Exchange contributions in $\alpha + {}^{9}Be$ elastic scattering

B. B. Srivastava and Archana Saxena
Physics Department, Meerut University, Meerut 250005, India
(Received 14 May 1984)

The relative importance of the various exchange terms in the single-channel resonating group method calculations of $\alpha + {}^{9}Be$ elastic scattering has been investigated by calculating in each case the $l=0$ norm kernel ranges, the $l=0$ to 7 phase shifts, and the differential cross sections in the c.m. incident energies in the range 0–8 MeV. A central two-nucleon interaction commonly taken in resonating group method calculations has been used, and for ${}^{9}Be$ a cluster model wave function has been taken. The results have been compared with one another and with the experimental data at 6.06 and 7.01 MeV. It is concluded that the resonating group method in the single-channel approximation can describe reasonably well the $\alpha + {}^{9}Be$ elastic scattering at incident energies of about 6 MeV and below. The one-nucleon exchange terms between the alpha particle and ${}^{9}Be$ are by far the most significant, followed by those involving the exchange of the whole alpha clusters.

I. INTRODUCTION

Microscopic studies of nucleus-nucleus elastic scattering can be successfully carried out using the resonating group method (RGM).[1,2] In the RGM, totally antisymmetric wave functions are used so that the Pauli principle is fully taken into account, a two-nucleon interaction which explains quite well the two-nucleon low-energy scattering and bound state data is used, the motion of the total center of mass is treated correctly, the cluster correlations are explicitly taken into consideration, and the nuclear bound state, scattering, and reaction problems are treated in a unified manner. But the application of the RGM, even in the single channel approximation, to systems having a large number of nucleons becomes very difficult mainly due to the large number of kernel terms that result from the antisymmetrization procedure. The generator coordinate techniques,[3–5] developed recently, have made the application of the RGM tractable in the case of systems involving a large number of nucleons. With the use of these techniques, a number of RGM calculations[6] on nucleus-nucleus elastic scattering have been carried out during the last few years. But even with the use of these techniques the calculations remain complex enough to call for investigations which could lead to further simplifications under reasonable assumptions. Recently, Baldock *et al.*,[7] using the natural boundary condition method, have investigated the effects on the phase shifts of consistently approximating the wave function by allowing only up to a few particle exchange between the colliding nuclei. For the ${}^{16}O$—${}^{16}O$ case they find that in general all exchange terms contribute significantly although the nature of these exchange effects is dependent upon the choice of the effective nucleon-nucleon interaction. However, following the usual approach of obtaining the RGM scattering equation and using a less tedious approximation method than that of Baldock *et al.*,[7] investigations[8] in some cases have revealed that many of the exchange kernel terms in it are relatively unimportant and can be neglected. Although, when only a few exchange terms are included, the approach of Baldock *et al.*[7] is formally more satisfactory, the earlier approaches[8] including a relatively larger number but not all of the exchange terms chosen judiciously on physical considerations may yield results of reasonable accuracy, since, in the limit of all exchanges being considered the two procedures should yield identical results. Further investigations from this point of view are therefore desirable since fully antisymmetrized calculations are extremely tedious for systems involving heavy nuclei, particularly, those far from closed shell structure. A systematic study of the RGM, with the purpose of determining as to which of the exchange kernel terms can be neglected in a particular case depending upon the structures of the nuclei involved and the incident energies considered, will allow one to obtain reasonably good estimates of the phase shifts and the differential cross sections keeping the RGM calculations within practical limits. The present investigation has been undertaken with this purpose in mind and attempts to determine the relative importance of the various exchange kernel terms in the elastic scattering of alpha particles from the ${}^{9}Be$ nucleus in single-channel RGM calculations. The kernel terms can be divided according to the number of nucleons exchanged between the projectile and the target. Both the direct potential, except for the Coulomb part, and the exchange kernel terms are short ranged due to the short-ranged nuclear forces in the case of the former and due to the finite range of the bound state functions in the case of the latter. The ranges of these terms can be defined under certain conditions[8] so that at lower energies only those terms which have the largest ranges will be important. This type of study is appropriate for the scattering of two nuclei with all nucleons strongly bound in the nuclear potential well. However, the situation becomes more complicated if at least one of the colliding nuclei has pronounced surface clustering. In such cases one can consider classes of nucleon permutations on the basis of the cluster structure of the nuclei involved and determine their relative importance. In the present investigation this latter approach will be used in detail since the ${}^{9}Be$ nucleus can, to a good approximation, be described as consisting of two alpha clusters plus a neutron.[9,10] The ranges of norm kernel terms will also be estimated to draw conclusions regarding the relative importance of the various exchange terms.

