Hyperspherical formalism for the photoeffect of the alpha particle

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In this work we employ the hyperspherical harmonics formalism to calculate the moments of the photoeffect of the alpha particle. The grand orbital quantum number is increased until well-converged results are obtained. The adiabatic approximation is used to solve the coupled differential equations. The calculations are performed for a central N-N potential with Serber exchange. Lastly, the moments are inverted to obtain the photodisintegration cross section. Comparisons with experimental observations are presented.

I. INTRODUCTION

The profusion of realistic N-N potentials testifies to the fact that we still do not have a complete understanding of the nature of the force between two nucleons. This has been the driving motivation for some important developments in the realms of few-nucleon problems. Lately we have realized that reproduction of few-nucleon binding energy is not a sufficient criterion for the accuracy of an N-N potential; potentials with very different shapes lead to nearly the same binding energy. Thus we are forced to look for a suitable set of observables, preferably depending explicitly on the potential used. The moments of the photoeffect (σ_p) are adequate for this purpose. The zeroth moment (σ_0) corresponds to the total photodisintegration cross section integrated over all incident photon energies. This quantity can be measured experimentally. σ_{-1} is proportional to the mean square charge radius, another experimentally measurable quantity. Moreover, we can calculate a few other moments and invert these to get the photodisintegration cross section as a function of the incident photon energy, which can also be compared with its experimental counterpart.

In this work we employ the hyperspherical harmonics (HH) formalism to calculate the moments of the photoeffect of the alpha particle. These moments were calculated for increasing values of the grand orbital quantum number K, until well converged values were obtained. The optimal subset approximation[1] was used for the expansion of the ground state wave function. However, obtaining an accurate wave function entails the solution of a coupled set of differential equations. This is a difficult numerical exercise. The adiabatic approximation[2,3] provides us with an adequate means of tackling this problem. Since the binding energy thus obtained is quite close to the exact ones,[4] we expect that the shape of the hyperradial functions will also be quite accurate. This technique has been used in the present work.

Recently Elminyawi and Levinger[5] calculated the moments of the photoeffect of the alpha particle, limiting the expansion of the wave function to the first term alone. They, however, did not consider the full contribution of the potential. We attempted to rectify it by considering all the potential multipoles.[6] As in Ref. 5, we too restricted the expansion of the ground state to the first term. Experience with the three-nucleon systems[7] shows that significant improvements occur when terms with $K>0$ are included in the wave function. Hence in this work we shall try to achieve well converged results.

In the next section we describe briefly the hyperspherical expansion technique. Section III contains derivations of expression for the moments of the photoeffect ($\sigma_p, p=0,\pm1$). Also outlined in Sec. III are the methods of inversion of the moments to obtain the cross section as a function of the incident photon energy (E_γ). In Sec. IV the results and the conclusions thereof are given.

II. THE HYPERSPHERICAL EXPANSION TECHNIQUE

In the HH technique we describe the wave function of an N particle system as a point moving in the $3N$ dimensional space [$(3N-3)$ dimensional if the center of mass coordinates are eliminated, as in this paper]. To eliminate the center of mass degrees of freedom we employ the Jacobi coordinates $\vec{x}_i$, defined as

$$\vec{x}_i=\left[\frac{2i}{i+1}\right]^{1/2}\left[\vec{r}_{i+1}-\frac{1}{i}\sum_{j=1}^{i}\vec{r}_j\right];\quad i=1-(N-1)\,. \tag{1}$$

Here $\vec{r}_i$ are the position vectors of each nucleon. The hyperspherical coordinates are defined in terms of the magnitudes and polar angles of the Jacobi vectors. Each of the vectors $\vec{x}_i$ will have two polar angles (θ_i,ϕ_i) associated with them. These lead to $(2N-2)$ independent coordinates. The remaining $(N-1)$ are related to the magnitude of $\vec{x}_i$ as

$$\begin{aligned}
&x_1=r\cos\alpha_{N-1}\,,\\
&x_i=r\sin\alpha_{N-1}\cdots\sin\alpha_{i+1}\cos\alpha_i\,,\\
&x_{N-1}=r\sin\alpha_{N-1}\cdots\sin\alpha_2\,,\\
&\cos\alpha_1=1\,.
\end{aligned} \tag{2}$$

Clearly

$$r^2=\sum_{i=1}^{N-1}x_i^2\,; \tag{3}$$

"r," as defined above, is the hyper-radius; and the angles α_i together with the polar angles are the hyperangles.

