Excited states of ^{4}He

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The energies of highly deformed "four-particle—four-hole" states of ^{4}He are calculated using Skyrme-like interactions. The $J=0$, 2, and 4 projected states are then considered. Deformed harmonic oscillator wave functions are used in which two oscillator lengths, b_x and b_0, are allowed to vary. Both the energies and the stability of the states are studied as a function of the amount of finite range energy in the interaction. Then a less restricted Hartree-Fock calculation is performed for the intrinsic state energies of both "four-particle—four-hole" states and "two-particle—two-hole" states. All the states considered come low enough in energy so as to be associated with the excited states of ^{4}He in the 20 to 35 MeV range.

INTRODUCTION

In a previous work[1] we considered the possibility of a "four-particle—four-hole" excitation in ^{4}He, in which four nucleons are excited from the $1s$ level with $N=1$ into the lowest deformed $N=2$ level. The $1s$ wave function was of the form $\exp(-r^2/2b^2)$ and the deformed wave function was

$$z\exp(-x^2/2b_x^2-y^2/2b_y^2-z^2/2b_z^2)\ .$$

In that work we constrained the product $b_0^3=bx$ by b_z to have the same value in the deformed state as in the ground state.

The interaction used was a combination of an attractive Gaussian interaction and a repulsive density dependent zero range interaction

$$V=-V_0e^{-r^2/a^2}+\frac{t_3}{6}\rho(R)\delta(\mathbf{r}_1-\mathbf{r}_2)\ .$$

In this work several modifications will be made. Most importantly, we will allow b_0 to be different in the excited state from what it is in the ground state. We will seek a minimum for the energy of the deformed state in which we vary both b_0 and b_z.

A second change is to use an interaction which is of the Skyrme type.[2] To do this, we simply take the expressions for the energy from the original model with the Gaussian attraction ($V_0e^{-r^2/a^2}$) and we keep terms up to order a^5. This truncation should not be thought of as an approximation, but rather as a way of getting the results for a different interaction, i.e., we replace $-V_0e^{-r^2/a^2}$ by

$$-t_0\delta(\mathbf{r})+\frac{t_1}{2}[k^2\delta(\mathbf{r})+\delta(\mathbf{r})k^2]\ ,$$

where k is equal to $-i(\nabla_1-\nabla_2)/2$.

The qualitative points that we make in this work do not depend very much on this change of interaction. The previously mentioned change, of allowing b_0 to vary, will turn out to be much more important.

THE PREVIOUS MODEL

The energy of the ground state $(1s)^4$ with the original interaction was

$$E(b)=\frac{3}{4}\left[\frac{3\hbar^2}{mb^2}\right]-6V_0\alpha^3+64T_3/b^6\ ,$$

where α is equal to

$$\frac{a}{\sqrt{2}b}(1+a^2/2b^2)^{-1/2}\ .$$

The intrinsic energy of the deformed state was

$$E(b_x,b_y,b_z)=\tfrac{3}{4}(\hbar^2/m)(1/b_x^2+1/b_y^2+3/b_z^2)$$
$$-\tfrac{1}{2}V_0\alpha_x\alpha_y\alpha_z(9-6\alpha_z^2+9\alpha_z^4)$$
$$+35.5555T_3/b^6\ .$$

In the limit that $a\rightarrow 0$ $V_0\alpha^3$ approaches

$$\frac{V_0a^3}{2\sqrt{2}b^3}(1-\tfrac{3}{4}a^2/b^2)\ .$$

In this limit $V_0e^{-r^2/a^2}$ approaches $a^3V_0\pi^{3/2}\delta(\mathbf{r})=t_0\delta(\mathbf{r})$. One can also show that the Skyrme parameter t_1 is given by $t_1=t_0a^2/2$. We should add that the parameters T_0 and T_3 were introduced for convenience with

$$T_0=\frac{3}{16\sqrt{2}}\frac{t_0}{\pi^{3/2}}\ ,\quad T_3=\frac{t_3}{48\sqrt{3}\pi^3}\ .$$

THE CURRENT MODEL

We now change the interaction by keeping terms to order a^2 (again, this is not an approximation, but is rather the correct expression for a different interaction).

