Hypertriton and hyperspherical harmonics

R. B. Clare*
U.S. Military Academy, West Point, New York 10996

J. S. Levinger
Rensselaer Polytechnic Institute, Troy, New York 12181
(Received 5 November 1984; revised manuscript received 13 March 1985)

We study the hypertriton using the formalism of hyperspherical harmonics, for spin-independent NN and NΛ two-body potentials of Gaussian shape. We use up to 19 Simonov harmonics for the Verma-Sural potential, and obtain a ground state energy 240 keV below their value of −2.35 MeV found for 9 Simonov harmonics. The partial waves are used to calculate various expectation values for the hypertriton. We readjust the strength of the ΛN potential to obtain agreement between the calculated energy and the experimental value.

The hypertriton, ${}^3_\Lambda$H, has been studied using variational techniques[1,2] and Faddeev calculations.[3] Verma and Sural (VS) used hyperspherical harmonics (HH) to study ${}^3_\Lambda$H for both model[4] and realistic potentials.[5] They solve a truncated set of coupled integral equations for the energy. We summarize the HH formalism for ${}^3_\Lambda$H giving a set of coupled differential equations (CDE). We also derive some expectation values relating to the shape and kinetic energy of ${}^3_\Lambda$H.

The CDE can be solved by searching for the eigenenergy and the ratios of the partial waves at the starting points of integration,[6] or by using various approximate techniques.[7,8] We summarize Gordon's integration strategy[9] which reduces the solution of the CDE to a one-dimensional eigenvalue search. We also review Johnson's renormalized Numerov method,[10] which is combined with Gordon's procedure. We also present the means of computing the partial waves.

We present our results of solving M CDE, for $M=1$–20, using central spin-independent two-body Gaussian potentials. We compare our results with those of VS and comment on the convergence of the HH formalism for this ${}^3_\Lambda$H problem. We also discuss some results we obtain with the partial waves.

The neutron and proton have position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ and mass m. The lambda particle's position vector is $\mathbf{r}_3$ and its mass is taken to be $\frac{6}{5}m$. We choose Jacobi coordinates

$$\boldsymbol{\eta}\equiv\alpha(\mathbf{r}_1-\mathbf{r}_2)\ ,\qquad \boldsymbol{\xi}\equiv\beta(\mathbf{r}_1+\mathbf{r}_2-2\mathbf{r}_3)\ , \tag{1}$$

where $\alpha=2^{-1/2}$ and $\beta=\alpha[m_\Lambda/(m_\Lambda+2m)]^{1/2}$. (VS[4] use another set.) We introduce six-space hyperspherical coordinates ρ and $\hat{\boldsymbol{\rho}}$ where

$$\rho^2\equiv\eta^2+\xi^2 \tag{2}$$

and $\hat{\boldsymbol{\rho}}$ is a unit vector uniquely defined by five appropriately chosen angles.

The kinetic energy operator is

$$\hat{T}=-(\hbar^2/2m)\left[\frac{1}{\rho^5}\frac{\partial}{\partial\rho}\left(\rho^5\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\hat{K}^2\right]\ . \tag{3}$$

The eigenfunctions of $\hat{K}^2$ are the Simonov hyperspherical harmonics (HH),[11] $Y_{K\nu}(A,\lambda)$, which obey the eigenvalue equation

$$[\hat{K}^2+K(K+4)]Y_{K\nu}(A,\lambda)=0;\ \ K=0,1,2,\ldots, \tag{4}$$

ν goes from $-K/2$ to $+K/2$ in steps of two. The HH with $\nu<0$ are not used for ${}^3_\Lambda$H since they are not invariant for a permutation of the identical nucleons. Furthermore, the even parity of the system considered restricts K to even values.

