PHYSICAL REVIEW C VOLUME 31, NUMBER 4 APRIL 1985

Physical boson basis states in the boson expansion theories

Y. K. Gambhir and J. A. Sheikh
Department of Physics, Indian Institute of Technology, Bombay-400 076, India

P. Ring
Theoretical Physics Division, Technical University, Münich, D-8046 Garching, Federal Republic of Germany

P. Schuck
Institut des Sciences Nucléaires, Université de Grenoble, F-38026 Grenoble Cedex, France
(Received 16 August 1984)

The use of physical boson basis states is stressed for the calculations in the boson space. The explicit form of physical boson basis states in terms of bosons is derived for the nonunitary boson mapping of Dyson. The ambiguity in the normalization introduced due to the use of bi-orthonormal basis states is satisfactorily resolved, resulting in a Hermitian matrix. This Hermitian matrix is found to coincide with the Hamiltonian matrix in the fermion space. The model cases where the use of a boson basis is justified are shown to be consistent with our view of using physical boson basis states.

I. INTRODUCTION

Recent years have witnessed a number of approaches, making use of bosons both at the phenomenological and microscopic levels, emerging for the description of the collective nuclear properties. The foremost and the most successful in the phenomenological category is the interacting boson model (IBM) and its extended versions introduced by Arima, Iachello, and co-workers.[1–5] On the microscopic front the formalisms are essentially approximate shell model theories making use of boson algebra, and are referred to by various names in the literature.[6–8] We shall refer to these as boson expansion theories (BET's). These microscopic theories[5,7,9–14] start from the shell model and generally proceed in two steps. In the first step the collective bi-fermion excitations ($Q^\dagger$) are generated using the shell model techniques [e.g., quasiparticle Tamm-Dancoff approximation (qpTDA), broken pair approximation (BPA) (Refs. 15–17), etc.]. The full problem, in principle, can be solved in the fermion space in the shell model sense. However, due to the prohibitively large dimensions of the configuration space, in practice one truncates the basis states so as to include only a small number of $Q^\dagger$ excitations. This truncation relies on the hope that the coupling between the retained collective and the neglected noncollective (less collective) excitations is small, which indeed may be the case in many practical cases. To tackle even this truncated problem is still nontrivial, apart from some simple cases. Therefore, approximate formalisms such as BET's are developed for this purpose. However, fermion calculations for such simple cases are nevertheless very useful especially for testing various approximate formalisms. The second step, which is really the crucial step, introduces a transformation to a boson space.[6,7,9] As a result, one expresses bi-fermion operators in terms of boson operators fully preserving the respective commutation algebra, and hence the name boson expansion theories. Both unitary[18–20] and nonunitary[21,22] transformations have been used. The former is termed as the Hermitian BET (H-BET) while the latter as the non-Hermitian BET (NH-BET). The H-BET is characterized in general by infinite boson expansions. However, for a finite dimensional space, which is always the case in practice, the reordered H-BET's do become finite expansions. On the other hand, NH-BET, which is mainly based on the work of Dyson,[23] is nonunitary but leads to a finite number of terms in the boson space, and besides, it is exact. The NH-BET will also be referred to here simply as Dyson mapping.

The fermion Hamiltonian (H_F) and other operators ($\hat{O}_F$) are now expressed in terms of boson operators using the transformation or mapping introduced in the second step. These transformed operators will formally be designated by suffix B in place of F (e.g., H_B or $\hat{O}_B$). The remaining task is then to diagonalize H_B in the physical boson basis (PBB) (Ref. 24) which is obtained by rewriting the orthonormal set of the original fermion basis in terms of the bosons using the boson mapping. For H-BET with Belyaev-Zelevinskii[19] or Marumori [Marumori, Yamamura, and Tokunaga (MYT)] (Ref. 20) mapping, this procedure is essentially equivalent to solving the full fermion problem unless one introduces further approximations. As a result, nothing is gained by the transformations to the boson space. A modified Marumori mapping advocated by Gambhir, Ring, and Schuck[9] (GRS) essentially assumes that the n-body fermion operator (in practice $n \leq 2$) transforms to $0-n$-body boson operators, i.e., the m-body boson terms with $m > n$ are ignored. To incorporate the effects of these neglected terms, the parameters of the transformed operators (H_B or $\hat{O}_B$) are now determined by equating the matrix elements of the original fermion operator (H_F or $\hat{O}_F$) with respect to the full N-fermion basis (e.g., BPA eigenstates) to the corresponding matrix elements of the boson operator between the respective boson states. This prescription has been shown to be reasonable in the degenerate case by Otsuka, Arima, and Iachello (OAI).[5] We refer to this method as the GRS prescription or OAI mapping. The resulting one- and two-body H_B is now to be diagonalized with PBB as basis states. A few calculations available in this field have in fact been carried out[10,25] using ordinary normalized boson basis (BB) states rather than the PBB states. This is rather serious. The BB states do contain nonphysical (spurious) states which appear because of the neglect of the Pauli principle and therefore must be removed before diagonalization.

In the Dyson mapping the use of BB states has also been advocated[26] and has been justified for some model cases. We show here that these model cases in fact belong to the situations where the PBB state is proportional to the BB state. Here, we strongly emphasize the use of PBB states in the absence of a reliable justification for the use of BB states.

In the present paper we discuss the aforementioned points. For the case of NH-BET or Dyson mapping we express[27] the fermion states (or PBB) in terms of BB states. The explicit expressions of the expansion coefficients are derived and the first few of them are listed. It turns out that for a single level case these coefficients correspond to the products of two particle fractional parentage coefficients (fpc's). This observation is general and is independent of the nature of the Hamiltonian. Due to the nonunitary nature of the Dyson mapping the Hermitian conjugate of ket is not equal to bra [i.e., $|i\rangle^{\dagger} \neq |i)$] and therefore bi-orthonormal basis states are to be used for solving the eigenvalue problem of the transformed Hamiltonian $(H_B \equiv H_D)$ with the normalization condition ${}_L(i\,|\,i)_R=1$. The Hamiltonian matrix is clearly non-Hermitian even if H_D is Hermitian. Furthermore, the normalization condition ${}_L(i\,|\,i)_R=1$ is well preserved even if $|\,i)_R$ is multiplied by γ_i and ${}_L(i\,|$ by γ_i^{-1}. This γ ambiguity disappears if we use the Hermitization condition of Gambhir and Basavaraju (GB).[28] This prescription, also advocated by us earlier, yields a Hermitian matrix