In this new model the ground state energy is

$$E(b)=\frac{3}{4}\left[\frac{3\hbar^2}{mb^2}\right]-\frac{16T_0}{b^3}(1-\tfrac{3}{4}a^2/b^2)+64T_3/b^6\ .$$

(The factor of $\frac{3}{4}$ in the kinetic energy term comes from removal of the center of mass contribution.) We must refit the values of T_0 and T_3 so that the ground state energy becomes $E=-28$ MeV and at equilibrium we demand that b is equal to 1.4 fm. The equilibrium condition is

$$-\frac{b\partial E(b)}{\partial b}=0 .$$

Hence

$$\frac{3}{4}\left[\frac{6\hbar^2}{\text{mb}^2}\right]-\frac{48T_0}{b^3}(1-\tfrac{5}{4}a^2/b^2)+384T_3/b^6=0 .$$

We thus have two equations in the two unknowns T_0 and T_3 (or t_0 and t_3). We can easily solve these.

We then use the same interaction parameter T_0 and T_3 to obtain the energy of the deformed state. The expression for the intrinsic states becomes

$$E_{\text{int}}=\frac{3}{4}\frac{\hbar^2}{m}\left[\frac{1}{b_x^2}+\frac{1}{b_y^2}+\frac{3}{b_z^2}\right]$$
$$-12T_0/b_0^3\left[1-a^2\left[\frac{1}{b_x^2}+\frac{1}{b_y^2}+\frac{7}{12b_z^2}\right]\right]$$
$$+35.55555T_3/b_0^6 .$$

FURTHER CALCULATIONS

We let $x=1/b_0^3$. We set $b_x^2=b_y^2=b_0^3/b_z$. The energy of the intrinsic state has the structure

$$E(b_0,b_z)=A(b_z)x^2+B(b_z)x+C(b_z)$$

with

$$A(b_z)=6a^2b_zT_0+35.5555T_3 ,$$

$$B(b_z)=\frac{3}{2}\frac{\hbar^2}{m}b_z-12T_0+T_0a^2/b_z^2 .$$

(We set $\hbar^2/m=41$):

$$C(b_z)=\tfrac{9}{4}\hbar^2/(mb_z^2)$$

The condition $\partial E/\partial x=0$ leads to $x=-B(b_z)/2A(b_z)$. Hence

$$E=-B^2(b_z)/4A(b_z)+C(b_z) .$$

The energy now is written explicitly in terms of only one variable b_z.

In Table I we plot the values of t_0 and t_3 for several values of a, as obtained by the condition that the ground state of ^{4}He has an energy of $E=-28$ MeV for $b=1,4$ fm.

We also list the effective mass parameter m^*/m defined by $m^*/m=(1+E_{\text{FR}}/E_k)^{-1}$, where E_{FR} is the finite range energy (i.e., terms in t_0a^2) and E_k the kinetic energy. The explicit expression for this parameter is

$$m^*/m=\left[1+\frac{12T_0a^2/b_0^5}{2.25\,\hbar^2/\text{mb}^2}\right]^{-1} .$$

We next consider the results for the intrinsic state energy, which are presented in Fig. 1 and Table II. We use the expression for the energy which is an explicit function of b_z alone, with b_0 having already been obtained from the condition $(\partial E/\partial x)_{b_z}=0$. We plot E as a function of b_z. On this curve we label special points of interest. The point A corresponds to a relative minimum energy. We note that such a minimum exists for $a<0.8$ but not beyond. As we increase b_z we find a relative maximum at point B, beyond which the energy decreases again. Beyond B the energy appears to decrease indefinitely. However there is a point C where the oscillator length parameters b_0, expressed as a function of b_z, undergo a singularity. This is shown in Fig. 2. That is, b_0

TABLE I. The values of m^*, t_0, and t_3 as a function of the range a.

a (fm)	m^*	t_0 (MeV fm^3)	t_3 (MeV fm^6)
0	1.000	855.371	13 249.78
0.1	0.990	856.463	13 157.78
0.2	0.963	859.757	12 880.38
0.3	0.919	865.304	12 413.28
0.4	0.864	873.191	11 749.12
0.5	0.800	883.545	10 877.21
0.6	0.733	896.538	9 783.03
0.7	0.664	912.395	8 447.69
0.8	0.598	931.404	6 847.00