Experimental measurements of the $\alpha+{}^9$Be elastic scattering have been carried out by a number of investigators.[11–15] However, most of the data are at somewhat higher energies. The data have so far been only analyzed using the macroscopic optical model[11,15] and no microscopic studies have been carried out. It is for these reasons that the $\alpha+{}^9$Be scattering has been microscopically studied in the present investigation. Calculations of the phase shifts and the differential cross sections have been carried out using different sets of exchange kernel terms in the single-channel RGM scattering equation at low energies and compared with the available experimental data to investigate their relative importance. This investigation will be of considerable interest, since on the basis of these analyses one may ignore the not-so-significant exchange terms from the beginning and significantly reduce the computational efforts in similar calculations.

II. FORMULATION

The translationally invariant Hamiltonian for the thirteen-nucleon system $\alpha+{}^9$Be is written as

$$H=\sum_{i=1}^{13}\left[-\frac{\hbar^2}{2M}\nabla_i^2\right]+\sum_{i<j=1}^{13}V_{ij}-T_{\text{c.m.}} , \qquad (1)$$

where $T_{\text{c.m.}}$ is the kinetic energy operator of the center of mass of the total system, M is the nucleon mass, and V_{ij} is the two-nucleon interaction which is taken to have the form[16]

$$V_{ij}=-V_0\exp(-kr_{ij}^2)(w+mP_{ij}^r+bP_{ij}^\sigma-hP_{ij}^\tau)+\theta_{ij}\frac{e^2}{r_{ij}} . \qquad (2)$$

In Eq. (2), P_{ij}^r, P_{ij}^σ, and P_{ij}^τ are the space, spin, and isospin exchange operators, respectively, and θ_{ij} is equal to unity if both i and j are protons and zero otherwise. The exchange parameters w, m, b, and h satisfy the conditions

$$\begin{aligned} &w+m+b+h=1 ,\\ &w+m-b-h=x , \end{aligned} \qquad (3)$$

and the constants V_0, k, and the ratio of singlet to triplet interaction x have been chosen to fit the low-energy two-nucleon data and are taken as

$$V_0=-72.98\ \text{MeV},\quad k=0.46\ \text{fm}^{-2},\quad x=0.63 . \qquad (4)$$

The values of w, m, b, and h are obtained from Eq. (3) assuming Serber exchange mixture, i.e., $w=m$, $b=h$.

The wave function for the system $\alpha+{}^9$Be in its center of mass in the single-channel approximation is taken as

$$\psi_T=\mathscr{A}[\phi_{\text{Be}}\phi_\alpha F(\underline{R})\xi(\sigma,\tau)] , \qquad (5)$$

where $\mathscr{A}$ is the total antisymmetrization operator; ϕ_{Be} and ϕ_α are the wave functions for the internal motions of ^{9}Be and incident alpha particle, respectively; F describes the relative motion of the two colliding nuclei with $\underline{R}$ as the distance between their centers of mass; and ξ is the spin-isospin function for the system.

