Total neutrino-scattering cross sections and total muon-capture rates in nuclei

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A simple relation between total inclusive neutrino-scattering cross sections and total muon-capture rates in nuclei is derived. The calculation makes use of the closure and the nonrelativistic impulse approximations. The relation is used to predict neutrino cross sections in ^{12}C for $E_\nu = 150$–250 MeV.

Recently there has been considerable interest in the reaction $\nu_\mu + i \to \mu^- + f$ from threshold to about 250 MeV in neutrino energy as a test for neutrino oscillations. The reaction $\nu_\mu + {}^{12}C \to X + \mu^-$ has been studied experimentally[1] in order to test for $\nu_e \leftrightarrow \nu_\mu$ oscillation. However, although no evidence for this was found, observed cross sections[2] were substantially smaller than expected, leaving open the question of $\nu_\mu \leftrightarrow \nu_\tau$ oscillation. In order to place limits on the crucial parameters, $\Delta m^2 = m^2(\nu_1) - m^2(\nu_2)$ and lepton mixing angles, theoretical estimates of neutrino cross sections must be available for comparison with experiment. However, such calculations are often model dependent.

In this Brief Report, we derive a simple relationship between the total muon-capture rate Γ and the total neutrino cross section σ in nuclei. The derivation is based on the observation that matrix elements for these two processes are of the same form when the closure approximation and proper averages over angular integrations and over final nuclear states are taken. Without these averages, one can only derive relationships[3] between the total muon-capture rates and differential neutrino-scattering cross sections. This relationship between Γ and σ has the advantage of being relatively independent of nuclear structure effects. The result, which we obtain here, will obviously be a function of the average excitation energy of the nucleus. While there has been much work on improving the closure approximation[4] to reduce this dependence, such treatments require more detailed assumptions concerning nuclear structure than we wish to make here. The accuracy which we can expect using a simple closure approximation should, nonetheless, be quite useful in an area where experimental work is just beginning.

We then apply the relation to the case of ^{12}C to calculate total neutrino-scattering cross sections as a function of the neutrino energy using the observed total muon-capture rate.

The matrix element for the reaction $\mu^- + i \to \nu_\mu + f$ is given by, in the standard notation,

$$M = \frac{G \cos\theta_c}{\sqrt{2}} \bar{u}(\nu_\alpha)\gamma_\mu(1+\gamma_5)u(\mu^-)\langle f | J_\alpha^{(-)}(0) | i \rangle \quad . \quad (1)$$

Denoting the momenta and masses involved in the reaction by

$$\mu^-(p_\mu, m_\mu) + i(P_i, M_i) \to \nu_\mu(p_\nu, 0) + f(P_f, M_f) \quad , \quad (2)$$

we obtain the total muon-capture rate[5]

$$\Gamma_i = \frac{G^2 \cos\theta_c^2}{4\pi^2} C(N_i)(\alpha Z)^3 m_\mu^5 \langle \eta^2 \rangle \left(\frac{m_\nu}{E_\nu} \right) \int \frac{d\Omega_\nu}{4\pi} L_{\alpha\beta} \sum_f \langle i | J_\beta^{(+)}(0) | f \rangle \langle f | J_\alpha^{(-)}(0) | i \rangle \quad ;$$

$$L_{\alpha\beta} = \frac{2}{m_\mu m_\nu} [(p_\mu)_\alpha (p_\nu)_\beta + (p_\nu)_\alpha (p_\mu)_\beta - \delta_{\alpha\beta}(p_\mu \cdot p_\nu) + \epsilon_{\alpha\beta\rho\sigma}(p_\mu)_\rho (p_\nu)_\sigma] \quad , \quad (3)$$

where $G = 10^{-5}/m_p^2$, θ_c is the Cabibbo angle, and Z the proton number of the initial state N_i. $C(N_i)$ is a correction factor arising from the nonpoint character of the charge distribution of N_i. Also $\langle \eta^2 \rangle$ denotes an appropriate weighted average over all possible final nuclear states of the quantity

$$\eta^2 = \left(\frac{E_\nu}{m_\mu} \right)^2 \left(1 - \frac{E_\nu}{M_i + m_\mu} \right) \quad . \quad (4)$$

In taking the average, the momentum of the final state N_f in Eq. (3) is replaced by $\langle \vec{p}_f \rangle = \langle \vec{q} \rangle$, with $\langle |\vec{p}_\nu| \rangle = \langle E_\nu \rangle$. For light-medium nuclei,[5] we have $\langle \eta^2 \rangle \simeq (0.75)^2$ and $\langle E_\nu \rangle \simeq 0.75 m_\mu$.