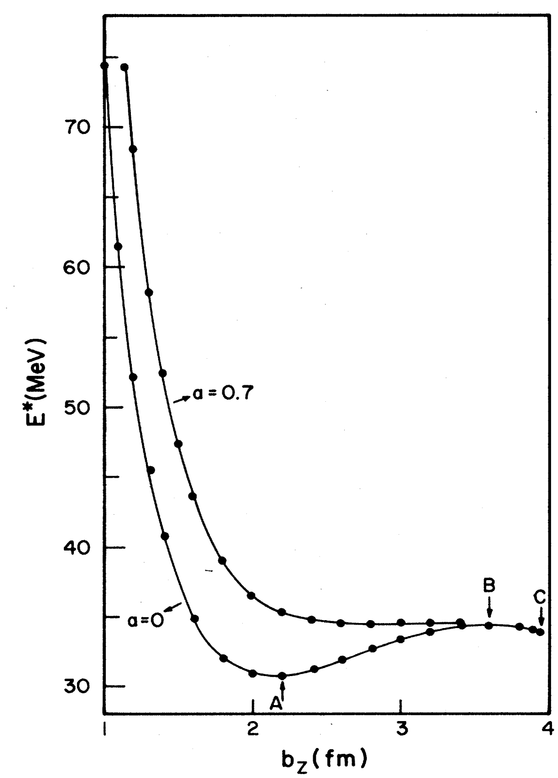

FIG. 1. The energy of the deformed state as a function of the deformation parameter b_z, for $a=0$ and 0.7 fm. For each b_z, the parameter b_0 is chosen to minimize the energy.

TABLE II. The values of b_z and b_0 for the minimum energy of the deformed state, as well as the relative maximum.

a (fm)	b_z (fm)	b_0 (fm)	$E_{\min}$ (MeV)	$E_{\max}$ (MeV)
0	2.13	1.48	30.74	34.39
0.1	2.13	1.48	30.87	34.39
0.2	2.15	1.50	31.24	34.40
0.3	2.19	1.53	31.81	34.40
0.4	2.25	1.57	32.52	34.41
0.5	2.35	1.64	33.27	34.42
0.6	2.49	1.74	33.97	34.44
0.7	2.77	1.94	34.48	34.52
0.8		No minimum		

becomes indefinitely large as we approach C from below, and beyond C it becomes negative, approaching minus infinite from above. Clearly a negative b_0 is unphysical so that the curve has no meaning beyond C. Note that as we approach the point C from the left the nucleus changes its slope from prolate to oblate, i.e., $b_0 \gg b_2$.

In Table II we present, for various values of a, the value of b_0 and b_z at which a minimum energy occurs, as well as the value of the minimum energy. We also list the value of the relative minimum energy which occurs for larger b_z.

For $a = 0$ there is a well-defined minimum at $b_z = 2.13$ fm and $b_0 = 1.48$ fm. The minimum energy is 30.74 MeV which is 3.65 MeV lower than the relative maximum energy at higher b_z. As we increase a the minimum energy rises and approaches $E_{\max}$, which is nearly constant as a function of the range parameter a, varying only between 34.39 and 34.52 MeV between $a = 0$ and $a \geq 0.8$ fm. For

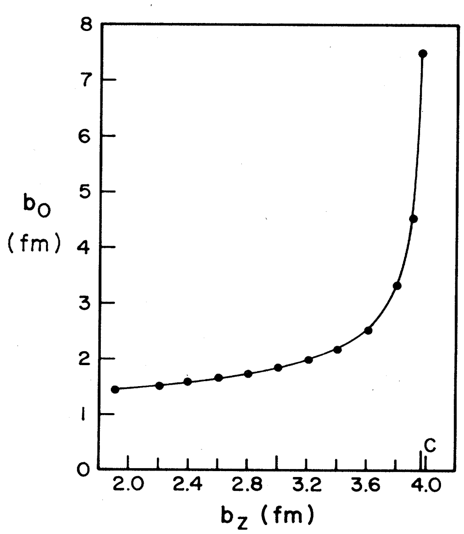

FIG. 2. The value of b_0 which minimizes the energy of the deformed state for a given b_z.

$a = 0.8$ fm and beyond there is no minimum anymore. In Fig. 2 for $a = 0.7$ fm the minimum is just barely there but the curve is fairly flat for large b_z.

Presumably, then the 4p-4h intrinsic state cannot be metastable for $a = 0.8$ fm. For lower values of a it is not clear if the minimum is deep enough to hold the four nucleons sufficiently long so that one has a clean cut resonance. However in addressing the problem of stability, we should be considering not the intrinsic state but rather the states of good angular momentum which are projected from the intrinsic state.

