Recoil effects in the relativistically corrected impulse approximation

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(Received 11 September 1984)

We invoke the formalism developed by Krajcik and Foldy to consistently treat recoil effects in the relativistically corrected impulse approximation in the presence of a local central potential. It is stressed that recoil effects in the case of the single-particle Dirac Hamiltonian are considerably different from those suitable for the Foldy-Wouthuysen diagonalized Hamiltonian. It is also found that, for the diagonalized Hamiltonian, intuitive arguments may result in an incorrect sign for certain terms as in a preceding paper. As an illustrative numerical example, we use the resultant formulae to determine the magnetic moment and charge radius of the ^{15}N nucleus.

I. INTRODUCTION

To describe electron scattering or other electroweak reactions at intermediate energies (e.g., at 1 GeV electron beam energy), it has become evident that relativistic effects associated with the target nucleus must be incorporated in a systematic manner. This turns out to be a formidable task, if not an impossible one, since a consistent description of a system of relativistic particles remains in its very infancy. On the other hand, it is inconceivable that the entire body of nonrelativistic nuclear physics, as shown over the last five decades to be highly successful in describing reactions at low energies, can be ignored altogether. It is likely that further developments in nuclear Dirac phenomenology or in quark physics will shed light on how to go beyond the low energy nuclear physics.

In an earlier paper,[1] one of us has attempted to derive a relativistically-corrected impulse approximation (RCIA) in the presence of a central local potential. Instead of the Foldy-Wouthuysen [FW] transformation,[2] in which an operator is generally considered separately from the initial and final wave functions, the Pauli-Breit reduction procedure[3] has been used to generate a *unique* nonrelativistic representation of a given matrix element. As shown by de Vries and Jonker,[4] this reduction method yields results identical with Eriksen's version[5] of the FW transformation. Accordingly, it appears as a proper choice to identify, in the context of the shell-model language, the normalized upper component of the four-component nucleon wave function in the FW-diagonalized picture with the nonrelativistic nucleon wave function. This identification is plausible for several reasons, viz.,

(1) The entire body of the low-energy nuclear physics is expected to remain more or less intact.

(2) The statistical interpretation of the wave function in nonrelativistic quantum mechanics is strictly reinforced.

(3) By letting the potential approach gradually to zero, one always maintains the connection between the Dirac picture and the nonrelativistic picture for the constituent nucleon.

(4) It serves as a model for allowing one to characterize relativistic corrections to the nucleon-only impulse approximation (NOIA).[6]

Of course, the v/c expansion obtained via the FW transformation does not necessarily converge rapidly for an arbitrary potential. The two potentials proposed in nuclear Dirac phenomenology[7] are fairly strong but, fortunately, we have demonstrated[8] that the convergence property of the v/c expansion remains appealingly good. Nevertheless, we are interested in the RCIA for a general central local potential and thereby do not intend to restrict ourselves to the Dirac phenomenology potential.[7]

The main purpose of this paper is to elucidate, within the framework of the RCIA, those (recoil) effects which arise from the fact that the initial and final nuclei in general cannot be both at rest. For technical reasons to be described below, we believe that this topic deserves further attention and certainly has not been addressed adequately in the previous paper.[1] We discuss in Sec. II how ambiguities may arise in an intuitive treatment of recoil effects as in the previous paper and then propose to use the formalism developed by Krajcik and Foldy[9] to treat recoil effects consistently and elegantly. As an illustrative numerical example, we employ in Sec. III the resultant formulae to evaluate the magnetic moment and charge radius of the ^{15}N nucleus.

