Phase ambiguities in the O(6) limit of the interacting boson model

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We present a generalization of the O(6) limit of the interacting boson model by considering an $E2$ transition operator which is different from the quadrupole operator used in the Hamiltonian. We analyze, both quantum mechanically and in the classical limit, under which conditions this more general O(6) limit is invariant under time reversal.

It is well known that there exists a sign ambiguity in the definition of the generators in both the SU(3) and O(6) limits of the interacting boson model (IBM). In the SU(3) limit the quadrupole operator is defined as $(s^\dagger\tilde{d}+d^\dagger s)^{(2)}+\chi(d^\dagger\tilde{d})^{(2)}$, where χ equals either $\sqrt{7}/2$ or $-\sqrt{7}/2$, both choices leading to a SU(3) Lie algebra. As long as one consistently uses the same quadrupole operator in the Hamiltonian and in the $E2$ transition operator, the properties of all physical observables will be independent of this sign choice. In the case of the SU(3) limit, Dieperink and Bijker have shown that the existence of this sign ambiguity can be elegantly exploited to unveil a new kind of SU(3) symmetry [which they denote as SU(3)*] in the neutron-proton interacting boson model (IBM-2), arising when the neutrons have one sign in the quadrupole operator and the protons the other.[1]

Given the important consequence of the sign choice in the SU(3) limit, it is natural to investigate the analogous problem in the O(6) limit. The problem can be shown to be different in two basic aspects. (i) In the O(6) limit there exists a *phase* ambiguity rather than a *sign* ambiguity. (ii) Attempting to generalize the O(6) limit of the IBM-2, by taking different phases for neutrons and protons, leads to a Hamiltonian which is, in general, complex and hence not invariant under time reversal.

In this paper, we analyze the problem of the phase ambiguity in the O(6) limit. Because of (ii) we shall make no distinction between neutrons and protons, i.e., we confine our analysis to the IBM-1. First, we present the problem in its full generality and, subsequently, we discuss under which conditions this generalized O(6) limit is invariant under time reversal. Much insight can be gained by studying the classical limit of this more general O(6) symmetry; in particular, we show that a generalized O(6) Hamiltonian necessarily leads to a phase in the boson condensate.

The generators of the O(6) subgroup of U(6) can be written as

$$(s^\dagger\tilde{d}+\xi d^\dagger s)^{(2)}_\mu,\quad \mu=-2,\ldots,2\ , \tag{1a}$$

$$(d^\dagger\tilde{d})^{(1)}_\mu,\quad \mu=-1,0,1\ , \tag{1b}$$

$$(d^\dagger\tilde{d})^{(3)}_\mu,\quad \mu=-3,\ldots,3\ . \tag{1c}$$

One usually puts $\xi=+1$ (Ref. 2) or $\xi=-1$ (Ref. 3). However, it can be shown that the generators (1) close under commutation for arbitrary ξ. If one requires that the Hamiltonian, which is written in terms of quadratic Casimir operators of the subgroups of the chain, is Hermitian, one obtains the condition $|\xi|^2=1$ and thus one may put $\xi=e^{i\phi}$ with $0\leqslant\phi\leqslant 2\pi$. One can also show that by taking the Hermitian form of the generators [i.e., $e^{-i\phi/2}(s^\dagger\tilde{d})^{(2)}_\mu+e^{i\phi/2}(d^\dagger s)^{(2)}_\mu$], the commutation rules of this generalized O(6) group reduce to the usual ones with $\phi=0$. The Casimir operator of O(6) corresponding to this choice of phase is then

$$C_{2O(6)}=N(N+4)-P^\dagger_\phi\cdot P_\phi\ , \tag{2}$$

where

$$P^\dagger_\phi=(s^\dagger s^\dagger-e^{i\phi}d^\dagger\cdot d^\dagger)\ . \tag{3}$$

The Casimir operators of O(5) and O(3) are not affected by the choice of phase. Note that the operator (2), though Hermitian, is in general complex.