$$h_{ij}=[{}_L(i\,|H_D\,|j)_R\;{}_L(j\,|\,(H_D)i)_R]^{1/2}$$

and is exact for the two-dimensional case. It is to be noted that the most important part of the interaction in H_D is the n-p part of the interaction as most of the identical part (p-p or n-n interaction) is supposed to be taken into account while generating $Q^{\dagger}$ for protons and/or for neutrons. The n-p part yields H_D which contains a product of a one-body boson term for protons and a one-body boson term for neutrons. As a result, only a few terms appearing in the expansion of physical boson states contribute. This along with the GB Hermitization procedure in fact yields h_{ij} identical to that of the original fermion problem. Therefore, this procedure with the PBB states can be regarded as an approximation to the seniority shell model (SSM). For the model cases[26,29] like the Lipkin model (LM) (Ref. 29) and pairing vibrations,[26] where the use of BB states in place of PBB states has been justified, it turns out that in these model cases the PBB and BB states are proportional. This is because only one type of boson exists in the LM and two commuting bosons exist in the pairing vibrations case.

The present approach is quite general and can be used to carry out approximate shell model calculations. In addition, the formalism may be quite useful in investigating, in particular, the following problems: (1) to study the convergence problem in the various truncation schemes like horizontal and vertical convergence and thereby to evolve an appropriate truncation scheme, and (2) to estimate the extent of spuriousness while working with the BB states.

In Sec. II, the formalism is presented. Two model cases, the Lipkin model and the pairing vibrations, are discussed in Sec. III. The concluding remarks are presented in Sec. IV.

II. FORMALISM

Consider a system of an even number of identical valence nucleons. A collective fermion pair excitation operator $Q^{\dagger}$ can be written in terms of a fermion (identical) pair creation operator $A^{\dagger}$ through

$$Q_{\mu}^{\dagger}=\sum_{a\leq b}\tilde{x}^{\mu}_{J_{\mu}}(ab)\tilde{A}^{\dagger}_{J_{\mu}M_{\mu}}(ab)\;, \tag{1}$$

with

$$\begin{aligned}\tilde{A}^{\dagger}_{JM}(ab)&=\frac{1}{\sqrt{(1+\delta_{ab})}}A^{\dagger}_{JM}(ab)\\&=(c_a^{\dagger}c_b^{\dagger})_{JM}/\sqrt{(1+\delta_{ab})}\;.\end{aligned} \tag{2}$$

Here $c^{\dagger}$ (c) are fermion creation (annihilation) operators, and the symbol "a" represents quantum numbers "nlj" of the single particle (sp) shell-model states. The expansion coefficients appearing in Eq. (1) can be suitably determined from the shell-model theories (e.g., BPA). In the lowest approximation BPA assumes that the ground state (g.s.) $|\,\Phi_0\rangle$ for p pairs of identical valence nucleons is obtained by repeated (p times) application of a distributed pair operator S_+ on particle vacuum (core),

$$|\,\Phi_0\rangle\equiv N_0S_+^p\,|\,0\rangle\equiv N_0\left[\sum_a\phi_a\frac{\hat{j}_a}{2}A^{\dagger}_{00}(aa)\right]^p|\,0\rangle, \tag{3}$$

where $\hat{j}_a=\sqrt{(2j_a+1)}$, and the parameters ϕ_a are determined by minimizing the energy of the ground state. Next, the basis states are constructed by replacing one S_+ operator in Eq. (3) by a two-particle creation operator $A^{\dagger}_{JM}(ab)$ and are written as

$$|\,\Phi_{JM}(ab)\rangle=N_J(ab)S_+^{p-1}A^{\dagger}_{JM}(ab)\,|\,0\rangle\;; \tag{4}$$

the Hamiltonian is then diagonalized in the space spanned by Eq. (4). The eigenstates $|\,\psi_{\mu}\rangle$ so obtained can be expressed as

$$|\,\psi_{\mu}\rangle=\sum_{a\leq b}\alpha^{\mu}_{J_{\mu}}(ab)\,|\,\Phi_{JM}(ab)\rangle\;. \tag{5}$$

Using the definition of $Q^{\dagger}$ [Eq. (1)] $|\,\psi_{\mu}\rangle$ can be rewritten as

$$|\,\psi_{\mu}\rangle=N^{\mu}_{J_{\mu}}Q_{\mu}^{\dagger}(S^{\dagger})^{p-1}\,|\,0\rangle\;. \tag{6}$$

The coefficients $\tilde{x}(ab)$ appearing in Eq. (1) are related to α's through

$$\tilde{x}^{\mu}_{J_\mu}(ab)=\alpha^{\mu}_{J_\mu}(ab)N^{\mu}_{J_\mu}(ab)\sqrt{(1+\delta_{ab})}/N^{\mu}_{J_\mu} = x^{\mu}_{J_\mu}(ab)/\sqrt{(1+\delta_{ab})}\ . \tag{7}$$

The coefficients x obey

$$x^{\mu}_{J_\mu}(ab)=(-)(-1)^{j_a+j_b-J_\mu}x^{\mu}_{J_\mu}(ba)\ .$$

For the lowest $J=0$ state the coefficients $\tilde{x}_0(aa)$ of Eq. (7) to a very good approximation can be taken to be

$$\tilde{x}_0(aa)=\phi_a\frac{\hat{j}_a}{\sqrt{2}}\bigg/\left[\sum_a(\phi_a^2\hat{j}_a^2/2)\right]^{1/2}\ . \tag{8}$$

The operators $Q^{\dagger}_{\mu}$ do not form an orthonormal set for the same J with different excitation energies and therefore an orthonormal set of $Q^{\dagger}_{\mu}$ is constructed from Eq. (1). Let these still be denoted by $Q^{\dagger}_{\mu}$. We shall reserve the symbol $S^{\dagger}$ ($Q^{\dagger}_{J_\mu=0_{\mathrm{g.s.}}}$) and $D^{\dagger}$ ($Q^{\dagger}_{J_\mu=2_1^+}$) for the lowest $J=0$ and $J=2$, respectively. The fermion states obtained by replacing one or more S_+ operators by an equal number of $Q^{\dagger}$ operators in Eq. (13) are in general nonorthogonal. Therefore, an orthonormal set of fermion states $|\,i\,\rangle$ is to be constructed. Symbolically, $|\,i\,\rangle$ is given by

$$|\,i\,\rangle_F=\mathcal{N}\hat{P}[(Q^{\dagger}_{\mu})^q S_+^{p-q}]\,|\,0\rangle\ . \tag{9}$$

Here $\mathcal{N}$ is the normalization and the projector $\hat{P}$ generates an orthonormal set of states. We shall now discuss the boson mappings, the next crucial step in this field.