Using the projection equation[2]

$$\langle\delta\psi_T\,|\,H-E_T\,|\,\psi_T\rangle=0 \qquad (6)$$

and writing the total antisymmetrization operator $\mathscr{A}$ as $\hat{\mathscr{A}}\mathscr{A}_1\mathscr{A}_2$, where $\mathscr{A}_1$ is the antisymmetrization operator with respect to the nucleons in ^{9}Be, $\mathscr{A}_2$ is that with respect to the nucleons in the incident alpha particle, and $\hat{\mathscr{A}}$ is the rest antisymmetrizer, one gets the integro-differential equation

$$\left[-\frac{\hbar^2}{2\mu}\nabla_R^2-E+V_D\right]F(\underline{R})+\int K(\underline{R},\underline{R}')F(\underline{R}')d\underline{R}'=0 \qquad (7)$$

for the relative motion function $F(\underline{R})$, where μ is the reduced mass for the $\alpha+{}^9$Be system and the energy of relative motion E, the direct potential $V_D(R)$ and the exchange kernel $K(\underline{R},\underline{R}')$ are given by

$$E=E_T-\langle(\mathscr{A}_1\phi_{\text{Be}})\,|\,H_{\text{Be}}\,|\,(\mathscr{A}_1\phi_{\text{Be}})\rangle-\langle(\mathscr{A}_2\phi_\alpha)\,|\,H_\alpha\,|\,(\mathscr{A}_2\phi_\alpha)\rangle , \qquad (8)$$

$$V_D(R)=\langle(\mathscr{A}_1\phi_{\text{Be}})(\mathscr{A}_2\phi_\alpha)\,|\,V_{\text{Be-}\alpha}\,|\,(\mathscr{A}_1\phi_{\text{Be}})(\mathscr{A}_2\phi_\alpha)\rangle , \qquad (9)$$

and

$$K(\underline{R},\underline{R}')=\langle(\mathscr{A}_1\phi_{\text{Be}})(\mathscr{A}_2\phi_\alpha)\,|\,(H-E_T)(\hat{\mathscr{A}}-1)\,|\,(\mathscr{A}_1\phi_{\text{Be}})(\mathscr{A}_2\phi_\alpha)\delta(\underline{R}-\underline{R}')\rangle . \qquad (10)$$

In Eqs. (8)–(10), $V_{\text{Be-}\alpha}$ is the potential energy for the interaction between ^{9}Be and the incident alpha particle,

$$V_{\text{Be-}\alpha}=\sum_{\substack{i\in\text{Be}\\ j\in\alpha}}V_{ij} , \qquad (11)$$

and H_{Be} and H_α are the Hamiltonian operators governing the internal motions of nuclei ^{9}Be and incident alpha particle, respectively. Here the spin-isospin functions of ^{9}Be and the alpha particle have been included in ϕ_{Be} and ϕ_α.

A. Wave function for the ^{9}Be nucleus

Following Tang *et al.*[9] the nucleus ^{9}Be is assumed to have a structure consisting of two alpha clusters moving about one another and a neutron moving about their center of mass. Since the spin parity of the ^{9}Be ground state is $J^\pi=\frac{3}{2}^-$

and that of the ^{8}Be ground state is $J^\pi=0^+$, the cluster model wave function for ^{9}Be is taken as

$$\phi_{(3/2)M}(^9\mathrm{Be})=\mathscr{A}_1\left\{\phi_{00}(^8\mathrm{Be})\left[\sum_{m_l+m_s=M}C(l,m_l,\tfrac{1}{2},m_s\mid\tfrac{3}{2}M)\chi_{lm_l}\xi_{(1/2)m_s}(\sigma_9,\tau_9)\right]\right\} \tag{12}$$

where $\phi_{00}(^8\mathrm{Be})$ is the ^{8}Be ground state wave function, χ_{lm_l} is the function for the relative motion of the neutron with respect to the center of mass of ^{8}Be, $\xi_{(1/2)m_s}$ is the spin-isospin function of the neutron, and $C(l,m_1,\frac{1}{2},m_s\mid\frac{3}{2}M)$ are the Clebsch-Gordan coefficients for the coupling of the spin of the neutron with its orbital angular momentum l about the center of mass of ^{8}Be. The functions $\phi_{00}(^8\mathrm{Be})$ and χ_{lm_l} are chosen to be the same as described in Ref. 9. This wave function was used in a variational procedure to obtain the binding energy of the ^{9}Be nucleus. It may be mentioned that these binding energy calculations also included the Coulomb potential energy terms whose expectation value was obtained using the procedure given by Kamimura and Matsuse.[17] This part of the binding energy was only crudely estimated in the earlier calculations.[9]