II. GENERAL FORMALISM

We are interested in the matrix elements of the polar vector and axial vector currents $V_\lambda(x)$ and $A_\lambda(x)$ between two nuclear states of definite four momenta $p^{(i)}$ and $p^{(f)}$,

$$M_\lambda^{(V)}=\langle N_f(p^{(f)})\,|\,V_\lambda(0)\,|\,N_i(p^{(i)})\rangle \ ,$$
$$M_\lambda^{(A)}=\langle N_f(p^{(f)}\,|\,A_\lambda(0)\,|\,N_i(p^{(i)})\rangle \ . \tag{1}$$

These matrix elements enter the transition amplitude T which allows us to calculate all physical observables of relevant interest. For instance, the transition amplitude T for electron scattering is given, in the one-photon-exchange approximation, by

$$T=-\frac{e^2}{q^2}iU_e^\dagger(p_e')\gamma_4\gamma_\lambda U_e(p_e) \times\langle N_f(p^{(f)})\,|\,J_\lambda(0)\,|\,N_i(p^{(i)})\rangle \ , \tag{2}$$

with $J_\lambda(x)$ the hadronic electromagnetic current and

$$q_\lambda\equiv(p^{(i)}-p^{(f)})_\lambda=(p_e'-p_e)_\lambda \ .$$

To evaluate any of these matrix elements, we need as the basic input: (1) the initial nuclear wave function of four-momentum $p^{(i)}$, (2) the final nuclear wave function of four-momentum $p^{(f)}$, and (3) the current operator $J_\lambda(x)$. All these input quantities must be defined in a single frame of reference, say the Breit frame in which the initial and final nuclei are treated symmetrically $(\mathbf{p}^{(i)}=-\mathbf{p}^{(f)}=\mathbf{q}/2)$, since the matrix element itself is a Lorentz four-vector. Accordingly, the space-time variables $(\mathbf{x},it)$ appearing in the current operator $J_\lambda(x)$ are defined in the Breit frame. On the other hand, the initial or final nuclear wave function is generally given in its own rest frame. Thus, we need to find the Breit-frame nucleon wave function $\Psi(\mathbf{r})$ when the nucleon wave function $\Psi^L(\mathbf{r}^L)$ obtained in the rest (laboratory) frame of the entire nucleus is given. This turns out to be a tricky question, for reasons to be elucidated below.

It is of practical interest to first obtain some intuitive understanding of the problem. To this end, we write

$$\Psi(\mathbf{r})=S_\Lambda\Psi^L(\mathbf{r}^L)=S_K\Psi^L(\mathbf{r}) \ , \tag{3}$$

where S_Λ is the Lorentz transformation determined by the relative velocity between the Breit-frame observer and the entire nucleus. To describe a system of A *identical* relativistic particles, we may assume that the overall wave function is already given in its own rest frame in terms of the space-time variables $(\mathbf{x}_1^L,it_1^L)$, $(\mathbf{x}_2^L,it_2^L),\ldots,$ and $(\mathbf{x}_A^L,it_A^L)$. We introduce

$$\mathbf{R}^L\equiv\frac{1}{A}\sum_{i=1}^{A}\mathbf{x}_i^L,\ \ T^L\equiv\frac{1}{A}\sum_{i=1}^{A}t_1{}^L \ . \tag{4}$$

If it is further *assumed* (with caution) that a *good* overall wave function depends on the ("laboratory") c.m. coordinates, $X^L\equiv(\mathbf{R}^L,iT^L)$, only through the trivial factor $\exp(i\mathbf{P}^L\cdot X^L)$, then the *internal* overall wave function depends on the relative coordinates,

$$\mathbf{r}_i^L\equiv\mathbf{x}_i^L-\mathbf{R}^L,\ \ \tau_i^L\equiv t_i^L-T^L \ . \tag{5}$$

In the spirit of the shell model, one may further assume that the ith constituent wave function depends on $(\mathbf{r}_i^L,i\tau_i^L)$. It is well known that the shell model suffers from the center-of-mass (c.m.) problem. The resolution to the c.m. problem has been discussed widely in the literature,[10] but mostly for a system of nonrelativistic particles. We wish to return to this aspect when we invoke the formalism developed by Krajcik and Foldy[9] to formulate recoil effects to second order in v/c.

