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Particle-hole symmetry in the interacting-boson model: Fermion and boson aspects

A. B. Johnson
Nuclear Radiation Division, National Bureau of Standards, Washington, D.C. 20234

C. M. Vincent
University of Pittsburgh, Pittsburgh, Pennsylvania 15260
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We show that the S-D subspaces, which are used in the Otsuka-Arima-Iachello microscopic derivation of the interacting-boson model, form a particle-hole-symmetric family. Consequently, there exist particle-hole-symmetric prescriptions for determining the structure of the S and D pairs. This result holds independently of whether the Hamiltonian conserves generalized seniority. Nevertheless, there are deviations from particle-hole symmetry when boson matrix elements involving more than two d bosons are calculated in lowest order using the boson mapping procedure of Otsuka, Arima, and Iachello. These deviations are used to estimate the inaccuracies introduced by the lowest-order mapping.

I. INTRODUCTION

An exact shell-model calculation in a full n-particle multishell Fock space must give physically equivalent results whether it is done in particle representation or in hole representation. However, if the space is truncated, this need no longer be true. The validity of particle-hole (p-h) symmetry for truncated spaces is of real practical interest in the Otsuka, Arima, and Iachello (OAI) method[1] for microscopic determination of the parameters of the interacting boson model (IBM). Pittel, Duval, and Barrett (PDB) (Refs. 2 and 3) have shown that in certain circumstances this method leads to different results depending on whether the fermion problem is treated in terms of particles or holes. Talmi has subsequently analyzed the question of p-h symmetry in the OAI approach to IBM, and concluded[4] that the discrepancies disappear if the fermion two-body interaction is constrained to conserve generalized seniority.[5]

The present work reinvestigates the question more generally than is done in Refs. 2—4. As far as possible we avoid special assumptions about the fermion and boson Hamiltonians and the dependence of the structure of the S and D pairs on the number of particles. In Sec. II we consider the fermion aspects of the problem, and show that truncation to the S-D space is consistent with p-h symmetry. Consequently, we find that for the energy-minimization S-D structure prescription the only surviving p-h asymmetries must be entirely due to the approximate nature of the lowest-order OAI boson mapping. In Sec. III we use this result to probe the accuracy of the mapping. Section IV compares our results with those of PDB, and with Talmi's analysis. Section V is a brief concluding summary.

II. MULTIFERMION KINEMATICS AND DYNAMICS

We consider only like nucleons. This is sufficient for the purposes of the IBM, because one can first discuss distinct neutron and proton bosons separately, and then treat their dynamical coupling. In terms of fermion creation operator tensor $a_j^\dagger$, we define pair creation operators as follows:

$$S=\sum_j \alpha_j\sqrt{\Omega_j/2}[a_j^\dagger\times a_j^\dagger]_0^0\ , \tag{1}$$

$$A_m=\sum_{ij}\beta_{ij}[a_i^\dagger\times a_j^\dagger]_m^2\ , \tag{2}$$

where $\Omega_j=(2j+1)/2$ and $\beta_{ij}=(-1)^{i+j}\beta_{ji}$ by convention. We define a seniority projection operator P_v to project onto states of good total seniority. As in Ref. 6, we define v to be the sum of the seniorities of all the shells, i.e., Racah seniority.[7] We can list the needed properties of P_v, which probably hold also for such other definitions of v as the "generalized seniority"[5] of Talmi.

$$P_v^2=P_v=P_v^\dagger\ , \tag{3a}$$

$$[P_v,N]=0\ , \tag{3b}$$

$$[P_v,S]=0\ , \tag{3c}$$

$$P_v\,|\,n\,\rangle=0,\quad (n<v)\ , \tag{3d}$$

$$\bar{P}_v=P_v\ . \tag{3e}$$

In (3d) $|\,n\,\rangle$ is any n-particle state. In (3e) $\bar{P}_v$ is defined to be constructed from hole operators in the same way (isomorphically) as P_v is constructed from particle operators. A proof of this relation, based on our definition of the hole creation operators

$$\tilde{a}_{jm}^\dagger\equiv(-1)^{j-m}a_{j-m}\ , \tag{4}$$

can be found in Ref. 6.

