Misconceptions regarding spin $\frac{3}{2}$

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Recent work involving virtual excitation of a spin $\frac{3}{2}$ baryon resonance is seen to contain two distinct problems. The Feynman propagator for spin $\frac{3}{2}$ from the Rarita-Schwinger formalism has often been mistaken for its on-mass-shell limit. In nonrelativistic work the direct channel exchange of a Δ resonance is normally included, but with the exclusion of a term for the intermediate anti-Δ. It is shown that both of these terms are of equal importance in the resonance region for the case of nucleon Compton scattering.

Two errors relating to calculations treating a spin $\frac{3}{2}$ particle in an intermediate state are presently propagating through the literature. One relates to a mistaken form of the Rarita-Schwinger spin $\frac{3}{2}$ propagator, which can be found in textbooks[1,2] and in journal articles.[3,4] The second concerns the form usually adopted for the $\gamma N\Delta$ vertex in a two-component (Pauli spinor) treatment for the nucleons.[3,5,6] This paper is an effort towards eliminating the errors.

The Rarita-Schwinger formalism allows one to treat higher spin particles (for present purposes, spin $\frac{3}{2}$) in the language and techniques of the four-component Dirac theory for spin $\frac{1}{2}$. In particular, the relativistic propagator for spin $\frac{3}{2}$ particles, $P^{\mu\nu}$, is a collection of 4×4 matrices which are most conveniently expressed in terms of the familiar γ matrices of the Dirac theory. The indices μ and ν are to be contracted with vector indices from the vertex functions which create and destroy the propagating particle; they are necessary to allow the number of degrees of freedom appropriate for particles of spin $\frac{3}{2}$. References 1–4 all present the following form for this propagator:

$$P^{\mu\nu}=D\frac{M+\not{p}}{2M}\left[g^{\mu\nu}-\tfrac{1}{3}\gamma^\mu\gamma^\nu-\frac{2p^\mu p^\nu}{3M^2}+\frac{p^\mu\gamma^\nu-p^\nu\gamma^\mu}{3M}\right]\ , \quad (1)$$

where M is the mass of the propagating particle and $p^\mu(=W,\mathbf{p})$ is its four momentum:

$$p^2=p_\mu p^\mu=W^2-\mathbf{p}^2;\ \ \not{p}=p_\mu\gamma^\mu\ ;$$

$$g^{00}=1,\ \ g^{11}=g^{22}=g^{33}=-1,\ \ g^{\mu\nu}=0\ \text{ if } \mu\neq\nu\ ;$$

$$D=1/(p^2-M^2)\ .$$

In contrast, Behrends and Fronsdal[7] present a simple derivation of this propagator, and show its form to be

$$P'^{\mu\nu}=D\frac{M+\not{p}}{2M}\left[g^{\mu\nu}-\tfrac{1}{3}\gamma^\mu\gamma^\nu-\frac{1}{3p^2}(\not{p}\gamma^\mu p^\nu+p^\mu\gamma^\nu\not{p})\right]\ . \quad (2)$$

To compare (1) and (2), use the relation $\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}$ to commute the last $\not{p}$ in (2) left of γ^ν, and then use $\not{p}\not{p}=p^2$ to arrive at the form

$$P'^{\mu\nu}=D\frac{M+\not{p}}{2M}\left[g^{\mu\nu}-\tfrac{1}{3}\gamma^\mu\gamma^\nu-\frac{2}{3}\frac{p^\mu p^\nu}{p^2}\right]+D\frac{M\not{p}+p^2}{6Mp^2}(p^\mu\gamma^\nu-p^\nu\gamma^\mu)\ . \quad (3)$$

From this form it is apparent that if the particle is on the mass shell, $(p^2=M^2)P'^{\mu\nu}$ reduces to $P^{\mu\nu}$, but this is not an appropriate limit for a relativistic propagator. The derivation[7] of $P'^{\mu\nu}$ is valid both on and off the mass shell. A general form for the indexed terms $(g^{\mu\nu}-\cdots)$ in the rest frame is generated, the constraints of various invariance properties are invoked, and the result is written in covariant form. This last step depends upon the identity in the rest frame

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}=g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\ .$$