Let us consider the quantity

$$S_{\mu\nu} \equiv \sum_f{}' \int \frac{d\vec{P}_f}{4\pi} \langle i | Q_\nu^{(+)}(\vec{q}) | f ; \vec{P}_f \rangle \langle f ; \vec{P}_f | Q_\mu^{(-)}(\vec{q}) | i \rangle \, , \quad (5)$$

where the prime on the summation indicates the sum over the final states with excitation energies up to $M_i - M_f + m_\mu$ and

$$Q_\mu^{(\pm)}(\vec{q}) \equiv \int J_\mu^{(\pm)}(\vec{x}, 0) e^{i\vec{q}\cdot\vec{x}} d\vec{x} \quad . \quad (6)$$

When the assumption is made that low-lying excited states with energies up to $M_i - M_f + m_\mu$ saturate the sum, the sum in Eq. (5) can be replaced by the sum over a complete set. Then, from the definition of closure, Eq. (5) becomes

$$S_{\nu\mu} = \langle i | Q_\nu^{(+)}(\vec{q}) Q_\mu^{(-)}(\vec{q}) | i \rangle \quad . \quad (7)$$

On the other hand, substituting Eq. (6) into Eq. (5), and using translational invariance, we find

$$S_{\nu\mu}=\sum_f{}'\int\frac{d\vec{P}_f}{(2\pi)^3V}[(2\pi)^3\delta^{(3)}(\vec{P}_f-\vec{q})]^2\langle i|J_\nu^{(+)}(0)|\vec{P}_f\rangle\langle\vec{P}_f|J_\mu^{(-)}(0)|i\rangle=\sum_f{}'\langle i|J_\nu^{(+)}(0)|\vec{P}_f=\vec{q}\rangle\langle\vec{q}=\vec{P}_f|J_\mu^{(-)}(0)|i\rangle\ , \tag{8}$$

and thus, from Eq. (7),

$$\sum_f{}'\langle i|J_\nu^{(+)}(0)|\vec{P}_f=\vec{q}\rangle\langle\vec{P}_f=\vec{q}|J_\mu^{(-)}(0)|i\rangle=\langle i|Q_\nu^{(+)}(\vec{q})Q_\mu^{(-)}(\vec{q})|i\rangle\ . \tag{9}$$

Substituting Eq. (9) into Eq. (3) and carrying out the sums over α and β, we find[6]

$$\Gamma_i=\frac{G^2\cos^2\theta_c}{2\pi^2}C(N_i)(\alpha Z)^3m_\mu^5\langle\eta^2\rangle\int\frac{d\Omega_\nu}{4\pi}[\langle i|\vec{Q}^{(+)}\cdot\vec{Q}^{(-)}|i\rangle+\langle i|Q_0^{(+)}Q_0^{(-)}|i\rangle]\ . \tag{10}$$

Similarly, for the neutrino-scattering process

$$\nu_\mu(p_\nu,0)+i(P_i,M_i)\rightarrow\mu^-(p_\mu,m_\mu)+f(\vec{P}_f,M_f)\ , \tag{11}$$

we obtain the following total cross section

$$\sigma_i(E_\nu)=\frac{G^2\cos^2\theta_c}{2\pi}\sum_f\int\frac{d\Omega_\mu}{4\pi}\frac{E_\mu|\vec{p}_\mu|(m_\mu m_\nu/E_\mu E_\nu)}{[1+(E_\mu/E_f)-(E_\mu E_\nu/|\vec{p}_\mu|E_f)\cos\theta]}L_{\alpha\beta}\langle i|J_\beta^{(-)}(0)|f\rangle\langle f|J_\alpha^{(+)}(0)|i\rangle\ , \tag{12}$$

where θ is the angle that the outgoing muon makes with the incoming neutrino. As in the case of muon-capture processes, we now replace the outgoing muon energy and momentum by their suitable averages, $\langle E_\mu\rangle$ and $\langle|\vec{p}_\mu|\rangle$. In the same spirit, $|\vec{q}|=|\vec{p}_f|$ is replaced by $E_\nu-\langle|\vec{p}_\mu|\rangle$. Furthermore, since we are interested in neutrino energy of the range 150 ~ 250 MeV, and $E_f\simeq M_f>>E_\nu$, the angular-dependent term in Eq. (12) may be neglected.[7] We note that because

$$\vec{q}^{\,2}=E_\nu^2+\langle|\vec{p}_\mu|\rangle^2-2E_\nu\langle|\vec{p}_\mu|\rangle\cos\langle\theta\rangle\ ,$$

where $\langle\theta\rangle$ is the average angle between the outgoing muon and the incident neutrino, our assumption that $|\vec{q}|=E_\nu-\langle|\vec{p}_\mu|\rangle$ is equivalent to choosing $\langle\theta\rangle=0°$. In fact, at the neutrino energies in question, the individual state transitions appear to be strongly peaked in the forward direction.[8] If a different choice is made, i.e., $\langle\theta\rangle=15°$, the neutrino cross section shows some increase (see Fig. 1).