As in Ref. 8, we define seniority-raising pair creation operators by

$$D_m=\sum_v P_{v+2}A_mP_v\ . \tag{5}$$

An S-D subspace of the n-fermion space (n even) is then spanned by states of the form

$$|S^pD^q\rangle_P \equiv S^pD^q|0\rangle, \quad (p+q=n/2), \tag{6}$$

where $|0\rangle$ is the vacuum state with no valence particles. Here and elsewhere we often omit magnetic quantum numbers, so that for example D^q stands for an unspecified homogeneous polynomial of degree q in the operators D_m ($m=2,1,0,-1,-2$). We define hole-pair creation operators corresponding to Eqs. (1), (2), and (5):

$$\bar{S} \equiv -\sum_j \bar{\alpha}_j\sqrt{\Omega_j/2}[\tilde{a}_j^\dagger \times \tilde{a}_j^\dagger]_0^0, \tag{7}$$

$$\bar{A}_m \equiv \sum_{ij} \bar{\beta}_{ij}[\tilde{a}_i^\dagger \times \tilde{a}_j^\dagger]_m^2, \tag{8}$$

$$\bar{D}_m \equiv \sum_v \bar{P}_{v+2}\bar{A}_m\bar{P}_v = \sum_v P_{v+2}\bar{A}_m P_v . \tag{9}$$

From these we can construct a basis for a hole version of the S-D space, called the $\bar{S}$-$\bar{D}$ space:

$$|\bar{S}^p\bar{D}^q\rangle_H \equiv \bar{S}^p\bar{D}^q|\bar{0}\rangle . \tag{10}$$

Here $|\bar{0}\rangle$ is the state in which all valence shells are filled. Our main task will be to show that there exists a choice of the $\bar{\alpha}_j$ and $\bar{\beta}_{ij}$ that makes the $\bar{S}$-$\bar{D}$ space identical to the S-D space.

We begin by considering some relevant commutators:

$$[S,\bar{S}] = \sum_j \alpha_j\bar{\alpha}_j\left(\Omega_j - \sum_m a_{jm}^\dagger a_{jm}\right), \tag{11}$$

$$[\bar{A}_m, S] = -2U_m , \tag{12}$$

$$[U_m, S] = -\sum_{ij} \alpha_i\bar{\beta}_{ij}\alpha_j[a_i^\dagger \times a_j^\dagger]_m^2 . \tag{13}$$

Here U_m is a one-body tensor operator of rank 2, given by

$$U_m = \sum_{ij} \alpha_i\bar{\beta}_{ij}[a_i^\dagger \times \tilde{a}_j^\dagger]_m^2 . \tag{14}$$

If we choose

$$\bar{\alpha}_j = 1/\alpha_j , \tag{15a}$$

$$\bar{\beta}_{ij} = \beta_{ij}/(\alpha_i\alpha_j) , \tag{15b}$$

the commutators (11)–(13) simplify as follows:

$$[S,\bar{S}] = \Omega - \hat{N}, \quad \hat{N} \equiv \sum_{jm} a_{jm}^\dagger a_{jm} , \tag{16a}$$

$$[\bar{A}_m, S] = -2U_m , \tag{16b}$$

$$[U_m, S] = -A_m . \tag{16c}$$

For completeness we include also the results

$$[S,\hat{N}] = -2S , \tag{16d}$$

$$[\bar{D}_m,\bar{S}] = [\bar{D}_m,\bar{D}_{m'}] = 0 . \tag{16e}$$

Talmi[4] has already shown the equivalence of particle and hole pictures for cases with at most one D pair. We can therefore begin by assuming the simplest of his results

$$|S^N\rangle_P \propto |\bar{S}^{\bar{N}}\rangle_H , \tag{17}$$

where

$$\bar{N} \equiv \Omega - N, \quad \Omega \equiv \sum_j \Omega_j . \tag{18}$$

Incidentally, we note that (provided none of the α_j vanishes) $N=\Omega$ gives

$$S^\Omega|0\rangle \propto |\bar{0}\rangle . \tag{19}$$

We can now extend Talmi's results to allow an arbitrary number of D pairs. From Eqs. (16) we can easily show by induction on p that