The Laplace equation in the hyperspherical coordinates is

$$\left[\frac{\partial^2}{\partial r^2}+\frac{3N-4}{r}\frac{\partial}{\partial r}+\frac{L^2(\Omega)}{r^2}\right]\Psi(r,\Omega)=0\ . \tag{4}$$

(Ω) is the set of hyperangles, $L^2(\Omega)$ is the grand orbital angular momentum operator. It satisfies the eigenvalue equation

$$L^2(\Omega)H_{[K]}(\Omega)=K(K+3N-5)H_{[K]}(\Omega)\ . \tag{5}$$

The function $H_{[K]}(\Omega)$ is the HH. $[K]$ denotes the full set of the $(3N-4)$ quantum numbers required to specify an HH completely. The subscript K, without any square bracket, represents the grand orbital quantum number. For the set of hyperspherical coordinates defined above, $H_{[K]}(\Omega)$ are

$$H_{[K]}(\Omega)=Y_{l_1}^{m_1}(\theta_1,\phi_1)\sum_{j=2}^{N-1}Y_{l_j}^{m_j}(\theta_j,\phi_j)\,{}^{(j)}P_{L_j(\alpha_j)}^{l_j,L_j-1}\ , \tag{6}$$

where

$$\begin{aligned}{}^{(j)}P_{L_j(\alpha_j)}^{l_j,L_j-1}=&\left[\frac{2\nu_j(\nu_j-n_j)n_j!}{\Gamma(\nu_j-n_n-l_j-\frac{1}{2})\Gamma(n_j+l_j+\frac{3}{2})}\right]^{1/2}\\&\times(\cos\alpha_j)^{l_j}(\sin\alpha_j)^{L_{j-1}}\\&\times P_{n_j}^{\nu_{j-1},l_j+(1/2)}(\cos2\alpha_j)\ ,\\ \nu_j=&\nu_{j-1}+2n_j+l_j+\tfrac{3}{2}=L_j+\frac{3j}{2}-1\\=&L_{j-1}+2n_j+l_j+\frac{3j}{2}-1\ .\end{aligned} \tag{7}$$

The HH obey the orthonormality condition

$$\int H_{[K]}^*(\Omega)H_{[K']}(\Omega)d\Omega=\delta_{[K],[K']}\ . \tag{8}$$

The δ function denotes that the HH are orthonormal in each of the $(3N-4)$ dimensions. We will use the notation $H_K(\Omega_{ij})$ for an HH in which (i) all quantum numbers, except the grand orbital quantum number, is zero, and (ii) $\vec{x}_i=(\vec{r}_i-\vec{r}_j)$. It should be noted that for a system of identical particles any permutation of the Jacobi coordinates given in (1) is valid.

The functions $H_K(\Omega_{ij})$ form a complete basis for the expansion of any function $f(r_{ij})$. A symmetric combination of $H_K(\Omega_{ij})$ would thus be a complete basis for the expansion of the magnitudes of the $N(N-1)/2$ particle pairs. Let us denote such a symmetric combination by $S_K(\Omega)$. Then

$$S_K(\Omega)=C_K\sum_{i,j>i}H_K(\Omega_{ij})\ . \tag{9}$$

C_K is a normalization constant such that

$$\int S_K^*(\Omega)S_{K'}(\Omega)d\Omega=\delta_{K,K'}\ .$$

Let us expand the ground state wave function $|i\rangle$ of a system of N fermions with positive parity in the $S_K(\Omega)$ basis

$$|i\rangle=\sum_K U_{2K}(r)S_{2K}(\Omega)A(s,t)r^{(4-3N)/2}\ . \tag{10}$$

$A(s,t)$ is a completely antisymmetric spin-isospin function, and $U_{2K}(r)$ are the expansion coefficients, to be determined. Substituting $|i\rangle$ in the Schrödinger equation,

$$H|i\rangle=E|i\rangle\ ,$$

multiplying by $S_{2K'}(\Omega)$ on the left, and integrating over the surface of a unit hypersphere, we obtain a system of coupled differential equations in $U_{2K}(r)$,

$$\begin{aligned}-\frac{\hbar^2}{m}&\left[\frac{d^2}{dr^2}-\frac{1}{r^2}(2K-3+3N/2)\right.\\&\left.\times(2K-2+3N/2)\right]U_{2K}(r)\\&+\sum_{K'}V_K^{K'}(r)U_{2K'}(r)=EU_{2K}(r)\ ,\end{aligned} \tag{11}$$

where

$$V_K^{K'}(r)=\langle S_{2K'}(\Omega)\,|\,V(r,\Omega)\,|\,S_{2K}(\Omega)\rangle\ . \tag{12}$$

$V(r,\Omega)$ is the total potential in the system, expressed in the hyperspherical coordinates. Since $S_{2K}(\Omega)$ and the potential are both symmetric with respect to exchange of nucleons, we can write

$$\begin{aligned}V_K^{K'}(r)=&f_Kf_{K'}\langle H_{2K'}(\Omega_{21})\,|\,V(r_{21})\,|\,H_{2K}(\Omega_{21})\rangle\\&+\left[1-\frac{2F_K^2}{N(N-1)}\right]V_0^0(r)\delta_{K,K'}\ ,\end{aligned} \tag{13}$$

with

$$\begin{aligned}&\langle H_{2K'}(\Omega_{21})\,|\,V(r_{21})\,|\,H_{2K}(\Omega_{21})\rangle\\&\quad=\sum_\chi\langle H_{2K}(\Omega_{21})\,|\,H_{2\chi}(\Omega_{21})\,|\,H_{2K'}(\Omega_{21})\rangle V_\chi^N(r)\ .\end{aligned}$$