A. Hermitian boson expansion theory

These H-BET's based on the spirit of the work of Holstein and Primakoff[18] preserve the Hermitian nature of the original fermion problem. There are mainly two approaches belonging to this class available in the literature,[6] one due to Belyaev-Zelevinskii (BZ) (Ref. 19) and the other due to Marumori *et al.*[20]

1. Belyaev-Zelevinskii approach

The BZ procedure starts with pure boson operators and then expands the boson image of bi-fermion operators in terms of these pure bosons. The expansion coefficients are determined by requiring that the original algebra of the bi-fermion operators is preserved. In practice, one follows the procedure up to a certain order, i.e., truncates the expansion. The Hamiltonian and the basis vectors written in terms of bi-fermion operators are then expressed in terms of bosons using the expansion coefficients determined earlier. One then solves the eigenvalue problem. The basis here does correspond to the physical boson basis. This procedure becomes quite cumbersome and involved as one goes beyond second and third order. We shall not discuss this mapping here any further.

2. Marumori mapping

This approach starts with a one-to-one mapping of fermion states to the physical boson states, which forms the physical subspace of the boson space. This correspondence of the basis vectors is related through a unitary transformation. Symbolically, it is represented as

$$|\,i\,\rangle_F\to|\,i)_B\ . \tag{10a}$$

The mapping of the fermion operator $\hat{O}_F$ is expressed by equating the expectation values, i.e.,

$$\langle\, i\,|\,\hat{O}_F\,|\,j\,\rangle_F=(i\,|\,\hat{O}_B\,|\,j)_B\ . \tag{10b}$$

For practical purposes, one writes the boson image of a fermion operator as a sum of one-body, two-body, three-body, etc. boson terms. The expansion coefficients are then determined by equating the matrix elements of the fermion operator with respect to one-, two-fermion, etc. paired states to the matrix element (m.e.) of the boson operators between the corresponding one-boson, two-boson, etc. states. Evidently, this requires the full $2p$-body fermion m.e., unless one truncates the boson image of the fermion operator. Therefore, nothing is gained, in this present form, through the transformation to the boson space.

3. Modified Marumori mapping

This procedure, which is also referred to as the GRS prescription or OAI mapping, tacitly truncates the boson image of an n-body fermion operator so as to include only one-body, two-body, . . . , n-body boson terms. The physical fermion operators are mainly one- and two-body—types (i.e., $n\le 2$); as a result the boson image contains maximum two-body boson terms. The expansion coefficients or the parameters of the boson image of $\hat{O}_F$ are now determined by equating the matrix element of $\hat{O}_F$ between the full p-pair fermion orthonormal basis to the corresponding matrix element of $\hat{O}_B$ between the respective p-boson basis states. This seems, so far, to be a reasonable prescription. The Hamiltonian so obtained is to be diagonalized in the physical boson states. The full p-boson basis does contain nonphysical (spurious) states, and therefore must be removed before the diagonalization of H_B. To illustrate the procedure we consider an n-p interaction of quadrupole-quadrupole—type

$$H_{\mathrm{int}}=\hat{Q}(\mathrm{p})\cdot\hat{Q}(\mathrm{n}) =\sum_M(-1)^M\hat{Q}_M(\mathrm{p})\hat{Q}_{-M}(\mathrm{n})\ , \tag{11}$$

where

$$\hat{Q}_M(\rho)=X_\rho\sum_{ab}\frac{\hat{j}_a}{\sqrt{4\pi}}\begin{bmatrix} j_a & 2 & j_b\\ \frac{1}{2} & 0 & \frac{1}{2}\end{bmatrix}\tfrac{1}{2}[1+(-1)^{l_a+l_b}] \times\langle r^2\rangle_{ab}(c_a^{\dagger}\tilde{c}_b)_{2,M}\ , \tag{12}$$

with $\rho = \text{p}$ or n. The symbol [ ] denotes the Clebsch-Gordan coefficient, X_ρ is the strength parameter, and $\langle r^2 \rangle_{ab}$ denotes the m.e. of the r^2 operator between the radial parts $|n_a l_a\rangle$ and $|n_b l_b\rangle$ of the sp states of $|a\rangle$ and $|b\rangle$. The operator

$$\widetilde{c}_{a,m_a} = (-1)^{j_a - m_a} c_{a,-m_a} . \tag{13}$$

The identical p-p and n-n parts of the interaction are assumed to be diagonal in the fermion basis of BPA type [Eq. (6)]. The boson form of the Hamiltonian is given by

$$H_B = E'\hat{n}_d + \hat{T}(\text{p})\cdot\hat{T}(\text{n}) . \tag{14}$$

The operator

$$\hat{T}(\rho) = K_\rho[(s^\dagger\widetilde{d} + d^\dagger\widetilde{s})_{2(\rho)} + \chi_\rho(d^\dagger\widetilde{d})_{2(\rho)}] , \tag{15}$$

$\hat{n}_d$ is the total (sum of protons and neutrons) number of nonzero coupled fermion pairs, and E' is related to the BPA excitation energies, corresponding to the $2(p_\text{n}+p_\text{p})$ valence nucleon system. The symbols $s^\dagger$ and $d^\dagger$ designate the boson operators for $J=0$ and $J=2$, respectively, and are analogous to $S^\dagger$ and $D^\dagger$ used in the fermion space. The parameters K_p (K_n) and χ_p (χ_n) of H_B appearing in Eq. (15) are now determined by equating the fermion m.e. of $\hat{Q}$ between the states Eq. (6), to the m.e. of $\hat{T}$ between the corresponding boson states. Inserting the values of the boson m.e. these reduce to the following equations:

$$\langle \psi_0 || \hat{Q}(\rho) || \psi_{J_\mu=2} \rangle = \sqrt{p_\rho} K_\rho \tag{16}$$

and

$$\langle \psi_{J_\mu=2} || \hat{Q}(\rho) || \psi_{J_\mu=2} \rangle = K_\rho \chi_\rho . \tag{17}$$

The fermion m.e. of $\hat{Q}$ appearing on the left-hand side of Eqs. (16) and (17) can easily be evaluated from the general expression for the m.e. of a multipole operator $\hat{O}^K_M$ of rank K and projection M between the BPA basis states [Eq. (5)] already derived.[15]

For a single level case Eq. (16) yields

$$K_\rho = X'_\rho \left[\frac{2(\Omega_a - p_a)}{\Omega_a(\Omega_a - 1)} \right]^{1/2} \tag{18}$$

and

$$K_\rho \chi_\rho = 10 X'_\rho \left[\frac{(\Omega_a - 2p_a)}{(\Omega_a - 2)} \right]^{1/2} \begin{Bmatrix} 2 & 2 & 2 \\ j_a & j_a & j_a \end{Bmatrix} , \tag{19}$$

where

$$\Omega_a = j_a + \tfrac{1}{2} ,$$

$$X'_\rho = X_\rho \langle r^2 \rangle_a \frac{\hat{j}_a}{\sqrt{4\pi}} \begin{bmatrix} j_a & 2 & j_a \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} ,$$

and the 6-j symbol is related to the W coefficient through a phase factor. These expressions are consistent with those of Otsuka *et al.*[5] The Hamiltonian H_B [Eq. (14)] so obtained is to be diagonalized using PBB states [Eq. (10a)].