With these approximations, we can now apply the technique used to derive Eqs. (10)–(12). The result is, dropping the terms which would vanish after the angular integration,

$$\sigma_i(E_\nu)=\frac{G^2\cos^2\theta_c\langle E_\mu\rangle\langle|\vec{p}_\mu|\rangle}{\pi}\int\frac{d\Omega_\mu}{4\pi}[\langle i|\vec{Q}^{(-)}(\langle\vec{q}\rangle)\cdot\vec{Q}^{(+)}(\langle\vec{q}\rangle)|i\rangle+\langle i|Q_0^{(-)}(\langle\vec{q}\rangle)Q_0^{(+)}(\langle\vec{q}\rangle)|i\rangle]\ . \tag{13}$$

Taking the ratio of Eqs. (10) and (13), we find

$$\sigma_i(E_\nu)=\frac{2\pi\Gamma_i\langle E_\mu\rangle\langle|\vec{p}_\mu|\rangle}{C(N_i)(\alpha Z)^3m_\mu^5\langle\eta^2\rangle}\left\{\frac{\int\frac{d\Omega_\mu}{4\pi}[\langle i|\vec{Q}^{(-)}(\langle\vec{q}\rangle)\cdot\vec{Q}^{(+)}(\langle\vec{q}\rangle)|i\rangle+\langle i|Q_0^{(-)}(\langle\vec{q}\rangle)Q_0^{(+)}(\langle\vec{q}\rangle)|i\rangle]}{\int\frac{d\Omega_\nu}{4\pi}[\langle i|\vec{Q}^{(+)}(-\langle\vec{P}_\nu\rangle)\cdot\vec{Q}^{(-)}(-\langle\vec{P}_\nu\rangle)|i\rangle+\langle i|Q_0^{(+)}(-\langle\vec{P}_\nu\rangle)Q_0^{(-)}(-\langle\vec{P}_\nu\rangle)|i\rangle]}\right\}\ . \tag{14}$$

This is the desired relationship between σ_i and Γ_i.

Note that since

$$\langle i|\vec{Q}^{(+)}\cdot\vec{Q}^{(-)}|i\rangle=\langle i|\vec{Q}^{(-)}\cdot\vec{Q}^{(+)}|i\rangle$$

and

$$\langle i|Q_0^{(+)}Q_0^{(-)}|i\rangle=\langle i|Q_0^{(+)}Q_0^{(-)}|i\rangle\quad\text{(all real)}\ ,$$

the numerator and denominator in the curly bracket have the same structure, but are evaluated at different values of $\langle\vec{q}\rangle$. The former is a function of $\langle\vec{q}\rangle^2=(E_\nu-\langle|\vec{p}_\mu|\rangle)^2$, whereas the argument of the latter takes a fixed value of

$$[-\langle\vec{q}_\nu\rangle]^2=\langle\vec{q}\rangle^2=\langle E_\nu\rangle^2\simeq(0.75m_\mu)^2\ .$$

For practical purposes, we take, based on the assumption that most transitions occur through giant dipole resonance states,

$$\langle E_\mu\rangle\simeq E_\nu-\delta,\quad\langle|\vec{p}_\mu|\rangle=[(E_\nu-\delta)^2-m_\mu^2]^{1/2}\ , \tag{15}$$

where $\delta\simeq M_f^*-M_i$, with M_f^* representing the location of giant dipole resonance states in the final nucleus. Also,

$$\langle\vec{q}\rangle^2\simeq\langle q^2\rangle\simeq(E_\nu-\langle|\vec{p}_\mu|\rangle)^2\ , \tag{16}$$

where q^2 is a four-vector squared.