For use in the scattering calculations an average of the ^{9}Be wave function for various possible M values was taken, i.e.,

$$\phi_{^9\mathrm{Be}}=\tfrac{1}{4}\mathscr{A}_1[\phi_{(3/2)(3/2)}(^9\mathrm{Be})+\phi_{(3/2)(1/2)}(^9\mathrm{Be})+\phi_{(3/2)(-1/2)}(^9\mathrm{Be})+\phi_{(3/2)(-3/2)}(^9\mathrm{Be})]\ . \tag{13}$$

The width parameter α for the alpha cluster wave functions was taken as 0.514 fm^{-2}, which matches the experimental rms radius of 1.48 fm for the free alpha particle and yields a binding energy $-$ 27.79 MeV which is also in good agreement with the experimental value of -28.3 MeV for the alpha particle. Minimization of the total energy of ^{9}Be with respect to the width parameters β_1 and β_2 of the oscillator functions describing the relative motion of the two alpha clusters in ^{8}Be and the motion of the neutron relative to the center of mass of ^{8}Be, respectively, yielded a value -57.9 MeV for $\beta_1=0.288$ fm^{-2} and $\beta_2=0.411$ fm^{-2}. This is in good agreement with the experimental value[16] of -58.2 MeV for the binding energy of ^{9}Be.

B. Exchange considerations in the antisymmetrization process

For the thirteen-nucleon system $\alpha+{}^9$Be, the antisymmetrization operator $\mathscr{A}$ will contain a very large number of exchange terms. A number of these terms were dropped from the outset in accordance with the following considerations based on the cluster structure of ^{9}Be. First, since the bound state relative motion functions between the clusters have a long tail, at the low incident energies considered, the incident alpha particle is likely to come very close to only one of the clusters of ^{9}Be at any time. Hence, terms including the simultaneous exchanges of two or more nucleons from the incident alpha particle and from at least two of the three clusters of the ^{9}Be nucleus were considered relatively less probable and were therefore ignored. Second, in the scattering of heavy ions with identical cores Clement *et al.*[8] have shown that the terms corresponding to the total exchange of the identical cores are relatively more significant. Hence, for the investigation of the relative importance of exchange contributions in $\alpha+{}^9$Be elastic scattering the following classes of exchanges were considered in the antisymmetrization process.

Case a: Exchanges considered include all exchanges of nucleons between the two alpha clusters of ^{9}Be in the antisymmetrization ($\mathscr{A}_1$) of the ^{9}Be wave function and those between the incident alpha particle and any one of the two alpha clusters of ^{9}Be in the rest antisymmetrization ($\hat{\mathscr{A}}$).

Case b: Exchanges considered include those corresponding to full antisymmetrization ($\mathscr{A}_1$) of the ^{9}Be wave function and those between the nucleons of the incident alpha particle and of any one of the three clusters of the ^{9}Be nucleus in the rest antisymmetrization ($\hat{\mathscr{A}}$).

Case b1: This is the same as b except that only the exchanges of nucleons of the incident alpha particle and those of any one of the two alpha clusters of ^{9}Be were included.

Case b2: This is the same as b1 except that in $\hat{\mathscr{A}}$ only one-nucleon exchanges between the nucleons of the incident alpha particle and those of the two alpha clusters of the ^{9}Be nucleus were included.

Case b3: This is the same as b1 except that in $\hat{\mathscr{A}}$ only two-nucleon exchanges between the nucleons of the incident alpha particle and those of the two alpha clusters of the ^{9}Be nucleus were included.

Case b4: This is the same as b1 except that in $\hat{\mathscr{A}}$ only three-nucleon exchanges between the nucleons of the incident alpha particle and those of the two alpha clusters of the ^{9}Be nucleus were included.

Case b5: This is the same as b1 except that in $\hat{\mathscr{A}}$ only the four-nucleon exchanges between the nucleons of the incident alpha particle and those of the two alpha clusters of the ^{9}Be nucleus were included.

Case b6: This is the same as b1 except that in $\hat{\mathscr{A}}$ only the exchange of the neutron cluster of ^{9}Be with the nucleons of the incident alpha particle was included.