The transformation of the coordinates between the Breit frame and the rest frame of the *initial* nucleus is characterized by[1,11]

$$\mathbf{x}^L=\mathbf{x}+(\mathbf{x}\cdot\mathbf{q})\mathbf{q}/(8A^2m_N^2)-\mathbf{q}t/(2Am_\mathrm{N}) +O(v^3/c^3) \ , \tag{6a}$$

$$t^L=t[1+|\mathbf{q}|^2/(8A^2m_\mathrm{N}^2)]-\mathbf{q}\cdot\mathbf{x}/(2Am_\mathrm{N}) +O(v^3/c^3) \ . \tag{6b}$$

Here m_N is the nucleon mass. Accordingly, we have

$$\mathbf{r}^L=\mathbf{r}+(\mathbf{r}\cdot\mathbf{q})\mathbf{q}/(8A^2m_\mathrm{N}^2)-\mathbf{q}\tau/(2Am_\mathrm{N}) +O(v^3/c^3) \ , \tag{7a}$$

$$\tau^L=\tau[1+|\mathbf{q}|^2/(8A^2m_\mathrm{N}^2)]-\mathbf{q}\cdot\mathbf{r}/(2Am_\mathrm{N}) +O(v^3/c^3) \ . \tag{7b}$$

It has been a standard practice to neglect any role played by the relative times τ_i^L. This approximation could be dangerous for a consistent treatment of recoil effects. If we assume that the given nuclear wave function defined in its own rest frame is in fact a *universal-time* wave function, then we have

$$\tau_i^L=0 \text{ for } i=1,\ldots,\ A \ . \tag{8a}$$

Equations (7a), (7b), and (8) yield

$$\mathbf{r}^L=\mathbf{r}-(\mathbf{r}\cdot\mathbf{q})\mathbf{q}/(8A^2m_\mathrm{N}^2)+O(v^3/c^3) \ . \tag{8b}$$

This result has been used in Ref. 1. Unfortunately, it is equally plausible to assume that the universal time has been specified in the Breit frame,

$$\tau_i=0 \text{ for } i=1,\ldots,\ A \ . \tag{9a}$$

Thus, Eq. (7a) yields

$$\mathbf{r}^L=\mathbf{r}+(\mathbf{r}\cdot\mathbf{q})\mathbf{q}/(8A^2m_\mathrm{N}^2)+O(v^3/c^3) \ , \tag{9b}$$

which predicts a recoil effect of the same magnitude but *opposite* in sign in contradistinction with Eq. (8b). To make a definitive choice, we need a consistent formalism such as that of Krajcik and Foldy.[9] It turns out that the sign implied by Eqs. (9a) and (9b) is the *right* choice according to the formalism to be described below.

The next delicate point has to do with the explicit form of the boost operator S_Λ appearing in Eq. (3). If we assume that the underlying Hamiltonian is

$$H=\boldsymbol{\alpha}\cdot\{\mathbf{p}-\mathbf{U}\}+U_0+\beta(m_\mathrm{N}+V) \ , \tag{10a}$$

then we may expect to use the standard form,[12]

$$S_\Lambda=\left[1+\frac{\mathbf{q}^2}{32A^2m_\mathrm{N}^2}\right]+\frac{\boldsymbol{\alpha}\cdot\mathbf{q}}{4Am_\mathrm{N}}+O(v^3/c^3) \ , \tag{10b}$$

for the transformation from the rest *frame* of the *initial* nucleus to the Breit frame. However, Eq. (10b) introduces a number of recoil effects to the order of $(1/A)(v/c)$. For reactions involving light nuclei, such as thermal neutron capture $^1H(n,\gamma)^2H$, these recoil terms present very serious problems since they are not negligible and, yet, have been ignored. (As an illustrative example, see Ref. 13.)

The key to resolve this last difficulty comes from the fact that we are dealing with the FW-diagonalized picture.[2,5] In this picture, the Hamiltonian is already diagonal. Intuitively, the term $\boldsymbol{\alpha}\cdot\mathbf{q}/(4Am_N)$ in Eq. (10b) characterizes a reorientation among the different components of a Dirac spinor when the Dirac spinor given in the rest of the initial nucleus is seen in the Breit frame. For the diagonalized single-particle Hamiltonian, we apply the formalism of Krajcik and Foldy[9] to the present problem and obtain

$$\zeta \to \left[1-\frac{i}{8Am_N^2}\sigma\times\mathbf{p}\cdot\mathbf{q}\right]\zeta\ , \tag{11}$$

with ζ a two-component Pauli spinor specified in the rest frame of the initial nucleus. Here the minus sign appearing in Eq. (11), as also different from what has been used in Ref. 1, presents another intriguing problem which calls for a consistent formalism.