The multipoles $V_\chi^N(r)$ are given by

$$\begin{aligned}V_\chi^N(r)=&\int_0^{\pi/2}V(r\cos\alpha_{N-1})^{(N-1)}P_\chi^{0,0}\\&\times(\sin\alpha_{N-1})^{3N-7}(\cos\alpha_{N-1})^2d\alpha_{N-1}\ .\end{aligned} \tag{14}$$

The coefficients f_K are defined as

$$f_K^2=\langle H_{2K}(\Omega_{21})\,|\sum_{i,j>i}H_{2K}(\Omega_{ij})\rangle\ . \tag{15}$$

These coefficients as well as the $3P$ coefficients [i.e.,

$$\langle H_{2K}(\Omega_{21})\,|\,H_{2\chi}(\Omega_{21})\,|\,H_{2K'}(\Omega_{21})\rangle]$$

are geometric coefficients independent of all extraneous factors like the shape of the potential. Detailed expressions for them appear in Ref. 1. The procedure for evaluating these geometrical coefficients and the potential multipoles (14) are given in the appendices, for $N=4$.

In actual calculations the potential used is of the form

$$V=\sum_{i,j>i}(1-x+xP_{ij})V(r_{ij}) . \tag{16}$$

P_{ij} is a Majorana exchange operator. Thus the potential is central, with exchange character.

III. THE METHOD FOR CALCULATING THE MOMENTS OF THE PHOTOEFFECT OF THE ALPHA PARTICLE

In this section we shall outline the method for calculating the moments of the photoeffect of the alpha particle. The number of particles in the system, N, is four. The moments are defined as

$$\sigma_p=\int_0^\infty E_\gamma^p\sigma(E_\gamma)dE_\gamma .$$

In the present work we shall confine ourselves to $p=0,\pm1$. The following two identities are of importance: (i) the Heisenberg relationship

$$E_\gamma\langle f\,|\,D\,|\,i\rangle=-\langle i\,|\,[H,D]\,|\,f\rangle=\langle f\,|\,[H,D]\,|\,i\rangle , \tag{17}$$

and (ii) the closure property

$$\sum_f A_{if}B_{fi}=(AB)_{ii} . \tag{18}$$

A_{if} and B_{fi} are transition matrices between states $|\,i\rangle$ and $|\,f\rangle$. Here $|\,f\rangle$ is any final state. D is the dipole operator and H is the Hamiltonian of the system. If we assume "1" and "2" to be the two protons, and the photon incident along the z axis (since there is no preferred direction of the alpha particle), then

$$D=e(\vec{r}_{12})_z=ex_{1z}=er\cos\alpha_3\cos\theta_1 . \tag{19}$$

θ_1 is a polar angle of $\vec{x}_1$, r is the hyper-radius, and α_3 is the hyperangle as defined in (2). Moreover, the cross section for the electric dipole transition from $|\,i\rangle$ to $|\,f\rangle$ will be given by[8]

$$\sigma_{if}=\frac{2\pi^2e^2}{9\hbar c}E_\gamma[\langle f\,|\,x_{1z}\,|\,i\rangle]^2 . \tag{20}$$

To obtain σ_{-1} we use the closure property (i.e., sum over all the final states)

$$\sigma_{-1}=\tfrac{16}{9}\pi^2\frac{e^2}{\hbar c}\langle i\,|\,R_c^2\,|\,i\rangle .$$

In the above R_c^2 is the mean square charge radius. Now since the space part of $|\,i\rangle$ is symmetric, the mean square charge radius will be equal to the mean square radius. Therefore

$$\sigma_{-1}=\frac{2\pi^2}{9}\frac{e^2}{\hbar c}\langle i\,|\,r^2\,|\,i\rangle . \tag{21}$$

It should be noted that σ_{-1} does not depend upon the potential explicitly.