$$[\bar{A}_m, S^p] = -2pS^{p-1}U_m + p(p-1)S^{p-2}A_m . \tag{20}$$

Apply Eqs. (5) and (9) (the definitions of D_m and $\bar{D}_m$) to Eq. (20), using the fact that P_v and S commute [Eq. (3c)], to get

$$[\bar{D}_m, S^N] = \sum_v P_{v+2}[-2NS^{N-1}U_m + N(N-1)S^{N-2}A_m]P_v . \tag{21}$$

Now let this result act on any maximum-seniority state, say $|n=v,v\rangle$ Because $U_m|n=v,v\rangle$ has only v particles, the contribution of the term in U_m will be annihilated by P_{v+2}. Hence,

$$[\bar{D}_m, S^N]|n=v,v\rangle = N(N-1)S^{N-2}D_m|n=v,v\rangle . \tag{22}$$

Moreover, since $\bar{D}_m|n=v,v\rangle = 0$, because $\bar{D}_m$ raises the seniority while lowering the particle number,

$$\bar{D}_m S^N|n=v,v\rangle = N(N-1)S^{N-2}D_m|n=v,v\rangle . \tag{23}$$

We now prove by induction on q that

$$|\bar{S}^{\bar{N}-q}\bar{D}^q\rangle_H \propto |S^{N-q}D^q\rangle_P , \tag{24}$$

beginning with the $q=0$ result, Eq. (17). If Eq. (24) holds for a given value of q,

$$\begin{aligned} |\bar{S}^{\bar{N}-(q+1)}\bar{D}^{(q+1)}\rangle_H &\propto \bar{D}\,|\bar{S}^{\bar{N}-q-1}\bar{D}^q\rangle_H , \\ &\propto \bar{D}\,|S^{N-q+1}D^q\rangle_P , \\ &\propto (\bar{D}S^{N-q+1})D^q|0\rangle . \end{aligned} \tag{25}$$

[In the first step Eqs. (10) and (16e) were used.] Now $D^q|0\rangle$ is a maximum-seniority state, so Eq. (23) applies. We get

$$\begin{aligned} |\bar{S}^{\bar{N}-(q+1)}\bar{D}^{(q+1)}\rangle_H &\propto S^{N-q-1}D^{q+1}|0\rangle , \\ &\propto S^{N-(q+1)}D^{(q+1)}|0\rangle , \end{aligned} \tag{26}$$

so that Eq. (24) holds for $q+1$ if it holds for q. This completes the proof of Eq. (24), which is our basic new result.

Equation (23) shows that the particle and hole (p and h) pictures are kinematically equivalent; the S-D basis and the $\bar{S}$-$\bar{D}$ basis are simply different parametrizations of the same family of bases. For any given S-D basis, characterized by the values of α_j and β_{ij}, there is a corresponding $\bar{S}$-$\bar{D}$ basis, with values of $\bar{\alpha}_j$ and $\bar{\beta}_{ij}$ given by Eq. (15), and this S-D basis and $\bar{S}$-$\bar{D}$ basis span the same space. In fact, corresponding vectors in the two bases are *equal* up to a multiplicative scalar.

It is now easy to see that a large class of methods for determining the structure amplitudes α_i and β_{ij} will lead to physically equivalent results in p and h pictures, provided that the same Hamiltonian is used in both pictures. For example, suppose α_i is determined by minimizing the expectation value, i.e.,

$$\delta\langle E_0\rangle=0,\ \ \langle E_0\rangle\equiv\langle S^N|H\,|S^N\rangle_P/\langle S^N|S^N\rangle\ . \tag{27a}$$