There is yet another problem which is related to the normalization of the overall nuclear wave functions. We already have[1]

$$\int d^3r^L \Psi^{L\dagger}(\mathbf{r}^L)\Psi^L(\mathbf{r}^L)=1+O(v^3/c^3)\ , \tag{12a}$$

so that

$$\int d^3r\, \Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})=1+\mathbf{q}^2/(8A^2m_N^2)+O(v^3/c^3)\ . \tag{12b}$$

If we introduce nuclear wave functions normalized in the Breit frame, then we need to make an additional substitution:

$$\Psi(\mathbf{r})\to[1-\mathbf{q}^2/(16A^2m_N^2)]\Psi(\mathbf{r})\ . \tag{12c}$$

This substitution gives rise to relativistic effects of trivial nature which, however, have often been neglected.

It is clear that the above considerations yield

$$S_K=1-\frac{i}{8Am_N^2}\sigma\times\mathbf{p}\cdot\mathbf{q}+\frac{i}{8A^2m_N^2}(\mathbf{q}\cdot\mathbf{r})(\mathbf{q}\cdot\mathbf{p}) +\frac{\mathbf{q}^2}{16A^2m_N^2}+S_K^{\rm res}\ , \tag{13}$$

with the residual $S_K^{\rm res}$ yet to be determined by the detailed formalism described below. Equation (13) differs from the corresponding formula in Ref. 1 [Eq. (9b)], where *intuitive* arguments have been invoked in the derivation. To unravel ambiguities such as those illustrated above, we find it useful to invoke the formalism developed by Krajcik and Foldy[9] to treat recoil effects in a consistent manner.

To maintain Lorentz covariance, it is essential that the ten infinitesimal generators of the proper inhomogeneous Lorentz group, i.e., the generators of the infinitesimal space translations $(P_1,P_2,P_3)=\mathbf{P}$, the generator of the infinitesimal time translation H, the generators of infinitesimal rotations $(J_1,J_2,J_3)=\mathbf{J}$, and the generators of infinitesimal Lorentz transformations $(K_1,K_2,K_3)=\mathbf{K}$, satisfy the well-known commutation relations:[9]

$$[P_i,P_j]=0\ , \tag{14a}$$

$$[P_i,H]=0\ , \tag{14b}$$

$$[J_i,H]=0\ , \tag{14c}$$

$$[J_i,J_j]=i\epsilon_{ijk}J_k\ , \tag{14d}$$

$$[J_i,P_j]=i\epsilon_{ijk}P_k\ , \tag{14e}$$

$$[J_i,K_j]=i\epsilon_{ijk}K_k\ , \tag{14f}$$

$$[H,K_j]=iP_j\ , \tag{14g}$$

$$[K_i,K_j]=-i\epsilon_{ijk}J_k\ , \tag{14h}$$

$$[P_i,K_j]=i\delta_{ij}H\ . \tag{14i}$$

For the sake of convenience, the above Lie algebra is to be referred to as the "Poincaré algebra" henceforth. The Poincaré algebra has been invoked in a classic paper of Krajcik and Foldy[9] for defining the relativistic center-of-mass variables for a system of relativistic particles with a *diagonalized* single-particle Hamiltonian. Recently, two of us[14] have also applied the formalism to the problem of recoil effects in a bag model. For a system of "noninteracting" Dirac particles with the single-particle Hamiltonian given by Eq.(10a), it is straightforward to choose the various generators,

$$H=\Sigma H^a\ , \tag{15a}$$

$$\mathbf{P}=\Sigma\mathbf{P}^a\ , \tag{15b}$$

$$\mathbf{J}=\Sigma\mathbf{J}^a\ , \tag{15c}$$

$$\mathbf{K}=\Sigma\mathbf{K}^a\ , \tag{15d}$$

where we have[14,9]