B. Non-Hermitian boson expansion theory or Dyson mapping

The respective algebra of bi-fermions and the bosons is preserved through the following transformation:

$$Q^\dagger_\mu \to \bar{B}^\dagger_\mu \equiv b^\dagger_\mu - \tfrac{1}{2} \sum_{\substack{\nu\rho\sigma \\ J}} \tau^{(\nu\rho)J}_{(\mu\sigma)J} [(b^\dagger_\nu b^\dagger_\rho)_J \otimes \widetilde{b}_\sigma]^\mu_{J_\mu, M_\mu} , \tag{20}$$

$$Q_\mu \to \bar{B}_\mu \equiv b_\mu , \tag{21}$$

and

$$(c^\dagger_a \widetilde{c}_b)_{J,M} = \sum_{\substack{\mu\nu \\ a'}} x^{*\mu}_{J_\mu}(aa') x^\nu_{J_\nu}(ba') \hat{J}_\mu \hat{J}_\nu W(j_a j_{a'} J j_\nu; J_\mu j_b)(b^\dagger_\mu \widetilde{b}_\nu)_{J,M} , \tag{22}$$

where

$$\tau^{(\nu\rho)J}_{(\mu\sigma)J} = \sum_{abcd} \hat{J}_\nu \hat{J}_\rho \hat{J}_\sigma \hat{J} \begin{Bmatrix} a & b & J_\nu \\ d & c & J_\rho \\ J_\sigma & J_\mu & J \end{Bmatrix} x^{*\nu}_{J_\nu}(ab) x^{*\rho}_{J_\rho}(dc) x^\mu_{J_\mu}(bc) x^\sigma_{J_\sigma}(ad) . \tag{23}$$

The { } represents the 9-j symbol. The function τ obeys the following symmetry relations:

$$\begin{aligned} \tau^{(\nu\rho)J}_{(\mu\sigma)J} &= (-1)^{J_\nu + J_\rho + J} \tau^{(\rho\nu)J}_{(\mu\sigma)J} \\ &= (-1)^{J_\mu + J_\sigma + J} \frac{\hat{J}_\sigma}{\hat{J}_\mu} \tau^{(\nu\rho)J}_{(\sigma\mu)J} \\ &= (-1)^{J_\mu + J_\nu + J_\rho + J_\sigma} \frac{\hat{J}_\nu}{\hat{J}_\mu} \tau^{*(\mu\sigma)J}_{(\nu\rho)J} . \end{aligned} \tag{24}$$

It is immediately clear from Eqs. (20)—(22) that the transformation is (a) nonunitary, i.e.,

$$(\bar{B}^\dagger)^\dagger \neq \bar{B} , \tag{25}$$

and (b) it is finite. As a consequence of (b) any fermion operator written in terms of the bi-fermion operator will have a finite number of terms in the boson space. The explicit form of the boson images of H_F and the basis states $|i\rangle_F$ designated by H_D and $|i_D\rangle$, respectively, can easily be obtained using Eqs. (20)—(22). The full problem can be solved in the boson space. However, due to (a) one is required to use the bi-orthonormal set of basis states with a normalization ${}_L(i_D | i_D)_R = 1$. The Hamiltonian matrix is clearly non-Hermitian even if H_D is Hermitian. Furthermore, the normalization condition ${}_L(i_D \mid i_D)_R = 1$ is preserved even if $\mid i_D)_R$ is multiplied by a factor γ_i and ${}_L(i_D \mid$ by γ_i^{-1}. The γ's can be fixed by requiring that the matrix element of H_D between the PBB states is equal to the m.e. of H_F between the corresponding fermion basis states. This obviously requires the evaluation of fermion matrix elements of the original problem. The γ ambiguity can be overcome by using the Hermitization prescription of GB. This yields a Hermitian matrix through

$$h_{ij} = [{}_L(i_D \mid H_D \mid j_D)_R \, {}_L(j_D \mid H_D \mid i_D)_R]^{1/2} . \tag{26}$$

It is shown that h_{ij} coincides with the fermion Hamiltonian matrix of the original shell-model problem. It can therefore be visualized that the boson parameter of a fermion operator obtained by using modified Marumori mapping or the GRS prescription will be exactly identical to the matrix element of the Dyson image of the fermion operator between the corresponding PBB states with the GB prescription. Before we discuss this important aspect anv further we observe that if all bi-fermion excitations are retained then the present approach is equivalent to solving the original shell-model problem. However, the use of all PBB states makes the calculations cumbersome. Nevertheless, for simple cases this approach is not difficult as we shall see in a short while. At the same time the present formalism is well suited in investigating the convergence problem, different truncation schemes, and their implications. The analysis of these investigations may pave the way for evolving a suitable truncation scheme. Besides, it will yield additional information—such as the extent of coupling between collective and less collective modes, the content of the spurious components in the calculated eigenstates when boson basis states are used, etc.