The result is a certain state in S-D space. Since this same state can be written either as $|S^N\rangle_P$, and parametrized by α_i, or as $|\bar{S}^N\rangle_H$, and parametrized by $\bar{\alpha}_i$, $|S^N\rangle_P$ and $|\bar{S}^N\rangle_H$ will *both* minimize $\langle E_0\rangle$. Next suppose we find β_{ij} by the rule

$$\delta\langle E_2\rangle=0\ ,$$
$$\langle E_2\rangle\equiv\langle S^{N-1}D\,|H\,|S^{N-1}D\rangle_P/\langle S^{N-1}D\,|S^{N-1}D\rangle_P\ , \tag{27b}$$

where the parameters of D are varied while those of S are fixed. Again this leads to a certain state, which can be parametrized equivalently in either of two ways (p or h), both corresponding to the same minimum value of $\langle E_2\rangle$. Moreover, all S-D space states constructed from these S and D operators will be physically equivalent. These variational prescriptions (27) for obtaining α_i and β_{ij} are jointly designated "MIN" in Ref. 6.

Obviously the p-h symmetry depends on our using the same fermion Hamiltonian

$$H=\sum_{jm}\epsilon_j a^\dagger_{jm}a_{jm}+\sum_{rstu}W^\Gamma_{rstu}[[a^\dagger_r\times a^\dagger_s]^\Gamma\times[a_t\times a_u]^\Gamma]^0_0 \tag{28}$$

in p and h pictures. In the h picture one would in practice use a different normal ordering

$$H=\mathscr{E}+\sum_{jm}\tilde{\epsilon}_j\tilde{a}^\dagger_{jm}\tilde{a}_{jm}+\sum_{rstu}W^\Gamma_{rstu}[[\tilde{a}^\dagger_r\times\tilde{a}^\dagger_s]^\Gamma\times[\tilde{a}_t\times\tilde{a}_u]^\Gamma]^0_0 \tag{29a}$$

with single-hole energies

$$\tilde{\epsilon}_r=-\epsilon_r-\frac{4}{2r+1}\sum_{\Gamma s}(2\Gamma+1)^{1/2}W^\Gamma_{rsrs} \tag{29b}$$

and a modified zero of energy

$$\mathscr{E}=\sum_r\epsilon_r(2r+1)-2\sum_{rs\Gamma}W^\Gamma_{rsrs}\ . \tag{29c}$$

Of course Eqs. (28) and (29) merely exhibit different *forms* of the *same* operator.

The p-h symmetry of the MIN prescription immediately generalizes to include every method that determines the structure coefficients by a rule expressible in the form

$$\delta\mathscr{F}=0,\ \ \mathscr{F}=\mathscr{F}[\,|\Psi\rangle,H] \tag{30}$$

where $\mathscr{F}$ is a definite functional of an S-D space state $|\Psi\rangle$. For example, one might seek a stationary-energy state by varying

$$\mathscr{F}\equiv\langle\Psi|H\,|\Psi\rangle \tag{31}$$

subject to $\langle\Psi|\Psi\rangle=1$, with $|\Psi\rangle$ in the S-D family of subspaces but otherwise unconstrained. Another example is suggested by the trace-variational principle of Klein[9] and others, which they discuss in the context of a boson problem obtained by boson mapping. If we instead apply their method to the S-D space, we obtain a trace generalization of the rule (31), which obviously must share its p-h symmetry properties.

In contrast, the generalized open-shell Tamm-Dancoff approximation (TDA) method of Johnson and Vincent[8] is not of the type (30). Supposing that α_j has already been determined by the rule (27a), the TDA rule for determining the β_{ij} can be written

$$\delta\mathscr{E}_2=0\ ,$$
$$\mathscr{E}_2\equiv\frac{1}{2}\frac{\langle S^N|\,2QHQ^\dagger-QQ^\dagger H-HQQ^\dagger\,|S^N\rangle}{\langle S^N|\,QQ^\dagger\,|S^N\rangle}\ . \tag{32}$$

Here $Q^\dagger$ is a seniority-raising one-body quadrupole operator, containing the unknown coefficients α_i and β_{ij}. The terms $QQ^\dagger H$ and $HQQ^\dagger$ prevent the direct application of the p-h symmetry result (24), because for example $QQ^\dagger|S^N\rangle$ (which depends on β_{ij}) generally lies outside the S-D family of subspaces. Consequently the p-h symmetry properties of the Tamm-Dancoff approximation (TDA) method are still undecided. This illustrates some of the subtlety of the dynamical p-h symmetry question.