$$H^a=\boldsymbol{\alpha}^a\cdot\nabla^a/i+U_0^a+\beta^a(m_N+V^a)\ , \tag{16a}$$

$$\mathbf{P}^a=\nabla^a/i\ , \tag{16b}$$

$$\mathbf{J}^a=\mathbf{x}^a\times\nabla^a/i+\boldsymbol{\sigma}^a/2\ , \tag{16c}$$

$$\mathbf{K}^a=t\nabla^a/i+i\boldsymbol{\alpha}^a/2-\mathbf{x}^aH^a\ . \tag{16d}$$

It is important to bear in mind that the validity of the Poincaré algebra implies Lorentz covariance of the entire system, and so, the existence of the relativistic center-of-mass (c.m.) variables. However, it is the basic assumption of a shell model that the single-particle generators $(H^a,\mathbf{P}^a,\mathbf{J}^a,\mathbf{K}^a)$ can be found without violation of the Poincaré algebra and the generators of the entire system are given as simple sums of single-particle generators [i.e., Eqs. (15a)–(15d)]. Owing to the complexities associated with nuclei, such an assumption appears to be a sensible working hypothesis (or a reasonable approximation). A further detailed synthesis[14,9] indicates that, provided that U_0^a and V^a can be considered as "internal interactions" (as already assumed in practice), corrections of higher order in v/c (which are in fact negligible for practical purposes) need to be added to the boost operator $\mathbf{K}$. Granting the validity of such an assumption, we note of the possibility of performing a unitary transformation U on all the generators while maintaining the Poincaré algebra, Eqs. (14a)–(14i). In particular, it is of great interest to perform a unitary transformation which diagonalizes the single-particle Hamiltonian, Eq. (16a). To this end, we choose[8], to second order in v/c,

$$U=1+\frac{1}{2}\beta\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{m_N}-\frac{1}{8}\frac{\mathbf{p}^2}{m_N^2}-\frac{3}{16}\frac{\beta}{m_N^3}\boldsymbol{\alpha}\cdot\mathbf{p}\mathbf{p}^2+\frac{11}{128}\frac{\mathbf{p}^4}{m_N^4}$$
$$+\left\{-\beta V\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{2m_N^2}+\frac{i}{4m_N^2}\beta\boldsymbol{\alpha}\cdot[\nabla(V+\beta U_0)]\right\}$$
$$+\left\{-\frac{1}{16m_N^3}[\nabla^2(V+\beta U_0)]-\frac{i}{4m_N^3}(\nabla V)\cdot\mathbf{p}+\frac{1}{4m_N^3}V\mathbf{p}^2+\frac{1}{8m_N^3}\boldsymbol{\sigma}\cdot[\nabla(V-\beta U_0)]\times\mathbf{p}\right\}. \tag{17}$$

Here the particle index a has been dropped everywhere for the sake of simplicity. We obtain[8]

$$H'=UHU^\dagger$$
$$=\beta m_N+\left[\beta\left[\frac{\mathbf{p}^2}{2m_N}+V\right]+U_0\right]$$
$$+\left\{-\frac{1}{8m_N^3}\beta\mathbf{p}^4-\frac{1}{4m_N^2}\beta(\mathbf{p}^2V+V\mathbf{p}^2)+\frac{1}{8m_N^2}[(\nabla^2U_0)-\beta(\nabla^2V)]-\frac{1}{4m_N^2}\boldsymbol{\sigma}\cdot[(\nabla U_0)-\beta(\nabla V)]\times\mathbf{p}\right\}. \tag{18}$$

We recall[9,14] that the wave function of three-momentum $\mathbf{p}$, i.e., $\psi(\mathbf{p})(\{\mathbf{x}^a\})$, can be obtained from the wave function at rest, i.e., $\psi^{(0)}(\{\mathbf{x}^a\})$, as follows:

$$\psi^{(\mathbf{p})}(\{\mathbf{x}^a\})=\exp(-i\theta\mathbf{v}\cdot\mathbf{K})\psi^{(0)}(\{\mathbf{x}^a\}), \tag{19}$$

with $\mathbf{v}=\mathbf{p}/|\mathbf{p}|$, $\tanh\theta=|\mathbf{p}|/E$, and $E=(|\mathbf{p}|^2+M^2)^{1/2}$.