Case b7: This is the same as b except that the two-nucleon and three-nucleon exchanges between the nucleons of the incident alpha particle and those of the two alpha clusters of ^{9}Be were neglected.

C. Calculations

For each of the preceding classes the integro-differential equation[7] for the scattering function was explicitly determined. The calculation of the Coulomb potential terms is quite tedious and hence the terms were calculated approximately as follows. The Coulomb part of the direct potential $V_C(R)$ was calculated with the ^{9}Be wave function in the antisymmetrization of which only the exchanges between the nucleons of the two alpha clusters were included. The kernel term $K^C(\underline{R},\underline{R}')$, corresponding to the Coulomb interaction was approximated as[18]

$$K^C(\underline{R},\underline{R}')=V_C\left[\frac{\underline{R}+\underline{R}'}{2}\right]K_N(\underline{R},\underline{R}')\ , \quad (14)$$

where $K_N(\underline{R},\underline{R}')$ is the normalization kernel. The expressions for the direct potential and kernel terms are quite lengthy and are given elsewhere.[19]

In each of the nine cases previously mentioned the integro-differential equation corresponding to the lth partial wave was obtained following the usual procedure and was solved numerically using a method similar to that described by Robertson[20] to obtain the nuclear phase shifts δ_l. The differential cross section $\sigma(\theta)$ was obtained from the relation[21]

$$\sigma(\theta)=|f_C(\theta)+f_N(\theta)|^2\ , \quad (15)$$

where $f_C(\theta)$ is the pure Coulomb scattering amplitude and $f_N(\theta)$ is the nuclear scattering amplitude. The required Coulomb phase shifts were calculated as usual except that the one for $l=0$ was calculated using the procedure described by Melkanoff *et al.*[22]

The ranges Δ of the norm kernel terms corresponding to the nine cases of exchanges described previously were estimated as[8]

$$\Delta\sim(A_i-|B_i|)^{-1/2} \quad (16)$$

if the norm kernel term $K_N(\underline{R},\underline{R}')$ behaved asymptotically as

$$K_N(\underline{R},\underline{R}')\sim\exp\{-[A_i(R^2+R'^2)+2B_i\underline{R}\cdot\underline{R}']\}\ . \quad (17)$$

III. RESULTS AND DISCUSSIONS

The $l=0$ to 7 nuclear phase shifts δ_l for $\alpha+{}^9$Be elastic scattering obtained for the c.m. incident energies in the range 0–8 MeV are shown in Figs. 1 and 2 for each of the nine classes of exchange terms considered. The $l=8$ phase shifts were found to be negligible near the top of this energy range. The differential cross sections calculated for the c.m. incident energies of 6.06 and 7.01 MeV at which the experimental data of Brady *et al.*[15] are available within the energy range considered are shown in Figs. 3 and 4. The estimated range of the $l=0$ norm kernel which corresponds to one-nucleon exchange between the neutron cluster of ^{9}Be and the incident alpha particle is found to be 2.69 fm, while those of the $l=0$ norm kernels corresponding to one-, two-, three-, and four-nucleon exchanges between two alpha clusters of ^{9}Be and the incident alpha particles are found to be 2.59, 1.70, 1.43, and 1.97 fm, respectively.

It is seen that the ranges of the one-nucleon exchange norm kernel terms are the largest indicating that the one-nucleon exchange terms are the most significant among all the exchange terms considered. This is in agreement with the results of earlier investigations.[23,24] Next in significance are those which correspond to the exchange of whole alpha clusters.