The Schrödinger equation can be written in the matrix-vector form:

$$\left([I]\frac{d^2}{d\rho^2}+[Q(\rho)]\right)\mathbf{u}(\rho)=0\ , \tag{5}$$

where $[I]$ is the unit matrix and the column vector $\mathbf{u}(\rho)$ contains the partial waves $u_K^\nu(\rho)$ as its components. The matrix $[Q(\rho)]$ is defined by

$$[Q(\rho)]=\{E[I]-[V_{\text{eff}}(\rho)]\}(2m/\hbar^2)\ , \tag{6}$$

where the components of the effective potential matrix are given by

$$[V_{\text{eff}}(\rho)]_{K\nu}^{K'\nu'}\equiv\langle K\nu|V_{123}|K'\nu'\rangle+(\hbar^2/2m)[K(K+4)+15/4]\rho^{-2}\delta_{KK'}\delta_{\nu\nu'}\ . \tag{7}$$

VS solve a set of coupled integral equations equivalent to a truncated form of Eq. (7). We find[12–14]

$$\langle K\nu|V_{123}|K'\nu\rangle=C_{KK'}^{\nu\nu'}\sum_{\substack{K''=0,2,\ldots\\ \nu''=\nu-\nu',\nu+\nu'}}\sum(-1)^{K''/2-\nu''}\frac{\left|\left\langle\frac{K}{4}\ \frac{\nu}{2}\ \frac{K'}{4}\ \frac{\nu'}{2}\left|\frac{K''}{4}\ \frac{\nu''}{2}\right.\right\rangle\right|^2}{K''+2}V_{K'',\nu''}(\rho)\ , \tag{8}$$

where the $\langle lml'm'|l''m''\rangle$ are Clebsch-Gordan coefficients and the constants $C_{KK'}^{\nu\nu'}$ are

$$C_{KK'}^{\nu\nu'}\equiv(-1)^{[K+K'-2(\nu+\nu')/4}\left(\frac{(K+2)(K'+2)}{(1+\delta_{\nu0})(1+\delta_{\nu'0})}\right)^{1/2}\ . \tag{9}$$

 ©1985 The American Physical Society

The hypermultipoles have the form

$$V_{K,\nu}(\rho)=V_K^{12}(\rho)+2\cos(\nu\gamma V_K^{13}(a\rho)\ , \tag{10}$$

where $a=(11/12)^{1/2}$ and the angle $\gamma=2.043$ rad. We use pair potentials with Gaussian shape, so the terms $V_K^{12}(\rho)$ and $V_K^{13}(a\rho)$ can be written[11] in analytical form.

We use[14] the first two terms in the HH expansion of the wave function to calculate some expectation values. The average interparticle separations are

$$\langle r_{12}^2\rangle=2\langle\eta^2\rangle\ ,$$

$$\langle r_{23}^2\rangle=\langle r_{13}^2\rangle=\tfrac{1}{2}\langle\eta^2\rangle+\tfrac{4}{3}\langle\xi^2\rangle+\frac{4}{\sqrt{6}}\langle\boldsymbol{\eta}\cdot\boldsymbol{\xi}\rangle\ . \tag{11}$$

$$\left\langle\left[\frac{\mathbf{r}_1+\mathbf{r}_2}{2}-\mathbf{r}_3\right]^2\right\rangle=\tfrac{4}{3}\langle\xi^2\rangle\ .$$

$$\langle\eta^2\rangle=\tfrac{1}{2}(\langle u_0|\rho^2|u_0\rangle+\langle u_2|\rho^2|u_2\rangle+\langle u_0|\rho^2|u_2\rangle)\ ,$$
$$\langle\xi^2\rangle=\tfrac{1}{2}(\langle u_0|\rho^2|u_0\rangle+\langle u_2|\rho^2|u_2\rangle-\langle u_0|\rho^2|u_2\rangle)\ . \tag{12}$$

Here the abbreviations $u_0^0(\rho)\equiv u_0$ and $u_2^1(\rho)\equiv u_2$ are used. We calculate the average kinetic energy of the system from the formula

$$\langle T\rangle=E-(\langle u_0|V_{00}^{00}|u_0\rangle+\langle u_2|V_{21}^{21}|u_2\rangle+\langle u_0|V_{00}^{21}|u_2\rangle)\ . \tag{13}$$