Since an additional power in p/m_N arises from the presence of θ, we need the boost operator $\mathbf{K}$ *only* to first order in v/c. To this end, we note

$$\mathbf{x}'\equiv U\mathbf{x}U^\dagger$$
$$=\mathbf{x}+(U,\mathbf{x})U^\dagger$$
$$=\mathbf{x}-\frac{i}{2m_N}\beta\boldsymbol{\alpha}-\frac{1}{4m_N^2}\boldsymbol{\sigma}\times\mathbf{p}+\frac{i}{2m_N}\frac{V}{m_N}\beta\boldsymbol{\alpha}+O(1/m_N^3), \tag{20a}$$

$$\mathbf{p}'\equiv U\mathbf{p}U^\dagger=\mathbf{p}[1+O(1/m_N^2)]. \tag{20b}$$

Accordingly, we find

$$\mathbf{K}'\equiv U\mathbf{K}U^\dagger=U[t\mathbf{p}-\tfrac{1}{2}(\mathbf{x}H+H\mathbf{x})]U^\dagger$$
$$=t\mathbf{p}'-\tfrac{1}{2}(\mathbf{x}'H'+H'\mathbf{x}'). \tag{21}$$

Or,

$$\mathbf{K}'=t\mathbf{p}-\mathbf{x}\left[\beta m_N+\beta\left[\frac{\mathbf{p}^2}{2m_N}+V\right]+U_0\right]+\frac{\beta}{4m_N}\boldsymbol{\sigma}\times\mathbf{p}+\frac{i}{2m_N}\beta\mathbf{p}+\frac{i}{2m_N}\beta\boldsymbol{\alpha}U_0+O(1/m_N^2). \tag{22}$$

It is of great interest to note that the term $i\boldsymbol{\alpha}/2$ appearing in Eq. (16d) has been removed upon diagonalization. This implies that recoil effects suitable for the Dirac Hamiltonian, Eq. (16a), are in fact very different from those anticipated for the diagonalized Hamiltonian, Eq. (18).

Finally, it is useful to rewrite Eq. (19) as follows:

$$\exp(-i\theta\mathbf{v}\cdot\mathbf{K}')\psi'^{(0)}(\{\mathbf{x}^a\})=\left[1-i\theta\mathbf{v}\cdot\mathbf{K}'-\frac{\theta^2}{2}(\mathbf{v}\cdot\mathbf{K}')^2+\frac{i\theta^3}{6}(\mathbf{v}\cdot\mathbf{K}')^3+\frac{\theta^4}{24}(\mathbf{v}\cdot\mathbf{K}')^4+\cdots\right]\psi'^{(0)}(\{\mathbf{x}^a\}), \tag{23a}$$

with $\psi'^{(0)}\equiv U\psi^{(0)}$.

Since the Hamiltonian has been diagonalized to the desired order in v/c, we may obtain an overall wave function $\psi^{(0)}(\{\mathbf{x}^a\})$ with each constituent wave function containing only the *normalized* upper component. In such a case, the matrix β can be neglected except for the $i\beta\boldsymbol{\alpha}$ term in Eq. (22) (which, in fact, is of order $m_N^{-3/2}$ and so can be neglected). Using the identity,

