$[[a(2)^\dagger \times \tilde{a}(2)]^{(2)} \times a(2)^\dagger]^{(0)} = [a(2)^\dagger \times [a(2)^\dagger \times \tilde{a}(2)]^{(2)}]^{(0)} . \quad \text{(B7)}$$

Equation (B5) must agree with Eq. (4.27). Equating coefficients of the same tensor in these two expressions, we obtain the following conditions:

The coefficient of $[a(2)^\dagger \times a(2)^\dagger]^0 a_0$ gives:

$$\epsilon_2 = -\frac{\sqrt{5}}{8} . \quad \text{(B8)}$$

The coefficient of $a_0^\dagger$ gives:

$$\epsilon_0 = \tfrac{5}{8} . \quad \text{(B9)}$$

The coefficient of $a_0^\dagger a_0^\dagger a_0$ gives:

$$x = 2 . \quad \text{(B10)}$$

The coefficients of

$$[a(2)^\dagger \times [a(2)^\dagger \times \tilde{a}(2)]^{(2)}]^{(0)}$$

and

$$a_0^\dagger [a(2)^\dagger \times \tilde{a}(2)]^{(0)}$$

provide equations which are already identities. Now Eq. (B4) provides

$$\epsilon_1 = \frac{\sqrt{3}}{8} . \quad \text{(B11)}$$

Agreement between Eqs. (5.2) and (4.28) can be checked in the same way. All tensors give results consistent with the values of the coefficients found above.

We still need to find the constant ϵ. In order to determine ϵ, we first act with the explicit form of ϵC_2 [which can be found from (5.5) after substituting $B_\mu^{(1)}$ and $B_\mu^{(2)}$ with their equals from Eqs. (4.11a) and (4.11b), respectively] on the state $a_0^\dagger |0\rangle$, where $|0\rangle$ is the boson vacuum. Putting the expression $C_2 a_0^\dagger$ in normal order, we remark that only the term

$$a_0^\dagger a_0 [\tilde{a}(2) \times a(2)^\dagger]^{(0)}$$

of C_2 makes a nonvanishing contribution when $C_2 a_0^\dagger$ acts on the boson vacuum. In fact we find that

$$\epsilon C_2 a_0^\dagger |0\rangle = \tfrac{4}{3}\epsilon_2 \sqrt{5} a_0^\dagger |0\rangle . \quad \text{(B12)}$$

On the other hand, the state $a_0^\dagger |0\rangle$ belongs to the (2,0) irrep of SU(3), thus using Eq. (6.22) we obtain

$$\epsilon C_2 a_0^\dagger |0\rangle = \epsilon \tfrac{20}{3} . \quad \text{(B13)}$$

Comparing (B12) and (B13) we get

$$\epsilon = -\tfrac{1}{8} . \quad \text{(B14)}$$

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