For clarity let us again consider the case of the n-p residual interaction of quadrupole-quadrupole type [Eq. (11)]. The Dyson boson image is given by [using Eq. (22)]

$$(H_{\rm int})_D \rightarrow (\hat{Q}(p))_D \cdot (\hat{Q}(n))_D , \tag{27}$$

where

$$(\hat{Q}_M(\rho))_D = X_\rho \sum_{\substack{abc\\ \mu\nu}} \frac{\hat{j}_a}{\sqrt{4\pi}} \langle r^2\rangle_{ab} \tfrac{1}{2}[1+(-1)^{1_a+1_b}] \begin{bmatrix} j_a & J & j_b \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \hat{J}_\mu \hat{J}_\nu x_{j_\mu}^{*\mu}(ac) x_{J_\nu}^{\nu}(bc) W(j_a j_c J J_\nu; J_\mu j_b) [b_\mu^\dagger \tilde{b}_\nu]_{J,M(\rho)} \tag{28}$$

for $\rho = $p or n. As stated earlier, the major part of the identical part of the interaction, i.e., p-p and n-n interaction, is assumed to be taken into account in generating collective bi-fermion excitations and therefore is assumed to be diagonal with states Eq. (5) or Eq. (6) of the BPA type for protons and for neutrons. The boson image of this identical part which will contribute to the diagonal m.e. can trivially be obtained by using Eq. (28) or Eqs. (20) and (21). The next step requires the evaluation of the m.e. of $(H_{\rm int})_D$ between the physical boson basis states. We discuss PBB states in the next subsection.

1. Physical boson basis states

The PBB states obtained from the fermion basis states [Eq. (9)] using the generalized Dyson transformation [Eqs. (20)—(22)], can be expressed as

$$(S^\dagger)^p \mid 0\rangle \equiv \mid p0\rangle \rightarrow \left[N_0(p)(s^\dagger)^p - \sum_{\nu\neq 0} N_0^\nu(p)[b_\nu^\dagger b_\nu^\dagger]_0 (s^\dagger)^{p-2} + \sum_{\nu\rho\sigma\neq 0} N_0^{(\nu\rho)\sigma}(p)[(b_\nu^\dagger b_\rho^\dagger)_\sigma \otimes b_\sigma^\dagger]_0 (s^\dagger)^{p-3} \cdots \right] \mid 0) , \tag{29}$$

$$\begin{aligned} Q_\mu^\dagger S^{\dagger p-1} \mid 0\rangle \equiv \mid p\mu\rangle \rightarrow \Big\{ & N_\mu(p-1) b_\mu^\dagger (s^\dagger)^{p-1} - \sum_{\nu\rho\neq 0} N_\mu^{(\nu\rho)\mu}(p-1)[b_\nu^\dagger b_\rho^\dagger]_\mu (s^\dagger)^{p-2} \\ & + \sum_{\nu\rho\sigma J_1\neq 0} N_\mu^{[(\nu\rho)J_1\otimes\sigma]\mu}(p-1)[(b_\nu^\dagger b_\rho^\dagger)_{J_1} \otimes b_\sigma^\dagger]_\mu (s^\dagger)^{p-3} + \cdots \Big\} \mid 0) , \end{aligned} \tag{30}$$

and similar equations for higher states, where

$$\begin{aligned}
N_0(k+1) &= (1-k\tau_0^0)N_0(k) \, , \\
N_0^{\mu}(k+1) &= k\tau_0^{\mu}N_0(k)+N_0^{\mu}(k)[1-(k-2)\tau_0^0-4\tau_{\mu}^{\mu}\hat{J}_{\mu}^{-1}] \, , \\
N_0^{(\mu\rho)\sigma}(k+1) &= N_0^{(\mu\rho)\sigma}(k)[1-(k-3)\tau_0^0-2\tau_{\sigma}^{\sigma}\hat{J}_{\sigma}^{-1}] \\
&\quad +\tau_{(0\sigma)\sigma}^{(\mu\rho)\sigma}N_0^{\sigma}(k)\hat{J}_{\sigma}^{-1} \\
&\quad -4\tau_{\sigma}^{\sigma}\hat{J}_{\sigma}^{-1}N_0^{(\mu\sigma)\rho}(k)(-1)^{J_\mu+J_\rho+J_\sigma} \\
&= (-1)^{J_\mu+J_\rho+J_\sigma}N_0^{(\rho\mu)\sigma}(k+1) \, ,
\end{aligned} \tag{31}$$

$$\begin{aligned}
N_{\mu}(k) &= N_0(k)(1-2\tau_{\mu}^{\mu}k\hat{J}_{\mu}^{-1})+2N_0^{\mu}(k)(\tau_0^{\mu})^2\hat{J}_{\mu}^{-2} \, , \\
N_{\mu}^{(\nu\rho)\mu}(k) &= 2kN_0(k)\tau_{(\mu 0)\mu}^{(\nu\rho)\mu}+4\tau_{(\mu\nu)\rho}^{(0\rho)\rho}(-1)^{J_\nu+J_\rho+J_\mu}N_0^{\nu}(k)\hat{J}_{\nu}^{-1} \\
&\quad +\hat{J}_{\mu}^{-1}\tau_{\mu}^{0}[N_0^{(\nu\rho)\mu}(k)+2N_0^{(\mu\nu)\rho}(k)] \\
&= (-1)^{J_\nu+J_\rho-J_\mu}N_{\mu}^{(\rho\nu)\mu}(k) \, ,
\end{aligned}$$

and

$$\begin{aligned}
\tau_0^0 &= \tfrac{1}{2}\tau_{(00)0}^{(00)0} \, , \\
\tau_0^{\mu} &= \tfrac{1}{2}\tau_{(00)0}^{(\mu\mu)0}=\hat{J}_{\mu}\tau_{\mu}^{0} \, , \\
\tau_{\mu} &= \tfrac{1}{2}\tau_{(0\mu)\mu}^{(0\mu)\mu} \, ,
\end{aligned} \tag{32}$$

with

$$\begin{aligned}
& N_0(1)=1, \quad N_0^{\mu}(1)=0, \quad N_0^{\mu}(2)=1 \, , \\
& (k-m)=(k-m) \text{ for } k \geq m \\
& \qquad =0 \text{ for } k \leq m \; ;
\end{aligned} \tag{33}$$