Next, we present an estimate of the ratio in the curly brackets in Eq. (14) in the nonrelativistic impulse approximation. In the standard impulse approximation, one assumes

$$J_\alpha^{(\pm)}(\vec{x},0)\simeq\sum_{a=1}^{A}(\Gamma_\alpha)_a\tau_a^{(\pm)}\delta(\vec{x}-\vec{r}_a)\ , \tag{17}$$

where $(\Gamma_\alpha)_a$ is the effective operator acting on the ath nucleon, and r_a is the location of the ath nucleon. For example, in the allowed (nonrelativistic) approximation,

$$(\Gamma_\alpha)_a=ig_\nu\delta_{\alpha,4}-(1-\delta_{\alpha,4})g_A(\sigma_\alpha)_a\ . \tag{18}$$

It has been shown that, in the above approximation[9]

$$\int\frac{d\Omega_q}{4\pi}[\langle i|\vec{Q}^{(+)}(\vec{q})\cdot\vec{Q}^{(-)}(\vec{q})|i\rangle+\langle i|Q_0^{(+)}(\vec{q})Q_0^{(-)}(\vec{q})|i\rangle]\simeq Z[G_V^2+3\Gamma_A^2]\left[1-\frac{A-Z}{2A}\delta(\vec{q}^{\,2})\right]\ , \tag{19}$$

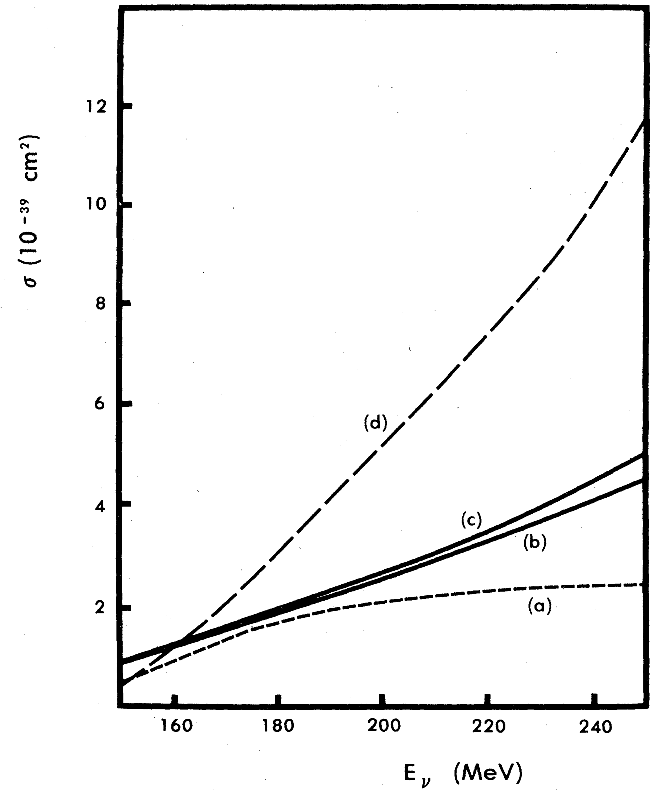

FIG. 1. Plot of the total inclusive neutrino cross section as a function of incident neutrino energy. Curve (a) is the result obtained by an impulse approximation calculation (see Ref. 7); curve (b) is the result obtained in this work for $|\vec{q}| = E_\nu - \langle |\vec{p}_\mu| \rangle$; curve (c) is the result obtained in this work for $\langle \theta \rangle = 15°$; and curve (d) is the result obtained by a Fermi gas model (see Ref. 6).

where

$$\begin{aligned} G_V &= g_V\left(1+\frac{E_\nu}{2m_P}\right) , \\ \Gamma_A &= G_A^2 + \tfrac{1}{3}(G_P^2 - 2G_A G_P) , \\ G_A &= g_A - g_V(1+\mu_P-\mu_n)\frac{E_\nu}{2m_P} , \\ G_P &= [g_P - g_A - g_V(1+\mu_P-\mu_n)]\frac{E_\nu}{2m_P} , \end{aligned} \tag{20}$$

with all form factors evaluated at $\vec{q}^{\,2}$. To a very good approximation, in particular, for $|\vec{q}| << 1$ GeV, one can write[9]

$$G_V^2 + 3\Gamma_A^2 \simeq (1+3g_A^2)R(\vec{q}^{\,2}) \simeq 1+3g_A^2 , \tag{21}$$

since $R(\vec{q}^{\,2})$ is a very slowly varying function of $\vec{q}^{\,2}$ with $R(0)=1$.

The only $\vec{q}^{\,2}$ dependence in Eq. (19) then comes from the quantity in the large parentheses in Eq. (19), which is due to the effects of the nucleon-nucleon correlation and the Pauli exclusion principle. With use of physically reasonable functional forms for nucleon-nucleon correlation density functions, $\delta(\vec{q}^{\,2})$ is given by, for light-medium nuclei,[9]

$$\delta(\vec{q}^{\,2}) = \left(\frac{d}{r_0}\right)^3\left(1-\frac{y^2}{10}+\cdots\right) , \tag{22}$$

where d is the radius of the Pauli correlation region, $r_0 = R/A^{1/3}$, and $y = qd$ with $q \equiv |\vec{q}|$.