$$\sum_a\mathbf{p}^a\psi'^{(0)}(\{\mathbf{x}^a\})=0, \tag{23b}$$

we obtain, from Eqs. (19), (22), and (23a),

$$\psi'^{(\mathbf{p})}(\{\mathbf{x}^a\})=\exp\left[+i\theta\sum_a m_{\rm N}\mathbf{v}\cdot\mathbf{x}^a\right]\left[1-\frac{\theta^2}{2}tiAm_{\rm N}+\cdots\right]$$
$$\times\left\{1+i\theta\mathbf{v}\cdot\sum_a\left[\mathbf{x}^a\left[\frac{\mathbf{p}^{a2}}{2m_{\rm N}}+V^a+U_0^a\right]-\frac{1}{4m_{\rm N}}\sigma^a\times\mathbf{p}^a\right]\right.$$
$$\left.+\frac{\theta^2}{4}i\sum_a(\mathbf{v}\cdot\mathbf{x}^a\mathbf{v}\cdot\mathbf{p}^a+\mathbf{v}\cdot\mathbf{p}^a\mathbf{v}\cdot\mathbf{x}^a)+\cdots\right\}\psi'^{(0)}(\{\mathbf{x}^a\})\;. \tag{24}$$

As pointed out earlier in another paper,[14] the time dependent factor,

$$\exp[-itp^2/(2Am_{\rm N})]=1-\frac{\theta^2}{2}tiAm_{\rm N}+\cdots \tag{25}$$

is precisely what is needed to give rise to the energy-conservation δ function. Furthermore, we may use either the expression,

$$\exp\left[i\theta\sum_a m_{\rm N}\mathbf{v}\cdot\mathbf{x}^a\right] \tag{26a}$$

or the expression

$$\exp\left[i\theta\sum_a\mathbf{v}\cdot\mathbf{x}^a\left[m_{\rm N}+\frac{\mathbf{p}^{a2}}{2m_{\rm N}}+V^a+U_0^a\right]\right] \tag{26b}$$

to generate the three-momentum-conservation δ function. Of course, these two choices, either of which is incredibly close to Eq. (4), will result in slightly different residual $S_K^{\rm res}$ [Eq. (13)]. The remaining terms in Eq. (24) yield, for the required transformation on the initial state ($\mathbf{p}^{(i)}=+\mathbf{q}/2$),

$$S_K=1-\frac{i}{8Am_{\rm N}^2}\sigma\times\mathbf{p}\cdot\mathbf{q}+\frac{i}{16A^2m_{\rm N}^2}\times[(\mathbf{q}\cdot\mathbf{r})(\mathbf{q}\cdot\mathbf{p})+(\mathbf{q}\cdot\mathbf{p})(\mathbf{q}\cdot\mathbf{r})]+S_K^{\rm res}\;, \tag{27}$$

which agrees exactly with Eq. (13) with the residual $S_K^{\rm res}$ determined from the *mismatch* between Eq. (4) and Eq. (26b). It is unlikely that such a mismatch will result in a residual $S_K^{\rm res}$ sizable compared to the terms explicitly given. Thus, we choose to neglect such an effect in our numerical example.

In summary, the formalism[9,14] developed on the basis of the Poincaré algebra provides a coherent language for treating recoil effects in the relativistically corrected impulse approximation.[1]

III. A NUMERICAL EXAMPLE

As an illustrative numerical example, we consider the ground state of the ^{15}N nucleus, which has the following properties:[15]

$$J^\pi=\tfrac{1}{2}^-,\quad T=\tfrac{1}{2}\;,$$
$$M=13\,972.632\ {\rm MeV}\;,$$
$$\mu=-0.283\,189\,\mu_{\rm N}\;, \tag{28}$$
$$\langle r^2\rangle^{1/2}=2.580\pm0.026\ {\rm fm}\;.$$

In view of the given spin structure, we define the matrix element of the electromagnetic current $J_\lambda(x)$ between the on-shell nuclear states of four-momenta p and p',

$$\langle{}^{15}{\rm N}(p')\,|\,J_\lambda(0)\,|\,{}^{15}{\rm N}(p)\rangle=iU^\dagger(p')\gamma_4\left[\gamma_\lambda F_1(q^2)+\frac{\sigma_{\lambda\eta}q_\eta}{2m_{\rm p}}F_2(q^2)\right]U(p)\;, \tag{29}$$

with

$$q_\lambda=(p-p')_\lambda,\gamma_\lambda^\dagger=\gamma_\lambda\;,$$
$$\gamma_\lambda\gamma_\eta+\gamma_\eta\gamma_\lambda=2\delta_{\lambda\eta}\;,$$
$$\sigma_{\lambda\eta}=(2i)^{-1}(\gamma_\lambda\gamma_\eta-\gamma_\eta\gamma_\lambda)\;,$$