$|0)$ represents the boson vacuum. Similar expansions can be written for higher fermion basis states [Eq. (9)] and the various coefficients determined. The expansion coefficients already obtained are unaffected by the inclusion of the higher fermion basis states. Furthermore, these expansion coefficients depend upon the detailed structure of the bi-fermion operators $Q^{\dagger}$ ($S^{\dagger}$), i.e., x coefficients, and therefore will depend upon the Hamiltonian and the number of valence particles considered. It therefore follows that for a single-level case, these expansion coefficients do not depend upon H. In fact, it is found that these expansion coefficients are just the corresponding products of fractional parentage coefficients (fpc's). This is not surprising and can be understood by rewriting the shell-model seniority (v) states of fermions in terms of paired states using two-particle fpc's. For example, for the $v=0$ state of p pairs of identical valence nucleons consider a term

$$|j_{1,2}^2(0),J_{3,4}^2(0),\ldots,j_{p-1,p}^2(0)\rangle \tag{34}$$

with all individual pairs coupled to zero. $j_{k-1,k}^2(0)$ in Eq. (34) is an antisymmetric state of $k-1$ and kth particles and no antisymmetry is required in Eq. (34) between different pairs which are orthonormal. Therefore, it can be visualized that $j_{k-1,k}^2(0)$ can be treated as some sort of a boson (analogous to $s^{\dagger}$) $\underline{s}^{\dagger}$ operator, because the boson operators $b^{\dagger}$ ($s^{\dagger}$) are obtained through the generalized Dyson transformation [Eqs. (20)—(22)] fully respecting the Pauli principle operating between the particles, say $k-1$ and k comprising $\underline{s}^{\dagger}$. The operators $s^{\dagger}$ or $b^{\dagger}$ do not depend upon the particle level and commute among themselves; this fact can be taken care of by rewriting Eq. (34) in the form of a normalized state with a symmetric combination of particle pair indices. As a result, the boson state will appear as a normalized state, e.g., for Eq. (34) $(s^{\dagger})^p/\sqrt{p!}$. As no antisymmetry is required among the paired states appearing in Eq. (34), therefore the expansion coefficients of Eqs. (29) and (30) for a single level case will reduce to the respective products of two-particle fpc's.

Due to the nonunitary nature of the mapping, one is required to introduce a bi-orthonormal basis for solving the eigenvalue problem. The ket states are given in Eqs. (29) and (30) and the corresponding bra states obtained by using the transformation Eq. (21) are

$$\langle 0|(S)^p \rightarrow (0|(s)^p=(p0| \, , \tag{35}$$

$$\langle 0|S^{p-1}Q_{\mu} \rightarrow (0|b_{\mu}s^{p-1}=(p\mu| \, , \tag{36}$$

and similar equations for higher states. The normalization constant to be used is obtained through

$$_L(i|i)_R=1 \, , \tag{37}$$

where suffixes L and R are used explicitly to distinguish the ket and bra states of the boson basis, i.e., only the square of the normalization constant is obtained from Eq. (37). As a result, the condition Eq. (37) is still preserved even if we multiply $|i)_R$ by a factor γ_i and $_L(i|$ by γ_i^{-1}. In other words, the ambiguity appears which in principle yields a different Hamiltonian matrix (in particular nondiagonal matrix elements) depending upon γ's. This γ ambiguity can be resolved through the GB prescription, in turn leading to a Hermitian matrix (h) given by

$$\begin{aligned}
h_{ij} &= \left[\frac{\gamma_j}{\gamma_i} {}_L(i|H_D|j)_R \; {}_L(j|H_D|i)_R \frac{\gamma_i}{\gamma_j} \right]^{1/2} \\
&= [(H_D)_{ij}(H_D)_{ji}]^{1/2} \, .
\end{aligned} \tag{38}$$

This prescription yields exact results in two dimensions. Furthermore, it is observed that h_{ij} obtained through Eq. (38) is identical to the corresponding fermion m.e. in the fermion space, for the cases studied. This implies that the boson problem with the Dyson transformation along with the GB prescription is equivalent to solving the corresponding shell-model problem. This, in fact, is very satisfying and hence can be used to investigate the following:

(1) The study of the convergence problem in various truncation schemes.

(2) To calculate the extent of spuriousness when the full boson basis is used, in place of PBB states. This in turn may pave the way for establishing an approximate formalism for solving the problem in boson space.

Some authors[26,30] determined γ factors by equating the calculated m.e. to the corresponding fermion m.e., i.e.,

$$\frac{\gamma_i}{\gamma_j} {}_L(j|H_D|i)_R=\langle j|H|i\rangle \, . \tag{39}$$

This prescription, though, again leads to the corresponding shell-model problem, but requires the calculation of the fermion m.e. $\langle j \mid H \mid i \rangle$, thus nullifying any gain achieved in working with boson representation. To show that the GB prescription yields the corresponding fermion m.e., consider again a single level case. The m.e. of the quadrupole operator [Eq. (28)] of the $\hat{Q}\cdot\hat{Q}$ interaction obtained between the PBB states can be expressed in the following form:

$$_L(p_\rho\mu \mid (\hat{Q}_M(\rho))_D \mid p_\rho 0)_R = X'_\rho \left[\frac{2p_\rho}{\Omega_a} \right]^{1/2} \frac{(\Omega_a - p_\rho)}{(\Omega_a - 1)} , \tag{40}$$

$$_L(p_\rho 0 \mid (\hat{Q}_M(\rho))_D \mid p_\rho\mu)_R = X'_\rho \left[\frac{2p_\rho}{\Omega_a} \right]^{1/2} . \tag{41}$$

To overcome the γ ambiguity, we find, using the GB prescription,

$$[_L(p_\rho\mu \mid (\hat{Q}_M(\rho))_D \mid p_\rho 0)_R \; _L(p_\rho 0 \mid (\hat{Q}_M(\rho))_D \mid p_\rho\mu)_R]^{1/2} = X'_\rho \left[\frac{2p_\rho(\Omega_a - p_\rho)}{\Omega_a(\Omega_a - 1)} \right]^{1/2} , \tag{42}$$

which is equal to the corresponding fermion m.e. [see the right-hand side (rhs) of Eq. (18)] of the quadrupole operator; similarly

$$_L(p_\rho\mu \mid (\hat{Q}_M(\rho))_D \mid p_\rho\mu)_R = 10X'_\rho \frac{(\Omega_a - 2p_\rho)}{(\Omega_a - 2)} W(22j_aj_a;2j_a) \tag{43}$$

is also equal to the fermion m.e. It is expected that this observation holds even for the general multilevel case and for other operators. This, in fact, has been proved using the group theoretical techniques and will be reported elsewhere (see Ref. 31).

Before closing this section, we once again emphasize the use of PBB states for the calculation in the boson space. The model cases where the use of the full boson basis states are advocated, are dealt with in the next section.

III. MODEL CASES

It is shown in this section that for the model cases like the Lipkin model[29] and pairing vibrations,[26] the physical basis states are the same as the corresponding normalized boson basis states apart from a multiplicative factor. This happens because only one type or two types of commuting bosons exist for these cases. It turns out that the multiplicative factor is just equal to γ_i for the ket $\mid i)_R$ and γ_i^{-1} for $_L(i \mid$, respectively. This then ensures that the Hermitian matrix obtained with the GB prescription from the m.e. evaluated with the bi-orthonormal physical boson basis states with normalization $_L(i \mid i)_R = 1$ is equal to that obtained from the matrix element calculated between the corresponding normalized boson states.