This is valid regardless of the value of p^2, but the substitution $p^2=M^2$ makes it valid only on shell. Since the denominator p^2 in Eq. (3) is generated in this way, the substitution required to make $P'^{\mu\nu}$ take the form $P^{\mu\nu}$ is one which causes the form $P^{\mu\nu}$ to be invalid as a propagator (except on shell). There is a well-documented ambiguity (point contact ambiguity) related to off-mass-shell extrapolations. The effect of this ambiguity in the Lagrangian upon the propagator is to add nonresonant terms. The difference between $P^{\mu\nu}$ and $P'^{\mu\nu}$ are terms with the resonant factor D, and thus cannot be point contact terms. $P'^{\mu\nu}$ is correct, but not unique; $P^{\mu\nu}$ is not correct. In practice, $P'^{\mu\nu}$ typically leads to simpler expressions than the often used $P^{\mu\nu}$. Its center-of-momentum form is

$$p'^{00}=p'^{0i}=p'^{i0}=0\ ,$$
$$p'^{ij}=-\frac{D}{6M}\begin{pmatrix}(W+M)S^iS^{\dagger j} & 0\\ 0 & (-W+M)S^iS^{\dagger j}\end{pmatrix}\ , \quad (4)$$

where $i,\ j=1,\ 2$, or 3, and S is the transition operator from spin $\frac{3}{2}$ to spin $\frac{1}{2}$, normalized such that $S\cdot S^\dagger=2$. The representation of the γ^μ assumed is that of Bjorken and Drell,[8] which produces free field solutions for spin $\frac{1}{2}$ which have the lower two components smaller than the upper two by $\sim p/m$.

To illustrate the use of $P'^{\mu\nu}$, I will present amplitudes for γN scattering, π photoproduction, and γN scattering through an intermediate s channel Δ resonance [spin $\frac{3}{2}$, isospin $\frac{3}{2}$, + parity, $M=1232$ MeV, $\Gamma=110$ MeV; $D=1/(p^2-M^2+iM\Gamma)$]. The vertex functions are as follows:

$$\pi N \rightarrow \Delta : G_\pi g_\mu \ , \quad \gamma N \rightarrow \Delta : -iG_{\gamma 1}\left(e_\nu - \frac{\not{e} k_\nu}{m+M} \right)\gamma_5 - iG_{\gamma 2}\frac{P_N \cdot e k_\nu - P_N \cdot k e_\nu}{(m+M)^2}\gamma_5 \ ,$$

where $P_{N\mu}$, q_μ, and k_μ are the nucleon, pion, and photon 4-momenta; $e_\nu = (0, -\mathbf{e})$ is the photon polarization; and m is the nucleon mass. Of the two gauge invariant electromagnetic couplings, the $G_{\gamma 2}$ term is of higher order in p_N/m than the $G_{\gamma 1}$, and will be omitted in the following formulas. The scattering matrix elements are

$$T_{\pi N \rightarrow \pi N} = -\frac{G_\pi^2 D}{6M}\bar{u}_f \begin{pmatrix} (W+M)\mathbf{q}' \cdot \mathbf{S}\mathbf{S}^\dagger \cdot \mathbf{q} & 0 \\ 0 & (-W+M)\mathbf{q}' \cdot \mathbf{S}\mathbf{S}^\dagger \cdot \mathbf{q} \end{pmatrix} u_i \ , \tag{5}$$

$$T_{\gamma N \rightarrow \pi N} = -\frac{iG_{\gamma 1}G_\pi D}{6M}\bar{u}_f \begin{pmatrix} \frac{i(W+M)}{2(M+m)}\mathbf{q}' \cdot \mathbf{S}\mathbf{S}^\dagger \cdot \mathbf{e}\times\mathbf{k} & (W+M)\mathbf{q}' \cdot \mathbf{S}\mathbf{S}^\dagger \cdot \mathbf{e} \\ (-W+M)\mathbf{q}' \cdot \mathbf{S}\mathbf{S}^\dagger \cdot \mathbf{e} & \frac{i(W-M)}{2(M+m)}\mathbf{q}' \cdot \mathbf{S}\mathbf{S}^\dagger \cdot \mathbf{e}\times\mathbf{k} \end{pmatrix} u_i \ , \tag{6}$$

$$T_{\gamma N \rightarrow \gamma N} = -\frac{G_{\gamma 1}^2 D}{6M}\bar{u}_f \begin{pmatrix} \left[\frac{-(W+M)}{4(M+m)^2}\mathbf{e}'\times\mathbf{k}' \cdot \mathbf{S}\mathbf{S}^\dagger \cdot \mathbf{e}\times\mathbf{k} + (-W+M)\mathbf{e}' \cdot \mathbf{S}\mathbf{S}^\dagger \cdot \mathbf{e}\right] & \left[\frac{i(W+M)}{2(M+m)}\mathbf{e}'\times\mathbf{k}' \cdot \mathbf{S}\mathbf{S}^\dagger \cdot \mathbf{e} + \frac{i(W-M)}{2(M+m)}\mathbf{e}' \cdot \mathbf{S}\mathbf{S}^\dagger \cdot \mathbf{e}\times\mathbf{k}\right] \\ \left[\frac{i(W+M)}{2(M+m)}\mathbf{e}' \cdot \mathbf{S}\mathbf{S}^\dagger \cdot \mathbf{e}\times\mathbf{k} + \frac{i(W-M)}{2(M+m)}\mathbf{e}'\times\mathbf{k}'\mathbf{S}\mathbf{S}^\dagger \cdot \mathbf{e}\right] & \left[\frac{-(-W+M)}{4(M+m)^2}\mathbf{e}'\times\mathbf{k}' \cdot \mathbf{S}\mathbf{S}^\dagger \cdot \mathbf{e}\times\mathbf{k} \times (M+W)\mathbf{e}' \cdot \mathbf{S}\mathbf{S}^\dagger \cdot \mathbf{e}\right] \end{pmatrix} u_i \ . \tag{7}$$