III. PARTICLE-HOLE SYMMETRIES INDUCED BY APPROXIMATE MAPPING

We will show that even if there is kinematical and dynamical equivalence between calculations done in the p and h pictures, physical discrepancies are still produced (except at midshell) when the fermion problem is mapped into a boson space using the approximate OAI method.

Recall the general nature of the OAI mapping.[1] The states $|S^pD^q\rangle$ of the S-D space are normalized to unity and put into one-to-one correspondence with abstract boson states

$$|s^pd^q\rangle=(s^\dagger)^p(d^\dagger)^q\,|0B\rangle/(\langle 0B\,|d^qs^p(s^\dagger)^p(d^\dagger)^q\,|0B\rangle)^{1/2}\ , \tag{33}$$

where

$$[s,s^\dagger]=1,\ \ [d_m,d^\dagger_{m'}]=\delta_{mm'}\ ,$$
$$[s,d^\dagger]=[s,d]=0\ , \tag{34}$$
$$s\,|0B\rangle=d\,|0B\rangle=0\ ,$$

in terms of boson annihilation operators s, d_m ($m=2,1,0,-1,-2$), and their Hermitian conjugates $s^\dagger,d^\dagger_m$. As for the D operators, $(d^\dagger)^q$ denotes an unspecified homogeneous polynomial of degree q in the operators $d^\dagger_m$.

Consider some restrictions related to the Pauli principle. The maximum seniority attainable with $n=2N$ particles is $v_{\max}=2\min(N,\bar{N})$. It follows that only states with at most $v_{\max}/2$ D pairs can be admitted into the S-D space, regardless of the choice (p or h) of picture. We assume that all such states of the form (6) are linearly independent. Then it follows that a boson state $(s^\dagger)^p(d^\dagger)^q|0B\rangle$ possesses a fermion counterpart if and only if $q\le\min(N,\bar{N})$. We say that such states belong to the *physical subspace* of boson space; in contrast we designate as *spurious* all boson states that lack fermion counterparts.

For exact OAI mapping to all orders, the spurious boson states cause no difficulty, because they can be completely decoupled from the physical boson subspace simply by setting equal to zero all boson matrix elements that connect the physical and spurious subspaces.

In practice one must use an approximate mapping. A standard choice is the "lowest order" OAI mapping. For reference we give the results of this mapping for the Hamiltonian and the quadrupole operator. The boson image of the fermion Hamiltonian is

$$H_B^{(1)}=\epsilon_0+\epsilon_d\sum_{m=-2}^{2}d_m^\dagger d_m+\tfrac{1}{2}\sum_{L=0}^{4}C_L(2L+1)^{1/2}[[d^\dagger\times d^\dagger]^L\times[\tilde{d}\times\tilde{d}]^L]_0^0+F\{[[d^\dagger\times d^\dagger]^2\times\tilde{d}s]_0^0+\text{H.c.}\}$$
$$+G\{[d^\dagger\times d^\dagger]^0 ss+\text{H.c.}\}\ . \qquad (35)$$

The lowest order OAI mapping expresses the parameters in terms of fermion matrix elements, as follows:

$$\begin{aligned}\epsilon_0&=\langle S^N|H|S^N\rangle\ ,\\ \epsilon_d&=\langle S^{N-1}D|H|S^{N-1}D\rangle-\epsilon_0\ ,\\ C_L&=\langle S^{N-2}D^2L|H|S^{N-2}D^2L\rangle-2\epsilon_d-\epsilon_0\ ,\\ F&=[\tfrac{5}{2}(N-1)]^{-1/2}\langle S^{N-1}D|H|S^{N-2}D^2\rangle\ ,\\ G&=[2N(N-1)]^{-1/2}\langle S^{N-2}D^2|H|S^N\rangle\ .\end{aligned}\qquad (36)$$