The zeroth moment of the photoeffect is obtained by using the Heisenberg relation and then the closure property in (20)

$$\sigma_0=\frac{2\pi^2}{\hbar c}\langle i\,|\,[D,[H,D]]\,|\,i\rangle$$

$$=\frac{2\pi^2}{\hbar c}[\langle i\,|\,[D,[T,D]]\,|\,i\rangle+\langle i\,|\,[D,[V,D]]\,|\,i\rangle] . \tag{22}$$

The first term is given by the Thomas-Reiche-Kuhn sum rule and is found to be 59.7 MeV mb.[9] The second term goes to zero if the dipole commutes with the potential. This is true if the potential contains no Majorana or Heisenberg exchange character. Thus we will get a nonzero contribution from the second term for potential (16). Now

$$[V,D]\,|\,i\rangle=ex\left\{\left[\sum_{i,j>i}V(r_{ij})P_{ij}x_{1z}\right]-x_{1z}\left[\sum_{i,j>i}V(r_{ij})P_{ij}\right]\right\}|\,i\rangle . \tag{23}$$

But

$$P_{ij}\,|\,i\rangle=|\,i\rangle \tag{24}$$

and $P_{ij}(x_{1z}\,|\,i\rangle)$ would have the same symmetry as x_{1z}. Therefore

$$[V,D]\,|\,i\rangle=-ex\left[\sum_n\sum_p V(r_{np})(\vec{r}_{np})_z\right]|\,i\rangle . \tag{25a}$$

Similarly

$$\langle i\,|\,[V,D]=ex\langle i\,|\left[\sum_n\sum_p V(r_{np})(\vec{r}_{np})_z\right] . \tag{25b}$$

The summation symbols "n" and "p" stand for neutrons and protons, respectively. The above leads to

$$\sigma_0=59.7-\frac{8\pi^2e^2x}{\hbar c}\langle i\,|\,V(r_{12})(\vec{r}_{12})_z^2\,|\,i\rangle . \tag{26}$$

In obtaining (26) a simplification was used, namely

$$V(r_{ij})=V(r_{12})\text{ for every pair }(ij) . \tag{27}$$

Again, since the space part of $|\,i\rangle$ is symmetric with respect to exchange of nucleons we could write,

$$\sigma_0=59.7-\frac{4\pi^2e^2x}{9\hbar c}\left[\sum_K\sum_{K'}f_Kf_{K'}\sum_\chi\langle H_{2K}(\Omega_{12})\,|\,H_{2\chi}(\Omega_{12})\,|\,H_{2K'}(\Omega_{12})\rangle\langle U_{2K}(r)\,|\,X_\chi(r)\,|\,U_{2K'}(r)\rangle\right]$$
$$-\frac{4\pi^2e^2x}{9\hbar c}\sum_K\left[1-\frac{f_k^2}{6}\right]\langle U_{2K}(r)\,|\,X_0^0(r)\,|\,U_{2K}(r)\rangle , \tag{28}$$

where

$$X_\chi(r)=r^2\int_0^{\pi/2}V(r\cos\alpha_3)\sin^5\alpha_3\cos^4\alpha_3\,{}^{(3)}P_\chi^{0,0}(\alpha_3)d\alpha_3 , \tag{29}$$

such that

$$X_K^{K'}(r)=f_K f_{K'}\sum_\chi \langle H_{2K}(\Omega_{12})\,|\,H_{2\chi}(\Omega_{12})\,|\,H_{2K'}(\Omega_{12})\rangle X_\chi(r)+\left[1-\frac{f_K^2}{6}\right]X_0^0(r)\delta_{K,K'}\,.$$

Thus, by calculating the "multipoles" $X_\chi(r)$ and performing the one-dimensional integrals over the hyper-radius, we can evaluate the zeroth moment of the photoeffect.

To find the first moment (σ_1) we use the Heisenberg relation twice and then the closure property to get

$$\begin{aligned}\sigma_1&=-\frac{4\pi^2}{\hbar c}\langle i\,|\,[H,D]^2\,|\,i\rangle\\&=-\frac{4\pi^2}{\hbar c}\{\langle i\,|\,[T,D]^2\,|\,i\rangle+\langle i\,|\,[T,D][V,D]+[V,D][T,D]\,|\,i\rangle+\langle i\,|\,[V,D]^2\,|\,i\rangle\}\\&=\sigma_1^0+\sigma_1^x+\sigma_1^{xx}\,.\end{aligned}\tag{30}$$

In the above, the last three quantities are the contributions to σ_1 from terms independent of x, terms proportional to x, and terms proportional to x^2, respectively. Now,

$$[T,D]=-\frac{e\hbar^2}{m}\frac{\partial}{\partial x_{1z}}\,.\tag{31}$$

Therefore

$$\sigma_1^0=\frac{4\pi^2}{9}\frac{e^2}{\hbar c}\frac{\hbar^2}{m}\langle i\,|\,T\,|\,i\rangle\,.$$

But $(T+V)\,|\,i\rangle=E\,|\,i\rangle$, so

$$\sigma_1^0=\tfrac{4}{9}\pi^2\frac{e^2}{\hbar c}\frac{\hbar^2}{m}(E-\langle i\,|\,V\,|\,i\rangle)\,.\tag{32}$$