A. Lipkin model

The Lipkin model is a two level model, these levels have the same j value and are equidistant from the Fermi level—one above and the other below. The model Hamiltonian is

$$H = \epsilon J_0 + \frac{V}{2}(J_+^2 + J_-^2) , \tag{44}$$

where

$$J_0 = \tfrac{1}{2} \sum_m (c_{m_+}^\dagger c_{m_+} - c_{m_-}^\dagger c_{m_-}) , \tag{45}$$

$$J_+ = \sum_m c_{m_+}^\dagger c_{m_-} , \quad J_- = (J_+)^\dagger , \tag{46}$$

ϵ being the energy separation between the two levels, and V being the interaction strength. The operator $c_{m_+}^\dagger$ ($c_{m_-}^\dagger$) creates a particle in the upper (lower) level, respectively. $J_0, J_\pm$ obey angular momentum commutation algebra.

For $V=0$, the g.s. corresponds to $J_0 = -N/2$, i.e., all N particles fill the lower level. Considering only the g.s. band with $J = N/2$, the corresponding basis states are

$$\mid J = N/2 \; J0 \rangle \equiv \mid n - J \rangle, \text{ with } n = 0,1,\ldots,N . \tag{47}$$

These can be generated by J_+^p operating on $\mid 0 \rangle$ for $p \le N$ where $\mid 0 \rangle \equiv \mid -N/2 \rangle$. The states of Eq. (47) will be simply denoted by $\mid n \rangle$. The model Hamiltonian Eq. (44) can easily be diagonalized in the space spanned by the basis states $\mid n \rangle$ [Eq. (47)]. The boson images of $J_0, J_\pm$ are now obtained through the Dyson transformation

$$J_+ \to \sqrt{2J}\, b^\dagger \left[1 - \frac{b^\dagger b}{2J} \right] , \qquad J_- \to \sqrt{2J}\, b , \tag{48}$$

and

$$J_0 \to b^\dagger b - J ,$$

where $b^\dagger$ (b) represents the boson creation (annihilation) operator. The boson image of the model Hamiltonian obtained using the transformation Eq. (48) is

$$(H_B) \to -\epsilon J + \epsilon b^\dagger b + VJ \left[\left[1 - \frac{1}{2J} \right] (b^\dagger)^2 + b^2 - \frac{1}{J} \left[1 - \frac{1}{2J} \right] (b^\dagger)^3 b + \frac{1}{4J^2} (b^\dagger)^4 b^2 \right] . \tag{49}$$

The physical boson states are

$$|p\rangle \equiv (J_+)^p|0\rangle \to (\sqrt{2J})^p\left[1-\frac{1}{2J}\right]\left[1-\frac{2}{2J}\right] \cdots \left[1-\frac{p-1}{2J}\right]\sqrt{p!}\frac{(b^\dagger)^p}{\sqrt{p!}}|0) \to \frac{\sqrt{p!}}{(2J)^{p/2-1}}\frac{(2J-1)!}{(2J-p)!}\frac{(b^\dagger)^p}{\sqrt{p!}}|0\rangle \equiv |p) , \tag{50}$$

$$\langle p| = \langle 0|(J_-)^p \to (2J)^{p/2}\sqrt{p!}(0|\frac{b^p}{\sqrt{p!}} \equiv (p| . \tag{51}$$

The bi-orthonormal physical basis states are

$$|p)_R = \left[\frac{(2J-1)!}{(2J-p)!(2J)^{p-1}}\right]^{1/2}\frac{(b^\dagger)^p}{\sqrt{p!}}|0) \equiv \gamma_p\frac{(b^\dagger)^p}{\sqrt{p!}}|0) , \tag{52}$$

$$_L(p| = \left[\frac{(2J)^{p-1}(2J-p)!}{(2J-1)!}\right]^{1/2}(0|\frac{b^p}{\sqrt{p!}} \equiv \gamma_p^{-1}(0|\frac{b^p}{\sqrt{p!}} , \tag{53}$$

with normalization $_L(p\,|p)_R=1$. Obviously, due to the γ ambiguity, the use of physical boson states [Eqs. (52) and (53)] with normalization $_L(p\,|p)_R=1$ will yield results identical to those obtained by using normalized boson basis states along with the GB prescription.

The Hermitian matrix h so obtained is

$$h_{pp'} = h_{p'p} = [(p\,|H_B|p')(p'|H_B|p)]^{1/2} . \tag{54}$$

It is easy to show that Hamiltonian h is identical to the original Hamiltonian matrix, i.e.,

$$h_{pp'} = \langle p\,|H\,|p'\rangle = \langle p'|H\,|p\rangle . \tag{55}$$

B. Pairing vibrations

Recently Hahne[26] advocated the use of the normalized boson basis in treating the case of pairing vibrations, in boson space, with Dyson mapping. The open shell states are divided into a set of particlelike (p) and a set of holelike (h) states. The operators $Q_p^\dagger$ $(Q_h^\dagger)$ create particle (hole) pair of angular momentum zero. The Hamiltonian can be written in terms of boson operators $(b_p^\dagger, b_h^\dagger)$ using Dyson transformation, i.e.,

$$Q_x^\dagger \to b_x^\dagger - \frac{1}{\Omega_x}b_x^\dagger b_x^\dagger b_x , \qquad Q_x \to b_x , \tag{56}$$

for $x=p$ or h with $\Omega_x=\hat{j}_x+\frac{1}{2}$. The physical boson states for x $(=p$ or $h)$ can be expressed by using Eq. (56) as

$$(Q_x^\dagger)^{p_x}|0\rangle \to \frac{\Omega_x!\sqrt{p_x!}}{\Omega_x^p(\Omega_x-p_x)!}\frac{(b_x^\dagger)^{p_x}}{\sqrt{p_x!}}|0) , \tag{57}$$

$$\langle 0|(Q_x)^{p_x} \to \sqrt{p_x!}(0|\frac{(b_x)^{p_x}}{\sqrt{p_x!}} . \tag{58}$$

The bi-orthonormal boson states with normalization $_L(p_x\,|p_x)_R=1$ for x $(=p$ or $h)$ are

$$|p_x)_R = \left[\frac{\Omega_x!}{\Omega_x^{p_x}(\Omega_x-p_x)!}\right]^{1/2}\frac{(b_x^\dagger)^{p_x}}{\sqrt{p_x!}}|0) = \gamma_{p_x}\frac{(b_x^\dagger)^{p_x}}{\sqrt{p_x!}}|0) \tag{59}$$

and

$$_L(p_x| = \left[\frac{\Omega_x^{p_x}(\Omega_x-p_x)!}{\Omega_x!}\right]^{1/2}(0|\frac{(b_x)^{p_x}}{\sqrt{p_x!}} = \gamma_{p_x}^{-1}(0|\frac{(b_x)^{p_x}}{\sqrt{p_x!}} . \tag{60}$$

These results are identical to those obtained [Eqs. (52) and (53)] for the Lipkin model. It is now trivial to conclude through the same line of arguments presented for the Lipkin model, that the use of boson basis states gives identical results as those obtained by using PBB states along with the GB prescription. This yields the Hermitian matrix identical to the Hamiltonian matrix of the original problem. It is to be pointed out that this conclusion is solely due to the fact that there exist only one or more, but independent, sets of boson operators. This conclusion therefore may not hold, in general, to collective bosons linked through the general Dyson transformation Eqs. (20)–(22).