We see that the parameters of $H_B^{(1)}$ depend on N, the number of fermion pairs, both explicitly and through α_i and β_{ij}. Similarly the boson image of the one-body quadrupole operator Q_m can be written approximately as

$$Q_m^B=Q_{sd}(s^\dagger\tilde{d}_m+sd_m^\dagger)+5^{-1/2}Q_{dd}[d^\dagger\times\tilde{d}]_m^2\ . \qquad (37)$$

The parameters are given by the lowest-order OAI mapping as

$$\begin{aligned}Q_{sd}&=\langle S^N||Q||S^{N-1}D\rangle/(5N)^{1/2}\ ,\\ Q_{dd}&=\langle S^{N-1}D||Q||S^{N-1}D\rangle\ .\end{aligned}\qquad (38)$$

Equations (36) and (38) assume that the particle picture is used, so that the S-D states $|S^pD^q\rangle$ are mapped onto boson states $s^pd^q|0B\rangle$ with $p=N-q$; we call this "p mapping." Alternatively we can use the hole picture and map the $\bar{S}$-$\bar{D}$ states $|\bar{S}^p\bar{D}^q\rangle$ onto boson states $s^pd^q|0B\rangle$ with $p=\bar{N}-q$; we call this option "h mapping." For h mapping one must replace S by $\bar{S}$, D by $\bar{D}$, and N by $\bar{N}$ in Eqs. (36) and (38), to define new parameters $\bar{\epsilon}_0$, $\bar{\epsilon}_d$, $\bar{C}_L$, $\bar{F}$, $\bar{G}$, $\bar{Q}_{sd}$, and $\bar{Q}_{dd}$. Both in p mapping and in h mapping the physical subspace of boson space is limited by $q\le\Omega/2$.

If the lowest-order boson Hamiltonian $H_B^{(1)}$ is taken literally and diagonalized in the full boson space, contamination of the eigenstates by spurious parts cannot be avoided (assuming of course that F and G do not both vanish). This is because its form [Eq. (35)] implies nonzero q-changing matrix elements of $H_B^{(1)}$ between multiboson states, and these matrix elements will always couple (at least) the highest-q states of the physical subspace to the spurious subspace. Fortunately it is easy to restrict $H_B^{(1)}$ to the physical boson subspace, simply by imposing the limitation $q\le\Omega/2$ on the basis states (for both p mapping and h mapping). The spurious contaminations are then of no concern. The eigenvalues of $H_B^{(1)}$ can be regarded as approximations to the results of diagonalizing H in the S-D space. (Indeed the OAI method can be thought of as a technique for extrapolating low-seniority matrix elements to higher seniorities.) The errors of the approximation must then be entirely due to the effects of the mapping procedure, since contaminations by spurious states are precluded.

Assume that one fermion Hamiltonian has been consistently used, and that the structure amplitudes have been determined by a method with dynamical p-h symmetry, such as MIN. Then some estimate of these mapping errors can be obtained by comparing the approximate p mapping and h mapping results for the (unknown) multifermion S-D space matrix elements. These approximations are just the multiboson matrix elements of the p mapped and h mapped versions of $H_B^{(1)}$. From Eqs. (36) and (38) it is clear that p mapping and h mapping will give different results for the q-changing IBM parameters F, G, and Q_{sd}, because the fermion matrix elements in these equations are p-h symmetric, while explicit N dependences occur in their relation to F, G, and Q_{sd}. In fact

$$\begin{aligned}F/\bar{F}&=\sqrt{(\bar{N}-1)/(N-1)}\ ,\\ G/\bar{G}&=\sqrt{\bar{N}(\bar{N}-1)/N(N-1)}\ ,\\ Q_{sd}/\bar{Q}_{sd}&=\sqrt{\bar{N}/N}\ .\end{aligned}\qquad (39)$$