It is now straightforward to repeat for this generalized Hamiltonian, the analysis of Arima and Iachello[2] or, equivalently, the one of Castaños, Chacón, Frank, and Moshinsky.[3] One finds that (i) the energy spectrum is independent of ϕ and (ii) the wave functions do depend on ϕ. If one expands the wave function in a vibrational basis, i.e.,

$$|[N]\sigma\tau\nu_\Delta LM\rangle_\phi=\sum_{n_d}\zeta^\sigma_{n_d,\tau}(\phi)\,|[N]n_d\tau\nu_\Delta LM\rangle\ , \tag{4}$$

where we use the notation of Ref. 2, then one finds for the transformation brackets (for $\sigma=N$)

$$\zeta^N_{n_d,\tau}(\phi)=e^{i(n_d-\tau)\phi/2}\left[\frac{(N-\tau)!(N+\tau+3)!(n_d-\tau+1)!!}{(N-n_d)!2^{N+1}(N+1)!(n_d-\tau+1)!(n_d+\tau+3)!!}\right]^{1/2}\ . \tag{5}$$

For arbitrary values of σ, the coefficients $\zeta^N_{n_d,\tau}(\phi)$ can be derived straightforwardly from the expressions given by Castaños *et al.*[3]

The next step is to generalize the $E2$ transition operator, which can be defined as

$$T(E2;\theta)=e^{-i\theta/2}(s^\dagger\tilde{d})^{(2)}+e^{i\theta/2}(d^\dagger s)^{(2)} \quad . \tag{6}$$

The reduced matrix element of this operator between the states (4) is given by

$$\begin{aligned}{}_\phi\langle [N]N\tau+1\nu'_\Delta L'||T(E2;\theta)||[N]N\tau\nu_\Delta L\rangle_\phi &= e^{i\theta/2}\frac{[(N-\tau)(N+\tau+4)]^{1/2}}{2(N+1)}[\,(N+\tau+3)(2\tau+5)^{-1/2}\langle\tau+1\tau+1\nu'_\Delta L'||d^\dagger||\tau\tau\nu_\Delta L\rangle \\ &\quad + e^{i(\phi-\theta)}2^{-1/2}(N-\tau-1)\langle\tau+1\tau+1\nu'_\Delta L'||\tilde{d}||\tau+2\tau\nu_\Delta L\rangle\,] \quad . \end{aligned} \tag{7}$$

This shows that, if one used the same quadrupole operator throughout ($\phi=\theta$), neither the energies nor the $B(E2)$ values depend on the choice of phase, as is the case in the SU(3) limit.

It should be emphasized that, for the model to be invariant under time reversal, additional restrictions have to be imposed on the values of ϕ and θ. A first restriction follows from the fact that a time-reversal invariant Hamiltonian should be real; this implies $\phi=0$ or $\phi=\pi$ in Eq. (2). Yet, this condition does not mean that the model becomes less general, since the energy spectrum does not depend on ϕ and, furthermore, Eq. (7) shows that the $B(E2)$ values only depend on the phase *difference* $\phi-\theta$. Secondly, time-reversal invariance also restricts the form of the electromagnetic transition operators, and this leads, in the case of $E2$ transitions, to the condition[4]

$$\langle\Psi_f||T(E2)||\Psi_i\rangle^*=\langle\Psi_f||T(E2)||\Psi_i\rangle \quad . \tag{8}$$