We use Johnson's renormalized Numerov method[10,14,15] to solve MCDE (5) for $\mathbf{u}(r)$. We define

$$[T_n]=[h^2/12][Q(nh)] \tag{14}$$

and

$$\mathbf{u}(r)=[U(r)]\cdot\mathbf{b}\ , \tag{15}$$

where $\mathbf{b}$ is a constant column vector and the M columns of $[U(r)]$ are linearly independent solutions of Eq. (5). The recurrence relation giving the "matrix wave function" $[U_{n+1}]$ is

$$([I]-[T_{n+1}])[U_{n+1}]-(2[I]+10[T_n])[U_n]+([I]-[T_{n-1}])[U_{n-1}]=[0]\ . \tag{16}$$

Here $n=0,1,\ldots$ N, N is the end point of integration, and h is the step size. We integrate the coupled Schrödinger equation (5) for the initial two guesses of the energy from the origin outward and from a large value $r=\mathrm{N}h$ inward to a matching point r_M. Take

$$[F_n]\equiv([I]-[T_n])[U_n]\ .$$

Define

$$[R_n]=[F_{n+1}][F_n^{-1}]\ ,$$

and

$$[\hat{R}_n]=[F_{n-1}]F_n^{-1}]\ .$$

Evaluate the determinant

$$D(E)\equiv|[R_M]-[\hat{R}_{M+1}^{-1}]|\ . \tag{17}$$

If we have the right energy we get a match, i.e., $D(E)=0$. After finding E, we can then find[10,14,15] the eigenfunction.

Beiner and Fabre de la Ripelle[16] use the optimal subset and find for triton calculations with the Gaussian Baker potential[17] that it was necessary to include terms up to $K=16$ to attain eigenvalue convergence. For ${}^3_\Lambda$H, where VS use similar potentials, we would expect to require even more grand orbitals.

To study convergence, the M CDE are solved for $M=1,2,\ldots,M_{\max}$. Each time the Hilbert space is enlarged by one term in the HH expansion, the corresponding decrease in the energy eigenvalue is noted. For the NN pair, we used Gaussian potentials with the VS strength and range parameters,

$$V_0=-72.5\ \text{MeV},\quad \beta=1.47\ \text{fm}\ . \tag{18}$$

For the ΛN pairs, VS use a spin-dependent Gaussian form where the intrinsic range for both the singlet and triplet interaction is taken to be $b=1.484$ fm (corresponding to the two-pion exchange mechanism). VS then solve nine coupled integral equations and use the hypertriton's measured binding energy, $E=-2.355$ MeV, to adjust the volume integral[18] yielding $V_2=719$ MeV fm^3, for the lambda-nucleon potential. In this work, we first use spin-independent Gaussian shapes for the ΛN pairs with parameters

$$V_0'=-58.4\ \text{MeV},\quad \beta'=1.034\ . \tag{19}$$

Using these potentials, we solve Eq. (5) by the renormalized Numerov method for $M=1$–19. The eigenvalues are presented in the third column of Table I. Comparing with the VS result of -2.355 MeV at $M=9$, we note a 20 keV discrepancy which is not unreasonable considering the differences in the calculational methods. However, we disagree with the VS inference that eigenvalue convergence is attained at $M=9$ ($K=8$) since Table I shows that E decreases by 240 keV with the addition of ten more terms in the HH expansion. Another reason to doubt that VS had

TABLE I. Energy $E(M)$ of hypertriton vs number M of CDE used in Eq. (5). V_0' is the assumed coefficient of the ΛN Gaussian potential.

		E_M (MeV)	
K,ν	M	$V_0'=-58.4$	$V_0'=-55.9$
0,0	1	−0.523	...
2,1	2	−1.072	−0.692
4,0	3	−1.641	...
4,2	4	−1.780	−1.398
6,1	5	−1.857	...
6,3	6	−2.216	−1.850
8,0	7	−2.289	...
8,2	8	−2.321	...
8,4	9	−2.376	−2.029
10,1	10	−2.411	...
10,3	11	−2.479	...
10,5	12	−2.499	−2.171
12,0	13	−2.515	...
12,2	14	−2.526	
12,4	15	−2.543	
12,6	16	−2.577	−2.263
14,1	17	−2.585	...
14,3	18	−2.609	...
14,5	19	−2.617	...
14,7	20	...	−2.326
	Infinite	(−2.71)	(−2.44)