Using $r_0 = \frac{1}{2}$ fm and $(d/r_0) \simeq 1.5$, one finds

$$\delta(\vec{q}^{\,2}) \simeq 3.30\left(1-0.091\frac{\vec{q}^{\,2}}{m_\mu^2}\right) , \tag{23}$$

which gives $\delta[\vec{q}^{\,2} = (0.75)^2 m_\mu^2] = 3.13$, consistent with the best fit value for the data on the total muon-capture rates for light-medium nuclei.

Substituting Eqs. (19) and (21) into Eq. (14), one finds

$$\begin{aligned} \sigma_i(E_\nu) &= \frac{2\pi\Gamma_i(E_\nu-\delta)[(E_\nu-\delta)^2-m_\mu^2]^{1/2}}{C(N_i)(\alpha Z)^3 m_\mu^5 \langle \eta^2 \rangle} \\ &\quad \times \frac{1-[(A-Z)/2A]\delta(\vec{q}^{\,2})}{1-3.13[(A-Z)/2A]} ; \\ \vec{q}^{\,2} &= \{E_\nu - [(E_\nu-\delta)^2-m_\mu^2]^{1/2}\}^2 , \end{aligned} \tag{24}$$

which gives $\sigma_i(E_\nu)$ as a function of E_ν.

For ^{12}C, Eq. (24) becomes

$$\begin{aligned} \sigma(^{12}\mathrm{C};E_\nu) &= \frac{2\pi\Gamma(^{12}\mathrm{C})(E_\nu-\delta)[(E_\nu-\delta)^2-m_\mu^2]^{1/2}}{0.84(6\alpha)^3 m_\mu^5 \langle \eta^2 \rangle} \\ &\quad \times 0.806\left(1+0.43\frac{\vec{q}^{\,2}}{m_\mu^2}\right) , \end{aligned} \tag{25}$$

which is plotted in Fig. 1 for $\Gamma_{\exp}(^{12}\mathrm{C}) = (3.97 \pm 0.10) \times 10^4$ sec^{-1} together with two previous model-dependent calculations, a Fermi gas model calculation[10] and the impulse approximation calculation with a specific nuclear model.[11]

Finally, we note that an elementary particle treatment[12] of this result, Eq. (14), may be undertaken. A sum of final states equivalent to Eq. (15) is set up, and closure is assumed leading to the replacement of the sum

$$\begin{aligned} \sum_f{}' \langle i|J_\nu^{(+)}(0)|\langle \vec{p}_f \rangle\rangle \langle\langle \vec{p}_f \rangle|J_\mu^{(-)}(0)|i\rangle &= \int d^4x e^{iq\cdot x}\langle i|J_\nu^{(+)}(0)J_\mu^{(-)}(x)|i\rangle \\ &= Q_{\nu\mu}(P_i,q) \end{aligned} \tag{26}$$

by a tensor $Q_{\mu\nu}(p_i,q)$, where q is the average momentum transferred and $\vec{q} = \langle \vec{p}_f \rangle$, and $q_0 = E_f - E_i$. The form of $Q_{\nu\mu}(p_i,q)$ is, in general,

$$\begin{aligned} Q_{\nu\mu}(P_i,q) &= \alpha g_{\nu\mu} + \frac{\beta}{M_i^2}(P_i)_\nu(P_i)_\mu + \frac{\gamma}{M_i^2}(P_i)_\nu q_\mu \\ &\quad + \frac{\delta}{M_i^2}(P_i)_\mu q_\nu + \frac{\rho}{M_i^2}\epsilon_{\nu\mu\rho\sigma}q_\rho(P_i)_\sigma , \end{aligned} \tag{27}$$

but by arguments equivalent to those used in deriving Eq. (19),

$$Q_{\nu\mu}(P_i,q) \simeq \alpha g_{\nu\mu} + \frac{\beta}{M_i^2}(P_i)_\nu(P_i)_\mu . \tag{28}$$

This result can then be used to derive expressions for both the total muon-capture rate and the total neutrino-scattering cross sections. Both are found to depend only on the combination $\beta - 2\alpha$, but evaluated at different values of q^2. This gives rise to the same relationship between Γ_i and σ_i noted above except that the q^2 dependence of the form factor combination $\beta - 2\alpha$ must be obtained from appropriate experimental data.

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