The problem with this condition is that it is valid only when the basis states Ψ_i and Ψ_f have the appropriate transformation properties under the time-reversal operation,[4] and that the effect of such an operation on the s and d bosons is not known. Therefore, we replace Eq. (8) by a condition (also following from time-reversal invariance) which is measurable and hence independent of the basis chosen. Denoting the $E2$ matrix element as

$$M_{mn}=\langle\Psi_n||T(E2)||\Psi_m\rangle \quad , \tag{9}$$

invariance under time reversal requires

$$M_{12}M_{23}\cdots M_{n-1,n}M_{n1}=M_{1n}M_{n,n-1}\cdots M_{32}M_{21} \quad , \tag{10}$$

i.e., the product of a *ring* of matrix elements is independent of its direction (clockwise or counterclockwise). With the help of Eq. (7), one finds in the generalized O(6) limit

$$M_{12}M_{23}\cdots M_{n-1,n}M_{n1}=(M_{1n}M_{n,n-1}\cdots M_{32}M_{21})^* \quad . \tag{11}$$

Both the Eqs. (10) and (11) are certainly satisfied if the matrix elements M_{mn} are real. However, because of the selection rule $\Delta\tau=\pm 1$, which is valid for $E2$ transitions in the O(6) limit, nonzero rings only exist for an *even* number of matrix elements and, consequently, Eqs. (10) and (11) can also be satisfied simultaneously if the matrix elements M_{mn} are purely imaginary. The conclusion of this analysis is that only two, nonequivalent forms of the O(6) limit are invariant under time reversal. (i) $\phi=0$ and $\theta=0$ (equivalent with $\phi=\pi$ and $\theta=\pi$). This is the "normal" O(6) limit,[2,3] which we denote as $O_+(6)$. (ii) $\phi=0$ and $\theta=\pi$ (equivalent with $\phi=\pi$ and $\theta=0$). This case will be denoted as $O_-(6)$.

The relation between the $B(E2)$ values in both limits is given by

$$B(E2;\tau+1L_i\rightarrow\tau L_f)_-=\left(\frac{\tau+2}{N+1}\right)^2 B(E2;\tau+1L_i\rightarrow\tau L_f)_+ \quad . \tag{12}$$

In Fig. 1, we show the $B(E2)$ values in the two cases, normalizing all $B(E2)$'s to $B(E2;2_1^+\rightarrow 0_1^+)$.

We now turn our attention to the analysis of the classical limit of the generalized Hamiltonian of Eq. (2), for which we will follow the methods and conventions of Refs. 5 and 6. Assuming a boson condensate with axial symmetry, i.e.,

$$|\Phi_N\rangle=(N!)^{-1/2}(\Gamma_0^\dagger)^N|0\rangle \tag{13}$$

with

$$\Gamma_0^\dagger=\eta_{0s}^* s^\dagger+\eta_{0d}^* d_0^\dagger \quad , \tag{14}$$

minimization of the expectation value of the generalized pairing Hamiltonian

$$H=AP_\phi^\dagger\cdot P_\phi \quad , \tag{15}$$

leads to the following Hartree equations for the variational parameters η_{0i} (with $i,j=s$, or d)

$$\sum_j h_{ij}\eta_{0j}=E_0\eta_{oi} \tag{16}$$

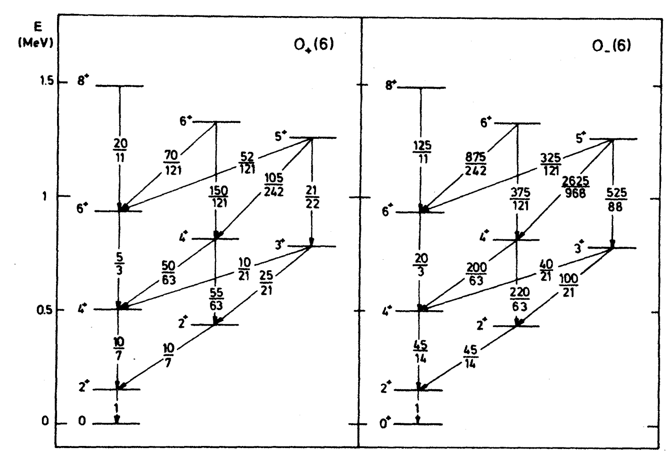

FIG. 1. The $B(E2)$ values, normalized to $B(E2;2_1^+\rightarrow 0_1^+)$, in the $O_+(6)$ and $O_-(6)$ limits of the IBM for $N\rightarrow\infty$. To find the expressions for finite N, one should multiply these values by $(N-\tau_f)(N+\tau_f+4)/N(N+4)$, where τ_f is the τ quantum number of the final state.