IV. CONCLUSION

We have analyzed some of the problems still unresolved, appearing in the boson expansion theories (BET's). A Hermitian type of BET is useful only if further approximations are introduced. These approximations can be tested either by using the present approach or by comparing the results obtained in the fermion case with the shell model techniques.

Due to the very nature of the nonunitary transformation in NH-BET, one is required to use bi-orthonormal physical boson basis states. There then appears γ ambiguity in the normalization of states. It is shown that the use of the bi-orthonormal PBB with the GB prescription yields a Hermitian matrix and resolves γ ambiguity amicably. In addition the Hermitian matrix turns out to be identical to the fermion Hamiltonian matrix. The present approach is, in fact, equivalent to solving the original shell model problem. Furthermore, it can be very useful in investigating various approximations or truncation schemes, which may be introduced in the BET. It is pointed out that the physical boson states are, in fact, proportional to normalized boson basis states, for the model cases where the use of normalized boson basis states is found to be justified. This therefore clarifies the reason for the justification and at the same time stresses the use of PBB states in actual calculations. We hope further investigations based on the present approach may pave the way for arriving at a suitable approximate scheme for carrying out the practical calculations in the boson space.

The authors thank T. Tamura, K. Allaart, C. R. Sarma, R. S. Nikam, and S. H. Patil for their interest in the work and useful suggestions.

[1]A. Arima and F. Iachello, Ann. Phys. (N.Y.) **99**, 253 (1976); **111**, 201 (1978); **123**, 468 (1979).
[2]*Interacting Bosons in Nuclear Physics,* edited by F. Iachello (Plenum, New York, 1979).
[3]O. Scholten, F. Iachello, and A. Arima, Ann. Phys. (N.Y.) **115**, 325 (1978).
[4]*Interacting Bose-Fermi Systems in Nuclei,* edited by F. Iachello (Plenum, New York, 1981).
[5]T. Otsuka, A. Arima, and F. Iachello, Nucl. Phys. **A309**, 1 (1978).
[6]P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer, New York, 1980), Chap. 9.
[7]See, for example, T. Tamura, Phys. Rev. C **28**, 2840 (1983), and references therein.
[8]Proceedings of the International Conference on Band Structure and Nuclear Dynamics, New Orleans, 1980, Nucl. Phys. **A347**, 253 (1980).
[9]Y. K. Gambhir, P. Ring, and P. Schuck, Phys. Rev. C **25**, 2858 (1982); Nucl. Phys. **A384**, 37 (1982).
[10]S. Pittel, P. D. Dual, and B. R. Barrett, Ann. Phys. (N.Y.) **144**, 168 (1982), and references therein.
[11]A. Van Egmond and K. Allaart, Nucl. Phys. **A394**, 173 (1983).
[12]M. R. Zirnbauer and D. M. Brink, Nucl. Phys. **A384**, 1 (1982).
[13]T. Kishimoto and T. Tamura, Phys. Rev. C **27**, 341 (1983).
[14]A. Klein and M. Vallieres, Phys. Lett. **98B**, 5 (1981).
[15]Y. K. Gambhir, A. Rimini, and T. Weber, Phys. Rev. **188**, 1573 (1969); Phys. Rev. C **3**, 1965 (1971); Y. K. Gambhir, S. Haq, and J. K. Suri, *ibid.* **16**, 2455 (1977); **20**, 381 (1979); **21**, 1124 (1980); Ann. Phys. (N.Y.) **133**, 154 (1981).
[16]B. Lorazo, Phys. Lett. **29B**, 150 (1969); Nucl. Phys. **A153**, 255 (1970).
[17]K. Allaart and E. Boeker, Nucl. Phys. **A168**, 631 (1971); **A198**, 33 (1972).
[18]T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).
[19]S. T. Belyaev and V. G. Zelevinskii, Nucl. Phys. **39**, 582 (1962).
[20]T. Marumori, M. Yamamura, and A. Tokunaga, Prog. Theor. Phys. **31**, 1009 (1964).
[21]D. Janssen, F. Dönau, S. Frauendorf, and R. V. Jolos, Nucl. Phys. **A172**, 145 (1971); F. Dönau and D. Janssen, *ibid.* **A209**, 109 (1973).
[22]H. B. Geyer and F. J. W. Hahne, Phys. Lett. **90B**, 6 (1980); **90B**, 1231 (1980).
[23]F. J. Dyson, Phys. Rev. **102**, 1217 (1956).
[24]E. R. Marshalek, Nucl. Phys. **A224**, 221 (1974); **A224**, 245 (1974); **A357**, 398 (1981).
[25]G. Bonsignori, K. Allaart, and A. Van Egmond, in *Collective Bands in Nuclei,* edited by D. H. Wilkinson (Pergamon, Oxford, 1982).
[26]F. J. W. Hahne, Phys. Rev. C **23**, 2305 (1981).
[27]Y. K. Gambhir, in Proceedings of the Nuclear Physics and Solid State Physics Symposium, Mysore, India, 1983, Invited Talk No. IN3; Y. K. Gambhir and J. A. Sheikh, *ibid.*, Contribution No. NC/2.
[28]Y. K. Gambhir and G. Basavaraju, Pramana **13**, 269 (1979).
[29]H. J. Lipkin, W. Meshkov, and A. J. Glick, Nucl. Phys. **62**, 188 (1965); **62**, 221 (1965).
[30]C. T. Li, Phys. Lett. **120B**, 251 (1983).
[31]Y. K. Gambhir, R. S. Nikam, C. R. Sarma, and J. A. Sheikh, J. Math. Phys. (to be published).