Since experimental phase shifts for $\alpha+{}^9$Be elastic

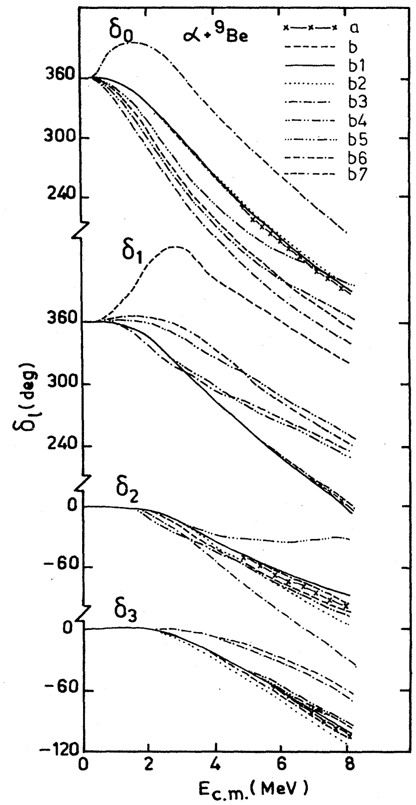

FIG. 1. The $l=0$–3 phase shifts (in degrees) for $\alpha+{}^9$Be elastic scattering from 0 to 8 MeV for the various cases of exchange terms considered.

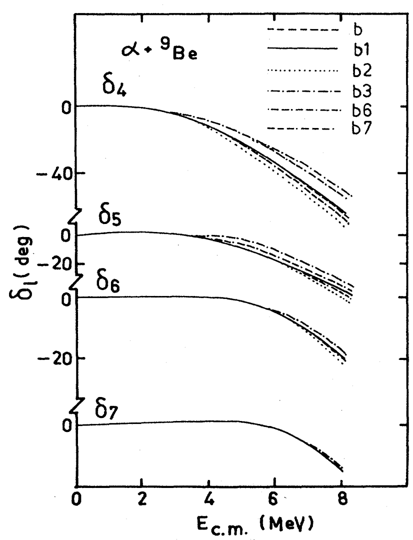

FIG. 2. The $l=4$–7 phase shifts (in degrees) for $\alpha+{}^9$Be elastic scattering from 0 to 8 MeV for the various cases of exchange terms considered. Curves for cases a, b4, and b5 coincide with those for case b1 and hence are not plotted.

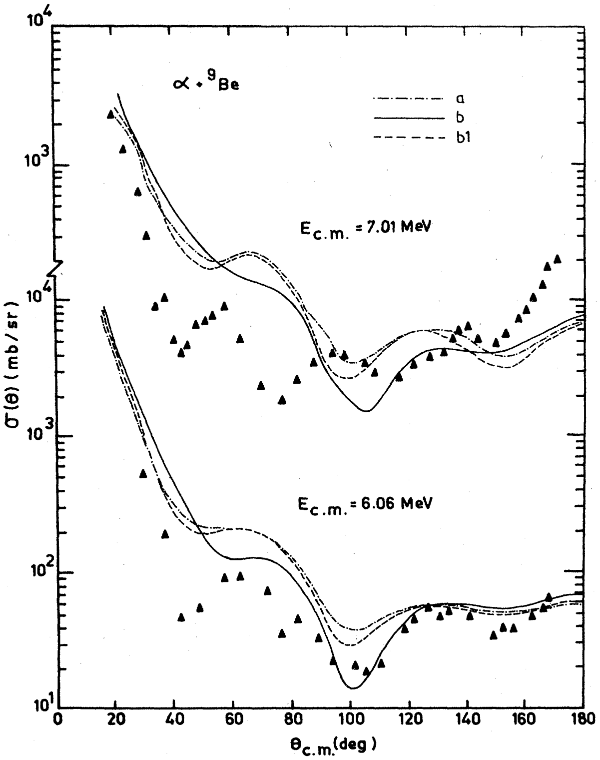

FIG. 3. Comparison of the calculated differential cross sections for $\alpha + {}^9$Be elastic scattering at 6.06 and 7.01 MeV for exchange cases a, b, and b1 and the experimental data (▲) of Ref. 15.

scattering are not available in the energy region of interest, the relative importance of the various categories of exchange terms with regard to the calculation of phase shifts may be discussed by comparison with those obtained for case b in which maximum antisymmetrization has been carried out. First, it is seen from Figs. 1 and 2