The expectation value of the potential is obtained by expanding the potential over the basis set $S_K(\Omega)$. Using relations (12)—(15) we obtain

$$\begin{aligned}\sigma_1^0=\tfrac{4}{9}\pi^2\frac{e^2}{\hbar c}\frac{\hbar^2}{m}\Bigg[&E-\sum_K\sum_{K'}f_K f_{K'}\sum_\chi\langle H_{2K}(\Omega_{12})\,|\,H_{2\chi}(\Omega_{12})\,|\,H_{2K'}(\Omega_{12})\rangle\langle U_{2K}(r)\,|\,V_\chi^4(r)\,|\,U_{2K'}(r)\rangle\\&-\sum_K\left[1-\frac{f_K^2}{6}\right]\langle U_{2K}(r)\,|\,V_0^0(r)\,|\,U_{2K}(r)\rangle\Bigg]\,.\end{aligned}\tag{33}$$

The contribution to σ_1 from terms proportional to x is given by

$$\sigma_1^x=-\frac{4\pi^2}{hc}\{\langle i\,|\,[T,D][V,D]+[V,D][T,D]\,|\,i\rangle\}\,.\tag{34}$$

Using (25a) and (25b) and then (31) we get

$$\sigma_1^x=4\pi^2\frac{e^2}{\hbar c}\frac{\hbar^2}{m}x\left\langle i\left|\left\{\frac{\partial}{\partial x_{1z}}\left[\sum_n\sum_p V(r_{np})(\vec{r}_{np})_z\right]\right\}\right|i\right\rangle\,.$$

The differential operator acts only on the potential terms and not on $|\,i\rangle$. Employing the simplification (27)

$$\sigma_1^x=4\pi^2\frac{e^2}{\hbar c}\frac{\hbar^2}{m}x\left[\langle i\,|\,V(x_1)\,|\,i\rangle+\left\langle i\left|x_{1z}\frac{\partial}{\partial x_{1z}}(V(x_1))\right|i\right\rangle\right]\,.$$

Since $x_{1z}=x_1\cos\theta_1$, we have

$$\frac{\partial V(x_1)}{\partial x_{1z}}=\frac{\partial V(x_1)}{\partial x_1}\frac{\partial x_1}{\partial x_{1z}}+\frac{\partial V(x_1)}{\partial\theta_1}\frac{\partial\theta_1}{\partial x_{1z}}\,.$$

But $V(x_1)$ does not depend on θ_1. Therefore the second term vanishes and we have

$$\sigma_1^x=4\pi^2\frac{e^2}{\hbar c}\frac{\hbar^2}{m}x\left[\left\langle i\left|V(x_1)+x_1\frac{\partial V(x_1)}{\partial x_1}\right|i\right\rangle\right]\,.$$

As in the earlier cases, the "operator" can be expanded in the $H_K(\Omega_{12})$ basis set

$$V(x_1)+x_1\frac{\partial V(x_1)}{\partial x_1}=\sum_\chi W_\chi(r)H_{2\chi}(\Omega_{12})\,,$$

such that

$$W_\chi(r)=\int_0^{\pi/2}{}^{(3)}P_\chi^{0,0}(\alpha_3)\left[V(r\cos\alpha_3)+r\cos\alpha_3\frac{\partial V(x_1)}{\partial x_1}\right]\times\sin^5\alpha_3\cos^2\alpha_3 d\alpha_3\,.$$

x_1 is set equal to $(r\cos\alpha_3)$ after the differentiation. In a notation similar to (14) we then write

$$\sigma_1^x=\tfrac{2}{3}\pi^2\frac{e^2}{\hbar c}\frac{\hbar^2}{m}x\left[\sum_K\sum_{K'}f_Kf_{K'}\sum_\chi\langle H_{2K}(\Omega_{12})\,|\,H_{2\chi}(\Omega_{12})\,|\,H_{2K'}(\Omega_{12})\rangle\langle U_{2K}(r)\,|\,W_\chi(r)\,|\,U_{2K'}(r)\rangle\right.$$
$$\left.+\sum_K\left[1-\frac{f_k^2}{6}\right]\langle U_{2K}(r)\,|\,W_0^0(r)\,|\,U_{2K}(r)\rangle\right]. \tag{35}$$

The term proportional to x^2 is given by

$$\sigma_1^{xx}=-\frac{4\pi^2}{\hbar c}\langle i\,|\,[V,D]^2\,|\,i\rangle\,. \tag{36}$$

Again using (25a) and (25b) and then the simplification (27), yields

$$\sigma_1^{xx}=\frac{16\pi^2e^2}{\hbar c}\langle i\,|\,(V(r_{12})(\vec{r}_{12})_z)^2\,|\,i\rangle\,.$$