However, the differences between the p and h versions of F, G, and Q_{sd} parameters do not have direct physical significance. Instead, one should compare the values of the *multiboson* matrix elements that result from p mapping and h mapping. We find the following ratios:

$$R_F\equiv\langle s^pd^q|H_B^{(p)}|s^{p-1}d^{q+1}\rangle/\langle s^{\bar{p}}d^q|H_B^{(h)}|s^{\bar{p}-1}d^{q+1}\rangle=\frac{F}{\bar{F}}\left[\frac{p}{\bar{p}}\right]^{1/2}=\left[\frac{(\bar{N}-1)(N-q)}{(\bar{N}-q)(N-1)}\right]^{1/2}, \qquad (40a)$$

$$R_G\equiv\langle s^pd^q|H_B^{(p)}|s^{p-2}d^{q+2}\rangle/\langle s^{\bar{p}}d^q|H_B^{(h)}|s^{\bar{p}-2}d^{q+2}\rangle=\left[\frac{\bar{N}}{\bar{N}-q}\,\frac{\bar{N}-1}{\bar{N}-1-q}\,\frac{N-q}{N}\,\frac{N-1-q}{N-1}\right]^{1/2}, \qquad (40b)$$

$$R_Q \equiv \langle s^p d^q L \mid Q^{(p)} \mid s^{p-1} d^{q+1} L' \rangle / \langle s^{\bar{p}} d^q L \mid Q^{(h)} \mid s^{\bar{p}-1} d^{q+1} L' \rangle = \left[\frac{N-q}{N} \frac{\bar{N}}{\bar{N}-q} \right]^{1/2} . \tag{40c}$$

Here

$$\bar{p} \equiv \bar{N} - q = \Omega - N - q = \Omega - 2N + p \tag{41}$$

and $|s^p d^q\rangle$ is defined in Eq. (33), while L and L' represent the multiboson angular momentum, where necessary. In all three cases, the parts of the matrix elements involving only d bosons cancel out.

The ratios (40) are all close to 1 when q is small. And at "midshell" ($N = \Omega/2 = \bar{N}$), dynamical p-h symmetry trivially guarantees that both mappings will lead to equal matrix elements, so the ratios are also 1 when $N = \Omega/2$. Table I shows the ratios for valence shells corresponding to the tin isotopes, whose $\Omega = 16$. Only results for $N = 7$ (i.e., one pair away from midshell) are shown. If q is comparable with p, the ratios depart considerably from 1. This shows that the p mappings and h mappings give appreciably different approximations to the N dependence of matrix elements involving high-seniority states. The importance of this is that the advantage of the OAI boson mapping scheme over direct shell-model calculations, truncated at low seniority, depends on its ability to simulate the high-seniority matrix elements.

If the errors of the p- and h-mapping results are equal, the exact result must lie either midway between them or very far from both. It seems reasonable to assume that p and h errors are at least roughly equal if, say, $\bar{N} - N = 2 \ll \Omega$. Then either the error of, say, the p-mapping result, must be about half the distance between the p- and h-mapping results [e.g., for Q_{sd} this is $(1-0.982)/2 = 0.9\%$], or else both mappings are in error by much more than this. More simply, one may say that the error of either mapping is at least about 0.9%. An error of this magnitude would not be large enough to have a serious effect on comparison with experiment—but of course the error may be *larger.*

IV. COMPARISON WITH OTHER TREATMENTS OF p-h SYMMETRY

PDB (Ref. 2) point out that standard OAI prescriptions do not always lead to p-h symmetry for observables. For example, they determine α_i and β_i by particle and hole versions of the favored-pair method. First, they diagonalize the two-particle matrix of the surface-delta interaction (SDI) plus midshell phenomenological single-particle (MSSP) energies ϵ_r ("MSSP prescription"). Next, they obtain $\bar{\alpha}_i$ and $\bar{\beta}_i$ by diagonalizing the two-hole matrix of the same surface-delta interaction with midshell single-hole (MSSH) energies $\bar{\epsilon}_f = \text{const.} - \epsilon_r$ ("MSSH prescription"). Of course the resulting structure coefficients are independent of N. The two prescriptions use fermion Hamiltonians that differ at most by an additive constant, because the SDI has the special property[10] that the Hartree-Fock term of Eq. (29b) does not affect the spacing of the single-hole energies. In spite of this, PDB find that the occupancies of individual shells is the multiparticle states of the forms (6) and (10) with $q \leq 1$ differ considerably in the two prescriptions. In our language, this shows that these two prescriptions lack dynamical particle-hole symmetry. This is not surprising, because requiring the structure coefficients to be independent of N prevents application of minimization principles such as Eq. (27). PDB emphasize the possibility of restoring p-h symmetry by including renormalization effects from outside the S-D space. This idea is made plausible by the fact that p-h symmetry always results if one exactly treats the same fermion Hamiltonian in particle and hole pictures. While we agree that this possibility exists, we would emphasize instead that p-h symmetry is restored more generally and quite simply if one allows α and β to depend on N, through appropriate energy minimization. PDB also find p-h asymmetries in the boson parameters Q_{sd} and Q_{dd}. They use $\min(N, \bar{N})$ in place of N in our Eq. (40), thus excluding the possibility of mapping-induced p-h asymmetries.