PROJECTION

We use the same method as we used previously to perform the projection of the $J = 0^+$, 2^+, and 4^+ states, the Elliot-Evans method.[3] We introduce a Lagrange multiplier and define

$$H' = H - \lambda J^2 \ .$$

The variational solution of H' is Φ. One adjusts the value of λ so that we obtain the right angular momentum on the average.

$$\langle \Phi J^2 \Phi \rangle = J(J+1) \ .$$

In our particular example the variational wave function is

$$\psi_\gamma = \cos\gamma \mid 001 \rangle + i \sin\gamma \mid 010 \rangle \ .$$

The expectation value of the Hamiltonian becomes

$$\langle H \rangle = \langle H \rangle_{\text{intrinsic}} + \tfrac{3}{2}\hbar^2/m \left[\frac{1}{b_y^2} - \frac{1}{b_z^2} \right] \sin^2\gamma + \left\{ 4T_0/b_0^3 \left[1 - a^2 \left[\frac{1}{4b_x^2} + \frac{1}{2b_y^2} + \frac{1}{b_z^2} \right] \right] - 21.3333 T_3/b_0^6 \right\} \sin^2 2\gamma \ .$$

The expectation value of J^2 is

$$\langle J^2 \rangle = J(J+1) = (3\sin^2 2\gamma + 2)(b_z^2/b_y^2 + b_y^2/b_z^2 + 2) \ .$$

If $p = \sin^2 2\gamma$, then $\sin^2\gamma = \frac{1}{2}(1 - \sqrt{1-p}\,)$.

We vary the energy of each state ($J = 0$, 2, and 4) *after* projection. The results are shown in Table III. For each J we give the minimum energy, the ratio $t = b_z/b_0$ and also b_0. This is done for several values of a.

The results are quite interesting. We find that we get a minimum for the $J = 0$ state for *all* values of a (or at least in the range we have looked at, from $a = 0$ to 2.1 fm); for $J = 2$ we get a minimum for some values of a ($a \leq 0.7$), but for the $J = 4$ state we do not get a minimum for any value of a. This suggests that only the low spin members of the rotational band might exist as metastable states.

When the nucleus does become unstable it appears to be due to the fact that b_0 becomes very large, not the ratio $t = b_z/b_0$. To gain insight into what might be happening we draw a picture of the intrinsic state in Fig. 3. We see that the system breaks up into two sets of two nucleons connected only at a point at the center. Such a system

TABLE III. The energies of projected states.

	a	0	0.1	0.3	0.5	0.7	1.0	1.3
$J=0$	E (MeV)	15.62	15.87	17.89	20.87	24.48	28.63	30.13
	t[a]	1.55	1.55	1.55	1.50	1.50	1.50	1.50
	b_0 (fm)	1.40	1.40	1.40	1.50	1.60	1.90	2.5
	Q_0[b] (b)	0.386	0.386	0.386	0.443	0.467	0.658	1.14
	rms[c] (fm)	2.164	2.164	2.164	2.318	2.412	2.864	3.769
$J=2$	E (MeV)	22.72	22.96	24.47	27.05	29.63	No minimum	
	t	1.70	1.70	1.65	1.65	1.60		
	b_0	1.40	1.40	1.50	1.60	1.70		
	Q_0	0.475	0.475	0.510	0.581	0.612		
	rms	2.329	2.329	2.436	2.598	2.694		
$J=4$				No minimum anywhere				

[a] $t=b_z/b_0$.
[b] Intrinsic mass quadrupole moment.
[c] Root mean square radius.

might easily break apart into two pieces. Under the projection the $J=0$ state looks more like Fig. 3(b)—a sphere with a hole in the middle. Such a system will be more stable to such a "fission."

HARTREE-FOCK CALCULATIONS WITH THE SKYRME INTERACTIONS

We have performed some Hartree-Fock calculations of the ground state, two-particle—two-hole and four-particle—four-hole states of ^{4}He. The interactions used were SI, SIII, and SII. We have, however, not as yet carried out the projections. The results, in which *seven* major shells were included, are shown in Table IV.