Integrating over the polar angles and using the by now familiar technique of expanding over the $H_K(\Omega_{12})$ basis set, we get,

$$\sigma_1^{xx}=\frac{16\pi^2}{3}\frac{e^2}{\hbar c}\left[\sum_K\sum_{K'}f_Kf_{K'}\sum_\chi\langle H_{2K}(\Omega_{12})\,|\,H_{2\chi}(\Omega_{12})\,|\,H_{2K'}(\Omega_{12})\rangle\langle U_{2K}(r)\,|\,Z_\chi(r)\,|\,U_{2K'}(r)\rangle\right.$$
$$\left.+\sum_K\left[1-\frac{f_K^2}{6}\right]\langle U_{2K}(r)\,|\,Z_0^0(r)\,|\,U_{2K}(r)\rangle\right], \tag{37}$$

such that

$$Z_\chi(r)=r^2\int_0^{\pi/2}{}^{(3)}P_\chi^{0,0}(\alpha_3)(V(r\cos\alpha_3))^2\sin^5\alpha_3\cos^4\alpha_3 d\alpha_3$$

and

$$Z_K^{K'}(r)=\langle H_{2K}(\Omega_{12})\,|\,(V(r\cos\alpha_3)r\cos\alpha_3)^2\,|\,H_{2K'}(\Omega_{12})\rangle\,,$$
$$Z_K^{K'}(r)=f_kf_{K'}\sum_\chi\langle H_{2K}(\Omega_{12})\,|\,H_{2\chi}(\Omega_{12})\,|\,H_{2K'}(\Omega_{12})\rangle Z_\chi(r)+\left[1-\frac{f_K^2}{6}\right]Z_0^0(r)\delta_{KK'}\,.$$

To recapitulate, in this section we have expressed the moments of the photoeffect (for $p=0,\pm1$) in the hyperspherical coordinates. Now we want to express the cross section $\sigma(E_\gamma)$ as a function of the incident photon energy E_γ. To achieve this end we invert the moments.[5,6]

Let us expand $\sigma(E_\gamma)$ in a complete orthonormal basis set as

$$\sigma(E_\gamma)=E_\gamma w^{1/2}\exp(-w^{1/2})\sum_n a_nB_n(w)\,, \tag{38}$$

where

$$w=(E_\gamma-E_0)/D'\,.$$

E_0 is the threshold for two-body breakup, $E_0=19.82$ MeV. D' is an adjustable parameter with energy units. $B_n(w)$ are a set of orthonormal polynomials,[5,6] and a_n are the expansion coefficients. The function $[w^{1/2}\exp(-w^{1/2})]$ is the weight function for $B_n(w)$, i.e.,

$$\int_0^\infty B_n(w)B_m(w)w^{1/2}\exp(-w^{1/2})dw=\delta_{n,m}\,.$$

Multiplying both sides of (38) by $B_m(w)/E_\gamma$ (for $m=0,1,2$) and integrating over w leads to

$$a_0=\sigma_{-1}/2D'\,,$$

$$a_1=\left[\left[1+\frac{E_0}{12D'}\right]\sigma_{-1}-\frac{\sigma_0}{12D}\right]/(\sqrt{6}D')\,,$$
$$a_2=\frac{0.359}{D'}\left[\sigma_{-1}\left[1+\frac{0.141E_0}{D'}+\frac{0.001\,92E_0^2}{D'^2}\right]\right.$$
$$-\frac{\sigma_0}{D'}\left[0.141+\frac{0.003\,84E_0}{D'}\right]$$
$$\left.+\frac{\sigma_1}{D'^2}0.001\,92\right]. \tag{39}$$

The parameter D' is obtained from the relation

$$\sum_n a_nB_n(0)=0\,.$$

Knowing the moments of the photoeffect, the coefficients a_n and the parameter D' can be evaluated. Substitution in the expression for $\sigma(E_\gamma)$ gives the desired result.

IV. RESULTS AND CONCLUSIONS

The coupled differential equations (11) are solved using the adiabatic approximation. Up to five coupled equations are considered. Even though the binding energy does not attain its saturation value, the quantities of in-

TABLE I. The binding energy and the moments of the photoeffect of the alpha particle for increasing values of K. The values of these quantities, for $K=0$, are taken from Ref. 6. Also, the values of the binding energy obtained by Ballot *et al.*, Ref. 4.

	$-E$ (MeV)				
K	Present work	Ref. 4	σ_{-1} (mb)	σ_0 (MeV mb)	σ_1 (MeV^2 mb)
0	28.6	28.58	2.83	99.3	
2	29.21	29.18	2.837	99.742	3925.3
3	29.74	29.72	2.829	100.21	3991.1
4	30.08	30.07	2.822	100.38	4011.1
5	30.21	30.19	2.818	100.44	4018.2
Expt.	28.4		2.7	103	

terest (i.e., the moments of the photoeffect) are well converged. In Table I we present the value of the moments of the photoeffect, together with the binding energy, for increasing values of the grand orbital quantum number K. The calculations were performed using the Volkov potential for $V(r_{ij})$.

$$V(r_{ij})=144.86\exp(-r_{ij}^2/0.82^2) -83.32\exp(-r_{ij}^2/1.60^2) .$$

Also given in Table I are the values of the binding energy calculated by Ballot *et al.*,[4] using the same uncoupled adiabatic approximation.