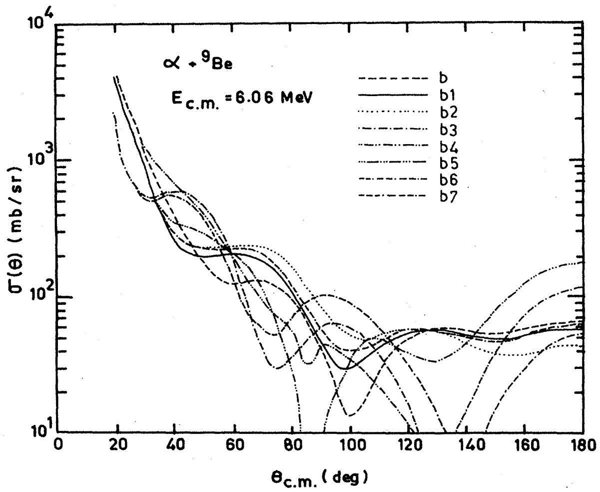

FIG. 4. Comparison of the calculated differential cross sections for $\alpha + {}^9$Be elastic scattering at 6.06 MeV for exchange cases b, b1, b2, b3, b4, b5, b6, and b7.

that all the phase shifts calculated for case a do not differ significantly from the corresponding phase shifts calculated for case b1. Hence, for these scattering calculations in the energy range of 0–8 MeV in the antisymmetrization of the ^{9}Be wave function, the exchanges involving the neutron cluster have a negligible effect compared to those involving the nucleons of the two alpha clusters. Next, comparing the results for cases b1 and b6 with those for case b, it is found that the $l = 0$–3 phase shifts differ significantly in both cases so that in the rest antisymmetrization one must consider the exchanges of nucleons of both the neutron cluster and the two alpha clusters of ^{9}Be with those of the incident alpha particle. Comparison of the results for case b7 with those for case b indicates that the two- and three-nucleon exchange terms between the target and the projectile appreciably affect only the $l = 0$ and 1 phase shifts. Finally, the comparison of the results for cases b2, b3, b4, and b5 with those for case b1 indicates that the one-nucleon exchange terms between the target and the projectile are the most significant. Next in significance are those which correspond to the exchange of the whole alpha clusters.

In Fig. 3 the differential cross sections (DCS's) calculated for cases b, b1, and a are compared with the experimental data of Brady *et al.*[15] shown by solid triangles. It is seen that the DCS's calculated for case b shown by solid curves are in reasonably good agreement with the experimental data at 6.06 MeV, while at 7.01 MeV the quality of agreement has become relatively poor. The DCS's calculated for case b1 shown by dashed curves shows nearly the same behavior as that of the results for case b, but though there seems to be better agreement at angles forward of 40° and in at the first valley, it overestimates to a large extent the DCS's in the angular region 50–130 deg. The dotted-dashed curve for case a resembles the dashed curve for case b1. For the incident energy of 7.01 MeV the three curves for cases b, b1, and a again show similar differences relative to one another. Comparison between curves b, b1, and a clearly indicates the importance of the exchange terms corresponding to the neutron cluster of ^{9}Be and the incident alpha particle. The result that the DCS's calculated for cases b1 and a are close to one another for most of the angular range supports the conclusion drawn earlier that the exchanges between the neutron cluster and the two alpha clusters of ^{9}Be are not of much significance.

In Fig. 4 the DCS's calculated for cases b2 to b7 are compared with those calculated for cases b and b1 to show the relative importance of the one-, two-, three-, and four-nucleon target-projectile exchange terms in DCS's calculations at an incident energy of 6.06 MeV. First, since the DCS's calculated in case b7 do not differ from those calculated for case b by any appreciable amount over the full angular range except for a small region around 100 deg, it may be inferred that the one-nucleon exchange terms along with the whole alpha cluster exchange terms are the most significant. Furthermore, since the DCS's calculated for case b6 are very much different from those calculated for case b, it is not adequate to consider only the exchange of the neutron cluster of ^{9}Be with the nucleons of the incident alpha particle. Finally, the comparison of the DCS's calculated for cases b2, b3, b4, and b5 with those calculated for case b1 shows that the one-nucleon exchange terms between the two alpha clusters of ^{9}Be and the incident alpha particle are the most significant ones, and next in significance are those involving the exchange of the whole alpha clusters. Since the $\alpha+{}^9$Be system does not seem to possess pronounced resonances in the energy region studied, no remarks can be made regarding the validity of these conclusions in their vicinity.