u_i, $\bar{u}_f$ are the initial and adjoint final state nucleon spinors.

In the spirit of Ref. 4, these same amplitudes can be evaluated in the higher spin formalism of Hayward,[9] using the vertex functions of Danos, Gillet, and Cauvin.[10] Since Ref. 4 used the incorrect propagator $P^{\mu\nu}$, the Rarita-Schwinger results presented there [Eqs. (12), (15), and (18)] are incorrect. The Hayward results were calculated correctly. The Hayward πN scattering results agree identically with Eq. (5) above. To lowest nonvanishing order in p/M, Hayward results can be made to agree with Eqs. (6) and (7), if both the current and anomalous moment contributions to the $\gamma N \Delta$ vertex are kept,[10] and the ratio of the anomalous moment coupling constant to that for the current is $\frac{1}{4}(M+m)$. Higher order terms (in k/m) which appear in the Hayward results correspond to the $G_{\gamma 2}$ Rarita-Schwinger coupling.

There is often a need to treat spin $\frac{3}{2}$ (Δ) degrees of freedom in the context of a two component nucleon, particularly in traditional nuclear physics. In this case the propagator becomes simply a spin projection sum (=1 through closure) and an energy difference denominator. The terms in the electromagnetic interaction Hamiltonian which excite the Δ are chosen to satisfy gauge invariance, parity, and angular momentum conservation, and normally only the lowest order in k/m (k=photon momentum) is retained. The term usually chosen to represent the $\gamma N \rightarrow \Delta$ excitation is

$$H_1 = a_1 \mathbf{S}^\dagger \cdot \mathbf{e}\times\mathbf{k} \ . \tag{8}$$

The $\mathbf{e}\times\mathbf{k}$ dependence reveals this to be an $M1$ excitation, in agreement with experiment, which has shown the $E2$ part of the photoexcited Δ peak to be small, and with the quark model which predicts the $E2$ contribution to vanish. In addition, a gauge invariant $E2$ addition to the Hamiltonian would be of higher order in k/M than H. Weber and Arenhovel[6] exhibit a contribution corresponding to an $E2$ term, first order in k/M,

$$H_2 = a_2[[\mathbf{S}^\dagger \sigma]^2[\mathbf{e}\mathbf{k}]^2]^0 \ , \tag{9}$$

but gauge invariance of this term can only be shown in the static limit $k_0 = M - m$ (the notation $[AB]^2$ represents two tensors, rank 1, coupled to rank 2 via Wigner coefficients[11]). An $E1$ term,

$$H_3 = a_3 M \mathbf{S}^\dagger \cdot \mathbf{e} \ , \tag{10}$$

is of the wrong parity to couple to the Δ. This term *does*, however, represent the coupling of $\gamma N \bar{\Delta}$, where all are incoming particles, since the antiparticle is of the opposite parity as the Δ itself. This interaction contributes to nucleon Compton scattering to the same order in k/m as H_1. Consider two time ordered diagrams (Fig. 1): Fig. 1(a), the "direct term," represents the creation and propagation of a real Δ; Fig. 1(b), the "Z graph," represents the creation of an anti-Δ at the vertex with the outgoing particles, and its absorption at the incoming vertex. The corresponding scattering amplitudes are

$$T_{1a} \propto a_1^2 \frac{\mathbf{e}'\times\mathbf{k}' \cdot \mathbf{S}\mathbf{S}^\dagger \cdot \mathbf{e}\times\mathbf{k}}{W - M + i\Gamma/2} \ , \tag{11}$$

$$T_{1b} \propto a_3^2 M^2 \frac{\mathbf{e}' \cdot \mathbf{S}\mathbf{S}^\dagger \cdot \mathbf{e}}{W+M} \ . \tag{12}$$

The denominator of T_{1a}, in the region of the resonance peak, is like a kinetic energy term, and thus of order k^2/M; the amplitude is of order M. T_{1b} has a nonresonant denominator of order M, so this amplitude is also of order M. Calculations show that both these terms are necessary to match Compton data in the Δ region, and that they are of the same order of magnitude.