For the calculation of the moments of the photoeffect, the fraction of Majorana exchange, x, was taken as $\frac{1}{2}$; that is, a Serber exchange was chosen. Again, in Table I, the values of σ_{-1} and σ_0 for $K=0$ have been taken from our earlier work.[6] The experimental values of σ_{-1} and σ_0, for the alpha particle, are from Refs. 10 and 11. For both these experimentally measurable quantities we can see that as the maximum allowed grand orbital quantum number increases, the theoretically calculated values converge towards the experimental ones. The agreement between the two improves. There is no experimental counterpart for σ_1.

In Fig. 1 we have plotted the photodisintegration cross section as a function of the incident photon energy. The experimental points are from Ref. 12. An inspection of the figure shows that the position of the peak, as well as the high energy portion are reproduced quite well. The height of the peak, however, leaves much to be desired. This discrepancy could be due to two reasons. Firstly, only three moments were considered in the expansion (38). It is probably that other moments can contribute significantly to the expansion. But from Table I we can see that the higher the moment, the larger the number of coupled differential equations required to obtain well-converged results. Thus, computing σ_2, etc., would require an excessive amount of computation. Secondly, it is also probable that the choice of the potential was faulty.

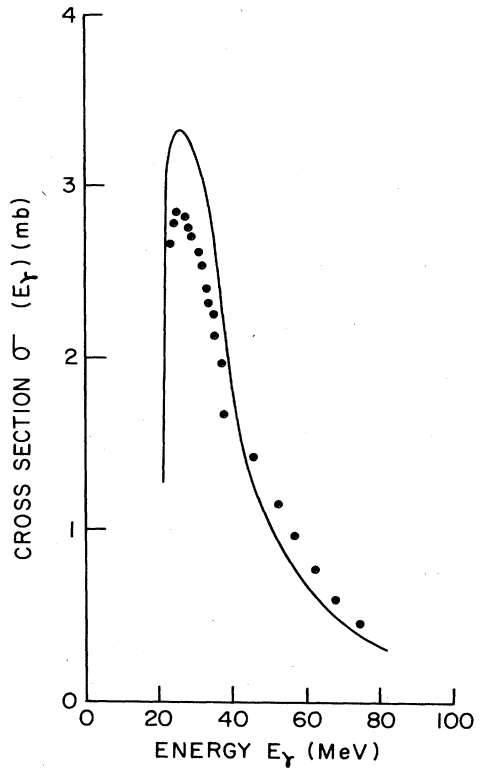

FIG. 1. The plot of $\sigma(E_\gamma)$ vs E_γ, obtained by inverting the moments of the photoeffect.

Perhaps a better method for calculating $\sigma(E_\gamma)$ is to find the final state wave function and compute the cross section directly. This technique has been employed by Levinger,[8] with the minimal subset approximation. However, in the general case it is necessary to solve the problem of mixed boundary condition, i.e., the initial and final states satisfy different boundary conditions. In a future work we intend using a guessed final state wave function with the correct asymptotic behavior. This, we hope, will give better results for the photodisintegration cross section.

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APPENDIX A: THE $P3$ COEFFICIENTS FOR $N=4$

Let the $P3$ coefficients be defined as

$$P3(K,\chi,K')=\int_{-1}^{1}(1-x)^2(1+x)^{1/2}P_K^{(2,1/2)}(x)P_\chi^{(2,1/2)}(x)P_{K'}^{(2,1/2)}(x)dx . \quad \text{(A1)}$$

These coefficients obey the triangular inequality, i.e., $P3(K,\chi,K')$ is nonzero only when $|K-K'| \le \chi \le (K+K')$. As given in Ref. 1, we start with $P3(K,K+K',K')$ and use the recurrence relationships to find the other coefficients; for a given K and K' we use the expression[1]

$$P3(K,K+K',K')=\frac{\pi}{16}\begin{bmatrix}2K+\frac{5}{2}\\ \frac{3}{2}\end{bmatrix}\begin{bmatrix}2K'+\frac{5}{2}\\ \frac{3}{2}\end{bmatrix}\begin{bmatrix}2(K+K'+3)\\ \frac{3}{2}\end{bmatrix}\left[\frac{2(K+K')+\frac{7}{2}}{2(K+K')+\frac{11}{2}}\frac{K+K'+\frac{7}{2}}{2(K+K'+3)}\frac{2(K+K')+3}{K+K'+1}\right]^{1/2} \tag{A2}$$