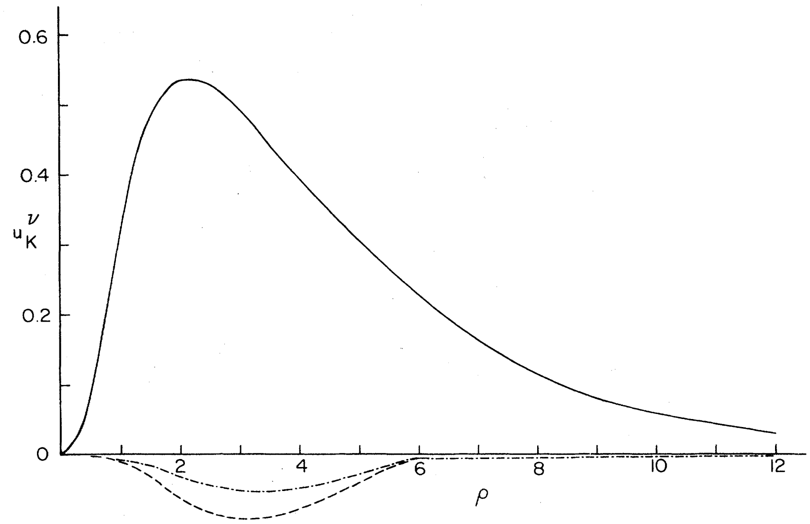

FIG. 1. Three partial waves $u_K^\nu(\rho)$ plotted as ordinate vs hyperradius ρ in fm. The solid curve is $u_0^0(\rho)$, the dashed curve is $u_2^1(\rho)$, and the dash-dot curve is $u_4^0(\rho)$. u_2^1 and u_4^0 agree to the accuracy of drawing for $\rho > 6$ fm.

achieved convergence is the magnitude of the VS volume integral, $V_2 = 719$ MeV fm^3. In contrast, Dalitz[2] gives the limits[1] $615 \leq V_2 \leq 685$ MeV fm^3.

We made a rough extrapolation of our $E(M)$ values of the third column of Table I to infinite M, using the technique of Beiner and Fabre de la Ripelle.[16] We find $E(\infty) \approx =2.71$ MeV with an estimated error of some 50 keV. Since our extrapolated value is 360 keV below the experimental hypertriton energy, we readjust the strength V_0' of the ΛN potential from the value -58.4 MeV of Eq. (19) to $V_0' = -55.9$ MeV. This readjustment gives the values of $E(M)$ in the right hand column of Table I. The extrapolated value of about -2.44 MeV is only a little below the experimental value, -2.335 MeV. This new value of the strength V_0 decreases the VS volume integral from their value of 719–688 MeV fm^3, barely outside Dalitz's limits.

In light of the convergence problems presented here, we stress the importance of presenting one's eigenvalue results in the form of Table I when utilizing the hyperspherical formalism.

Although eigenvalue convergence is slow in the HH formalism, the wave function converges rapidly because the effective centrifugal term in $[V_{\text{eff}}(\rho)]$ of Eq. (7) suppresses higher order partial waves. For the model potential with $V_0' = -58.4$ MeV we found from the solution of four CDE that the lowest partial wave dominates with a norm of 97.1%.

Figure 1 shows that the lowest partial wave, $u_0^0(\rho)$ is nodeless. This nodeless property is a general feature for the lowest partial wave of the ground state solution of an infinite set of CDE corresponding to a local but noncentral potential.[15,19] The figure thus provides a useful check on the accuracy of our numerical work.

Retaining two terms in the wave function expansion, we compute the expectation values discussed above. The hyperradial integrals have values

$$\langle u_0|\rho^2|u_0\rangle = 14.89\ \text{fm}^2,\quad \langle u_2|\rho^2|u_2\rangle = 0.302\ \text{fm}^2,\quad \langle u_0|\rho^2|u_2\rangle = -1.908\ \text{fm}^2\ , \tag{20}$$

$$\langle u_0|V_{00}^{00}(\rho)|u_0\rangle = -21.38\ \text{MeV},\quad \langle u_0|V_{21}^{00}(\rho)|u_2\rangle = -0.568\ \text{MeV},\quad \langle u_2|V_{21}^{21}(\rho)|u_2\rangle = -0.346\ \text{MeV}\ . \tag{21}$$