The relative strengths (a_1^2, a_3^2) with which these contri-

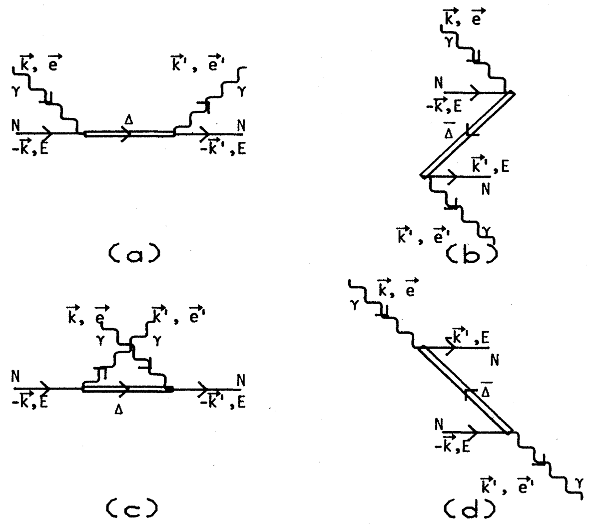

FIG. 1. Contributions to nucleon Compton scattering with intermediate Δ; (a) direct term; (b) Z graph; (c) crossed photon term; (d) crossed Z graph.

bute to Δ excitation, can be determined from a relativistic calculation, since the time ordered graphs of Figs. 1(a) and 1(b) are both contained in the Feynman graph for s-channel Δ exchange. A second Z graph, Fig. 1(d), comes from the antiparticles part of the Feynman graph for u-channel Δ exchange (crossed photon diagram). The process of Fig. 1(d) gives an $\mathbf{e}'\cdot\mathbf{S}\mathbf{S}^{\dagger}\cdot\mathbf{e}$ contribution which, in the nonrelativistic limit, is identical in sign and size to that of Fig. 1(b). Other contributions from Fig. 1(d) are of different multipolarities. Contributions from Fig. 1(c), the real Δ crossed photon graph, are ignored, since the Δ is excited by the interaction H_1 and is roughly 500 MeV off the mass shell in the resonance region. A relativistic evaluation of terms corresponding to Figs. 1(a), 1(c), and 1(d) leads to the conclusion $(a_3/a_1)^2=0.062$.

A term with the same spin and polarization dependence as T_{1b} is treated in Ref. 12 [Eq. (108)]. This term, in distinction to T_{1b}, has a coefficient proportional to $k'k$. Such a form comes from a nonrelativistic reduction of the two Z graph diagrams [Figs. 1(b) and 1(d)] which includes the limit $k_0=M-m$ and the eventual neglect of the baryon mass difference, $M-m$. Such a procedure produces a term which obeys the low energy theorem (at threshold, the cross section must exactly equal the Thompson cross section), but which is not valid near resonance. By inspection of the $1-1$ element of Eq. (7), T_{1b} comes from the cancellation of $W-M$ terms in numerator and denominator, and it is these terms which are treated approximately and unsymmetrically in the nonrelativistic limit. T_{1b} does not satisfy the low energy theorem, a fact related to the truncation of Hilbert space, and which can be compensated for by use of energy dependent vertex functions which vanish near threshold (see Ref. 4).

When developing a nonrelativistic $\gamma N\Delta$ interaction, ordinary practice has been to include a term like H_1 [Eq. (8)] and refer to it as Δ production, and either to ignore H_3 [Eq. (10)] or to include it as a necessary term of unspecified origin. Reference 6 develops a four component $\gamma N\Delta$ interaction current which includes only H_1 and H_2 [Eq. (9)], and then argues to the smallness of the quadrupole contribution H_2. A recent treatment of photon scattering on the nucleon in the resonance region can be found in Ref. 5. Therein it is shown that to adequately fit the data, terms such as both T_{1a} and T_{1b} must be included; the above shows the dynamic origin of the two and their relationship. Furthermore, Ref. 5 shows that reasonable data fits obtain only when both such terms take on additional energy dependence: T_{1a} from an energy dependent resonance width, and T_{1b} from a multiplicative energy dependent term. These corrections partially account for other resonance and t channel exchange processes, and are necessary to ensure unitarity. A forthcoming calculation by the present author using relativistic dynamics and a diagrammatic approach will explore to what degree these corrections can be accounted for in a less phenomenological way.

Because of the particular form of the $\gamma N\Delta$ vertex, the Z graphs contribute only to higher order in k/m relative to the direct term in pion nucleon scattering and in pion photoproduction. The situation described above for Compton scattering, however, should apply to photoproduction of vector mesons (ρ,ω, e.g.), a process particularly interesting in photon processes in nuclei.

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