Talmi, in discussing the results of Ref. 2, has already proved Eq. (24), though only for the cases $q \leq 1$ that are needed to discuss the PDB calculation. He uses generalized seniority projection rather than Racah-seniority projection to define the S-D space; however, this distinction disappears if there is no more than one D pair. His main conclusion (regarding p-h symmetry) is that even if the structure coefficients are constrained to be independent of N, p-h symmetry will still result, provided that the fermion Hamiltonian conserves generalized seniority.[5] This is certainly true, because for a Hamiltonian that conserves generalized seniority the states (6) and (10) are exact eigenvectors, and the same eigenvectors must result in both p and h pictures. Thus Talmi agrees with PDB in suggesting that conformity to p-h symmetry is most appropriately obtained by ensuring that the S-D states are eigenstates of the Hamiltonian; however, Talmi's preference is to bring this about by using a generalized-seniority-conserving fermion Hamiltonian. In contrast,

TABLE I. Particle-hole ratios of N-boson matrix elements of lowest-order boson images [Eq. (40)]. All results correspond to $\Omega = 16$, $N = 7$.

q	0	1	2	3	4	5	6	7
R_F		1.000	0.976	0.943	0.894	0.816	0.667	0
R_G	1.000	0.958	0.904	0.828	0.717	0.535	0	
R_Q	1.000	0.982	0.958	0.926	0.878	0.802	0.655	0

we would prefer to regard the OAI method as having some general domain of applicability, and not as limited to one type of Hamiltonian. Therefore, again we emphasize the possibility of obtaining p-h symmetry by freeing α and β to depend on N through a suitable variational principle.

V. CONCLUSIONS

We have extended Talmi's result on the kinematic p-h symmetry of the family of S-D states to an arbitrary number of D pairs. In agreement with Talmi we find that the discrepancy between the nondegenerate multishell results calculated in the particle and hole pictures by PDB can be explained by their use of N-independent structure coefficients with a Hamiltonian for which the S-D states are not eigenstates. We show that the kinematic p-h symmetry of S-D spaces allows one to restore dynamical p-h symmetry without requiring the S-D states to be eigenstates of the Hamiltonian. This is achieved by making use of an appropriate variational principle to determine the structure of the S and D pairs, which must of course be allowed to depend on N. Since our proof of kinematical p-h symmetry applies for any number of D pairs, p-h symmetry is now available for Fermion states of arbitrary seniority. In numerical work described elsewhere,[6] we have actually exploited the computational convenience of p-h symmetry for two D pairs, after verifying its validity in specific cases.

Some p-h asymmetries in calculated IBM parameters can arise even if the underlying Fermion calculation has exact p-h symmetry. Such "mapping asymmetries" can be revealed only by considering in particle (hole) formalism cases where the valence shell is more than half-filled with particles (holes). (PDB avoid mapping asymmetries by a special prescription.) We exploit the mapping asymmetries to obtain rough lower limits, of the order of 1%, on the errors of the lowest-order OAI mapping for the Hamiltonian and the quadrupole operator.

Our estimates of the mapping errors are only lower limits, and unfortunately relate only to the N dependence of matrix elements that involve high seniorities. It would be useful to have exact calculations of the high-seniority fermion matrix elements for comparison with the predictions of the lowest-order-OAI boson mapping. An excellent study of this question in the single-shell case has already been reported in Ref. 1.

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