Equations (11), (12), (20), and (21) give interparticle separations

$$\langle r_{12}^2\rangle = 13.3\ \text{fm}^2,\quad \langle r_{13}^2\rangle = \langle r_{23}^2\rangle = 14.7\ \text{fm}^2\ ,$$
$$\left\langle \left| \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} - \mathbf{r}_3 \right|^2 \right\rangle = 11.4\ \text{fm}^2\ . \tag{22}$$

We see that it is incorrect to view ${}^3_\Lambda$H as a Λ particle spending much of its time far from the deuteron core since the squared NN distance $\langle r_{12}^2\rangle$ is almost as large as $\langle r_{13}^2\rangle$. We also calculate the average kinetic energy of the system using Eqs. (13) and (21), giving

$$\langle T\rangle = 19.9\ \text{MeV}\ . \tag{23}$$

Compare the same quantity calculated[20] for the trinucleon using a Volkov potential: $\langle T\rangle = 22.2$ MeV.

To conclude, we comment on methods of solving CDE. The renormalized Numerov method is nearly as easy to employ as adiabatic approximation techniques.[7,8] The Simonov HH expansion gives useful results for hypertriton properties with our simple model potentials, but convergence is slow. It would be desirable to develop Fabre's optimal subset for the hypertriton, so convergence could be obtained for more complicated potentials.

Part of this work was done by R.B.C. as part of his Doctoral Thesis, Rensselaer Polytechnic Institute, August 1983.

*Now at T-9, MS-B279, Los Alamos National Laboratory, Los Alamos, NM 87545.

[1]R. C. Herndon and Y. C. Tang, Phys. Rev. **153**, 1091 (1967); **165**, 1093 (1968).

[2]R. H. Dalitz, R. C. Herndon, and Y. C. Tang, Nucl. Phys. **B47**, 109 (1972).

[3]B. F. Gibson and D. R. Lehman, Phys. Rev. C **18**, 1042 (1978); **22**, 2024 (1980); Phys. Lett. **83B**, 289 (1979); Nucl. Phys. **A329**, 308 (1979).

[4]S. P. Verma and D. P. Sural, Phys. Rev. C **20**, 782 (1979); **22**, 229 (1980).

[5]S. P. Verma and D. P. Sural, J. Phys. G **8**, 73 (1982).

[6]E. Rost, Phys. Rev. **154**, 994 (1967).

[7]J. L. Ballot, M. Fabre de la Rippelle, and J. S. Levinger, Phys. Rev. C **26**, 2301 (1982).

[8]T. K. Das, H. T. Coelho, and M. Fabre de la Ripelle, Phys. Rev. C **26**, 2281 (1982).

[9]R. G. Gordon, J. Chem. Phys. **51**, 14 (1969); A. M. Dunder and R. G. Gordon, *ibid.* **64**, 4984 (1976).

[10]B. R. Johnson, J. Chem. Phys. **69**, 4678 (1978).

[11]Y. A. Simonov, Sov. J. Nucl. Phys. **3**, 461 (1966); **3**, 755 (1966).

[12]G. Erens, J. L. Visschers, and R. van Wageningen, Ann. Phys. **67**, 461 (1971).

[13]V. B. Mandelzweig, Phys. Lett. **80A**, 361 (1980).

[14]R. B. Clare, Ph.D. thesis, Rensselaer Polytechnic Institute, 1983.

[15]I. Elminyawi and J. S. Levinger, J. Chem. Phys. **82**, 905 (1985).

[16]M. Beiner and M. Fabre de la Ripelle, Lett. Nuovo Cimento **1**, 583 (1971).

[17]G. A. Baker, J. L. Gammel, B. J. Hill, and J. G. Wills, Phys. Rev. **125**, 1754 (1962).

[18]R. H. Dalitz, *Nuclear Interactions of the Hyperons* (Oxford Univ. Press, London, 1965).

[19]J. S. Levinger, J. Math. Phys. (to be published).

[20]K. Myers, K. K. Fang, and J. S. Levinger, Phys. Rev. C **15**, 1215 (1977).