We note from these results that the four-particle—four-hole intrinsic states are very close in energy to the two-particle—two-hole intrinsic states. For example, for SIII the four-particle—four-hole energy is *26.95 MeV,* the energy of the 2p-2h state in which two neutrons have been excited from the s to the p shell is *26.52 MeV,* and the energy of the 2p-2h state in which a neutron and proton have been excited is *26.15 MeV.*

We next compare these less restricted calculations with the deformed oscillator calculations for 4p-4h states. We find that in the larger space calculation the energies are a few MeV *lower* than in the deformed oscillator calculation. For example, with SIII using six major shells the *absolute energy* of 4p-4h is 0.62 MeV, whereas the energy

TABLE IV. The energies, intrinsic mass quadrupole moments, and rms radii of states in ^{4}He obtained in Hartree-Fock calculations, using seven major shells.

		SI	SIII	SII
Ground state	E (MeV)	−28.75	−26.33	−26.07
	Q_0 (b)	0	0	0
	rms (fm)	1.822	1.901	1.873
4p-4h	E	27.67	26.95	29.43
	Q_0	1.026	1.054	1.032
	rms	4.017(4.019)[a]	4.119(4.122)	4.111(4.114)
2p-2h nn	E	27.23	26.52	27.25
pp	Q_0	0.585	0.555	0.274
	rms	3.217(3.335)	3.203(3.400)	2.742(2.751)
2p-2h np	E	24.38	26.15	27.26
np	Q_0	0.138	0.216	0.269
	rms	2.215(2.224)	2.508(2.517)	2.730(2.738)

[a] Neutron (proton).

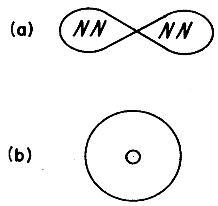

FIG. 3. (a) The shape of the deformed intrinsic state in which the four nucleons are placed in the $1p$ shell. (b) The projected $J=0$ state.

TABLE V. The energy intrinsic mass quadrupole moment, and rms radius of the 4p-4h state in ^{4}He, using the SIII interaction versus the number of major shells.

N_0[a]	1	6	9	12
E (MeV)	3.84	0.67	−0.04	−0.11
$Q_0(b)$	0.843	1.107	1.355	1.364
rms (fm)	3.849	4.183	4.578	4.594

[a]The number of shells included is N_0+1.

in the deformed oscillator case (see Table II) for $a=0.6$ (which corresponds to $m^*=0.733$, close to that of SIII) is 5.97 MeV. The lower energy in the better calculation makes the case for 4p-4h states in the 20—30 MeV region more convincing.

What is perhaps most surprising in this comparison is the fact that the rms radius is much larger in the unrestricted calculation. For the cases mentioned in the preceding paragraph the value of rms is 4.1 fm for SIII with seven major shells, but is only 2.4 fm in the deformed oscillator model.

One may worry though that as the calculation gets better and better (more major shells), the radius will become larger and larger, indicating a radial instability. We have therefore obtained the results for SIII for a number of different shells. We designate the number of shells ꞅ N_0+1 (so that $N_0=0$ corresponds to the $1s$ shell). In Table V we list the results for SIII for $N_0=1$, 6, 9, and 12.

For $N_0=1$ (two major shells), which is the least number of shells possible, the absolute energy is 3.84 MeV, which is closer to the value of our deformed oscillator model (5.97 MeV) than are the results for larger N_0. By the time we get to $N_0=12$ the energy has decreased to −0.11 MeV.

The rms radius does increase as we increase N_0. It changes from 3.849 fm for $N_0=1$ to 4.59 fm for $N_0=12$. Although we cannot be absolutely sure, the results do seem to be converging. For $N_0=9$ the value of rms is 4.578, very close to that for $N_0=12$.

CONCLUSIONS

What we have shown in this work is that by using interactions which have given reasonable results in heavier nuclei, we have no difficulty in obtaining the energy of the "four-particle—four-hole" intrinsic state in ^{4}He at a sufficiently low energy so that the projected $J=0$ state could be associated with low lying excited 0^+ states in ^{4}He.

We further addressed the problem of stability in a soluble model. We found that the projected $J=0$ state was always stable, the $J=4$ state was never stable, and the $J=2$ state was stable provided the range a was not too large, or alternatively, the effective mass parameter m^*/m was not too low.

The less restricted Hartree-Fock program yielded 2p-2h intrinsic states very close in energy, perhaps even slightly lower, to the 4p-4h state. Thus these states should also be important components of the low lying states of ^{4}He between 20 and 35 MeV excitation. The problem of stability cannot be addresssed with complete satisfaction with this Hartree-Fock program. That is why the soluble model was important.

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