and then

$$\langle {}^{(3)}P^{0,0}_{2K}\,|\,{}^{(3)}P^{0,0}_{2\chi}\,|\,{}^{(3)}P^{0,0}_{2K'}\rangle+\left[\frac{2\chi+\frac{7}{2}}{2\chi+\frac{1}{2}}\frac{2\chi+4}{\chi+\frac{5}{2}}\frac{\chi}{2\chi+1}\right]^{1/2}\langle {}^{(3)}P^{0,0}_{2K}\,|\,{}^{(3)}P^{0,0}_{2(\chi-1)}\,|\,{}^{(3)}P^{0,0}_{2K'}\rangle$$

$$=\frac{2\chi+\frac{5}{2}}{2\chi+1}g(\tfrac{5}{2},K)g(\tfrac{5}{2},\chi)g(\tfrac{5}{2},K')D_1(\tfrac{5}{2},2K+1,2K'+1,2\chi)\,, \tag{A3}$$

where

$$D_1(\tfrac{5}{2},2K+1,2K'+1,2\chi)=\frac{\pi}{16}\begin{bmatrix}S-2K+\frac{3}{2}\\ \frac{3}{2}\end{bmatrix}\begin{bmatrix}S-2K'+\frac{3}{2}\\ \frac{3}{2}\end{bmatrix}\begin{bmatrix}S-2\chi+\frac{5}{2}\\ \frac{3}{2}\end{bmatrix}\begin{bmatrix}S+5\\ \frac{3}{2}\end{bmatrix}, \tag{A4}$$

$$S=K+K'+\chi\,,$$

and

$$g(\tfrac{5}{2},K)=4\sqrt{5/2}\left[\frac{2}{\pi}\frac{(2K+\frac{7}{2})(2K+1)}{(2K+5)}\right]^{1/2}. \tag{A5}$$

APPENDIX B: THE COEFFICIENTS f_K FOR $N=4$

The coefficients are defined as (15)

$$f_K^2=\langle H_{2K}(\Omega_{21})\,|\sum_{i,j>i}H_{2K}(\Omega_{ij})\rangle$$

$$=\sum_{i,j>i}\frac{P_K^{(2,1/2)}(\cos 2\phi_{ij})}{P_K^{(2,1/2)}(1)}\,, \tag{B1}$$

where

$$\cos 2\phi_{21}=1\,,$$

$$\cos 2\phi_{43}=-1\,,$$

$$\cos 2\phi_{ij}=-\tfrac{1}{2}\text{ for }(ij)=(13),(14),(23),(24)\,.$$

APPENDIX C: POTENTIAL MULTIPOLES

The potential multipoles (14) are given as

$$V^4_\chi(r)=\int_0^{\pi/2}V(r\cos\alpha_3)\,{}^{(3)}P^{0,0}_\chi(\alpha_3)\times\sin^5\alpha_3\cos^2\alpha_3 d\alpha_3\,. \tag{C1}$$

The form of $V(r\cos\alpha_3)$ chosen in the Volkov potential

$$V(r\cos\alpha_3)=144.86\exp(-r^2\cos^2\alpha_3/0.82^2)-83.34\exp(-r^2\cos^2\alpha_3/1.60^2)\,. \tag{C2}$$

The integral (C1) cannot be evaluated analytically for potentials made up of a sum of Gaussians, as in (C2). Analytical evaluations exist for Yukawa, exponential, harmonic oscillator, etc., type of potentials. For Gaussian potentials (C7) reduces to the error function $\mathrm{erf}(r/\mu)$. For example,

$$V^4_0(r)=(\tfrac{105}{8})^{1/2}\sum_{i=1}^{2}V_i\left\{\left[\frac{1}{2}\left[\frac{\mu_i}{r}\right]^4-\frac{15}{4}\left[\frac{\mu_i}{r}\right]^6\right]\exp(-r_i^2/\mu_i^2)+\left[\frac{1}{2}\left[\frac{\mu_i}{r}\right]^2-\frac{3}{2}\left[\frac{\mu_i}{r}\right]^4+\frac{15}{8}\left[\frac{\mu_i}{r}\right]^6\right]\frac{\sqrt{\pi}}{2}\mathrm{erf}(r/\mu_i)\right\}, \tag{C3}$$

where

$$V_1=144.86,\quad \mu_1=1.60\,,$$

$$V_2=-83.32,\quad \mu_2=0.82\,.$$

Defining the integral I_{2n} as

$$I_{2n}=\int_0^1 \exp(-r^2t^2/\mu^2)t^{2n}dt \ ,$$

it is easy to show that

$$I_{2n}=\frac{-\mu^2}{2r^2}\exp(-r^2/\mu^2)+\frac{(2n-1)\mu^2}{2r^2}I_{2n-2}, \text{ for } n\geq 1 \ . \tag{C4}$$

Using the above recurrence relationship it is easy to write expressions for the higher multipoles. This has been used in the present work. Standard methods were employed to calculate the error function, numerically.

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