with

$$h_{ss}=2A(N-1)\eta_{0s}^{*}\eta_{0s} , \tag{17a}$$

$$h_{sd}=h_{ds}^{*}=-2A(N-1)e^{i\phi}\eta_{0s}^{*}\eta_{0d} , \tag{17b}$$

$$h_{dd}=2A(N-1)\eta_{0d}^{*}\eta_{0d} , \tag{17c}$$

$$E_0=2A(N-1)[(\eta_{0s}^{*}\eta_{0s})^2-e^{i\phi}(\eta_{0d}^{*}\eta_{0s})^2 - e^{i\phi}(\eta_{0s}^{*}\eta_{0d})^2+(\eta_{0d}^{*}\eta_{0d})^2] . \tag{17d}$$

Neglecting an overall phase in the boson condensate and taking into account the normalization of η_{0s} and η_{0d}, we make the substitution

$$\eta_{0s}^{*}=(1+\beta^2)^{-1/2}e^{i\alpha} , \tag{18a}$$

$$\eta_{0d}^{*}=(1+\beta^2)^{-1/2}\beta , \tag{18b}$$

which transforms Eq. (16) into

$$(1+\beta^2)(1-e^{i(\phi+2\alpha)}\beta^2)=[1-2\beta^2\cos(\phi+2\alpha)+\beta^4] . \tag{19}$$

Solving for the real and imaginary parts of this equation gives as a solution for the boson condensate

$$\Gamma_0^{\dagger}=(1+\beta^2)^{-1/2}(e^{-i\phi/2}s^{\dagger}+\beta d_0^{\dagger}) , \tag{20}$$

with $\beta=0, \pm1$. Calculation of the expectation value of the Hamiltonian (15) in the boson condensate (13) yields

$$\langle\Phi_N|H|\Phi_N\rangle=AN(N-1)\left(\frac{1-\beta^2}{1+\beta^2}\right)^2 , \tag{21}$$

which shows that, for $A>0$, the solutions $\beta=\pm1$ correspond to a minimum (either prolate or oblate) whereas $\beta=0$ corresponds to a maximum. It is important to note that also for $\phi=\pi$, when the Hamiltonian (15) is real and hence not forbidden by time-reversal considerations, the boson condensate (20) contains an imaginary phase. It provides an explicit illustration of the fact, which was already noted before,[7] that the variables in the boson condensate are complex in general. Finally, for the expectation value of the quadrupole operator (6) in the boson condensate, one finds

$$\langle\Phi_N|T(E2;\theta)|\Phi_N\rangle=\frac{2\beta N}{(1+\beta^2)}\cos[\tfrac{1}{2}(\phi-\theta)]$$

$$=\begin{cases}\dfrac{2\beta N}{(1+\beta^2)} \text{ for } O_+(6) , & (22a)\\ 0 \text{ for } O_-(6) . & (22b)\end{cases}$$

The fact that, in the classical limit, the expectation value of the quadrupole operator vanishes in the $O_-(6)$ limit, agrees with the result of Eq. (12). It also illustrates that in the $O_-(6)$ limit the eigenstates are not coherent with respect to electric quadrupole transitions.

In conclusion, we presented a generalization of the O(6) limit of the IBM and we analyzed under which conditions this generalization is invariant under time reversal. The analysis was performed both quantum mechanically and in the classical limit. The main conclusion is that only two formulations of the O(6) limit are possible, which differ in their electric quadrupole properties.

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