The results of the present investigation indicate that the RGM in the single-channel approximation is capable of describing reasonably well the elastic scattering of alpha particles by the ^{9}Be nucleus at energies of 6.06 MeV and below. Unfortunately, experimental DCS's are not available below 6 MeV to clearly illustrate the latter point. At 7.01 MeV the agreement worsens particularly at backward angles showing that at this energy more exchange terms should be included in the calculations. Also, at higher energies the effect of the nonelastic channels must be taken into account. The agreement between the calculated and the experimental results is likely to become still better if a more accurate wave function for ^{9}Be and two-nucleon interaction is taken. This investigation is of considerable interest in spite of these deficiencies as it is the only microscopic study of the $\alpha+{}^9$Be elastic scattering. For more definite conclusions regarding the relative importance of exchange kernel terms more studies of this type should be carried out for scattering systems involving nuclei with pronounced cluster structure.

One of the authors (A.S.) gratefully acknowledges the financial assistance from the University Grants Commission, New Delhi, during the course of this work.

[1]K. Wildermuth and W. McClure, *Cluster Representation of Nuclei* (Springer, Berlin, 1966).
[2]K. Wildermuth and Y. C. Tang, *A Unified Theory of the Nucleus* (Vieweg, Braunschweig, 1977).
[3]D. L. Hill and J. A. Wheeler, Phys. Rev. **89**, 1102 (1953); J. J. Griffin and J. A. Wheeler, *ibid.* **108**, 311 (1957).
[4]W. Sünkel and K. Wildermuth, Phys. Lett. **41B**, 439 (1972).
[5]D. R. Thompson and Y. C. Tang, Phys. Rev. C **12**, 1432 (1975).
[6]O. D. Sharma and B. B. Srivastava, J. Phys. G **6**, 209 (1980), and references therein.
[7]R. A. Baldock, B. A. Robson, and R. F. Barrett, Nucl. Phys. **A351**, 157 (1981).
[8]D. Clement, E. J. Kanellopoulos, and K. Wildermuth, Acta Phys. Austriaca **42**, 29 (1975).
[9]Y. C. Tang, F. C. Khanna, R. C. Herndon, and K. Wildermuth, Nucl. Phys. **35**, 421 (1962).
[10]J. Hiura and R. Tamagaki, Prog. Theor. Phys. (Suppl.) **52**, 25 (1972).
[11]R. B. Taylor, N. R. Fletcher, and R. H. Davis, Nucl. Phys. **65**, 318 (1965).
[12]B. T. Lucas, S. W. Cosper, and O. E. Johnson, Phys. Rev. **133**, B963 (1964).
[13]G. Gregoire and P. C. Macq, Phys. Lett. **8**, 328 (1964).
[14]R. G. Summers-Gill, Phys. Rev. **109**, 1591 (1958).
[15]F. P. Brady, J. A. Jungerman, and J. C. Young, Nucl. Phys. **A98**, 241 (1967).
[16]D. R. Thompson and Y. C. Tang, Phys. Rev. C **4**, 306 (1971); **4**, 1979 (1971).
[17]M. Kamimura and T. Matsuse, Prog. Theor. Phys. **51**, 438 (1974).
[18]H. Friedrich and K. Langanke, Nucl. Phys. **A252**, 47 (1975).
[19]Archana Saxena, Ph.D. thesis, Meerut University, Meerut, 1984 (unpublished).
[20]H. H. Robertson, Proc. Cambridge Philos. Soc. **52**, 538 (1956).
[21]L. I. Schiff, *Quantum Mechanics,* 3rd ed. (McGraw-Hill, New York, 1968), p. 145.
[22]M. A. Melkanoff, T. Sawada, and J. Raynal, in *Methods in Computational Physics,* edited by B. Alder, S. Fernbach, and M. Rotenberg (Academic, New York, 1966), Vol. 6.
[23]W. Sunkel and Y. C. Tang, Nucl. Phys. **A329**, 10 (1979).
[24]M. LeMere, D. J. Stubeda, H. Horiuchi, and Y. C. Tang, Nucl. Phys. **A320**, 449 (1979).