and $m_{\rm p}$ is the proton mass. $F_1(q^2)$ and $F_2(q^2)$ are, respectively, the nuclear charge and anomalous-magnetic-moment form factors for the ^{15}N nucleus. Furthermore, we impose the normalization condition on the *nuclear* Dirac spinors,

$$U^\dagger(p)U(p)=U^\dagger(p')U(p')=1\;, \tag{30}$$

so that nuclear wave functions need to be normalized accordingly.

We follow the procedure developed by Hwang and Ernst[16] to calculate the nuclear electromagnetic form factors. We define

$$I_0\equiv\frac{2E}{E+M}\langle{}^{15}{\rm N}(\mathbf{p}'=-\mathbf{q}/2;\uparrow)\,|\,J_0(0)\,|\,{}^{15}{\rm N}(\mathbf{p}=\mathbf{q}/2;\uparrow)\rangle\;, \tag{31a}$$

$$I_x\equiv\frac{2E}{E+M}\frac{2m_{\rm p}}{iq}\left\langle{}^{15}{\rm N}\left[\mathbf{p}'=-\frac{q\mathbf{y}}{2};\uparrow\right]\,|\,J_x(0)\,|\,{}^{15}{\rm N}\left[\mathbf{p}=\frac{q\mathbf{y}}{2};\uparrow\right]\right\rangle\;, \tag{31b}$$

with $E=[M+(q^2/4)]^{1/2}$ and $q^2 \equiv \mathbf{q}^2-q_0^2=\mathbf{q}^2$. Both I_0 and I_x can be evaluated for a given ^{15}N nuclear wave function. Following Hwang and Ernst,[16] it is possible to relate the Sachs form factors to the above matrix elements and their q^2-dependence behaviors,

$$I_0=I_0^0+I_0^1\frac{q^2}{m_p^2}+\cdots , \tag{32a}$$

$$I_x=I_x^0+I_x^1\frac{q^2}{m_p^2}+\cdots , \tag{32b}$$

$$G_E(0)=I_0^0 \tag{33a}$$

$$G_M(0)=I_x^0 , \tag{33b}$$

$$\langle r_E^2\rangle=-m_p^{-2}\left[\frac{3}{8}\frac{m_p^2}{M^2}+6\frac{I_0^1}{I_0^0}\right] , \tag{33c}$$

$$\langle r_M^2\rangle=-m_p^{-2}\left[\frac{3}{8}\frac{m_p^2}{M^2}+6\frac{I_x^1}{I_x^0}\right] . \tag{33d}$$

We make the usual identification,

$$\mu=G_M(0),\quad \langle r^2\rangle=\langle r_E^2\rangle . \tag{34}$$

In the remainder of this paper, we wish to invoke the RCIA,[1] modified to incorporate recoil effects addressed in this paper, to evaluate both μ and $\langle r^2\rangle$.

With recoil effects specified by Eq. (27) instead of Eq. (9b) in Ref. 1, the relativistically-corrected impulse approximation (RCIA) for the nuclear electromagnetic current is given by[1]

$$\langle N_f(p')\,|\,\mathbf{J}(0)\,|\,N_i(p)\rangle\,|_{\text{RCIA}}=\left\langle\psi_f\left|\sum_{a=1}^{A}e^{-i\mathbf{q}\cdot\mathbf{r}^a}\left[f_V^a(q^2)\left[\frac{F(\mathbf{r}^a)}{2m_N}\left[2\zeta\frac{\nabla^a}{i}-\zeta\mathbf{q}+i\mathbf{q}\times\boldsymbol{\sigma}^a\right]\right.\right.\right.\right.$$
$$\left.-\frac{i}{2m_N}\{[\nabla F(\mathbf{r}^a)]+i\boldsymbol{\sigma}^a\times[\nabla F(\mathbf{r}^a)]\}\right]$$
$$\left.\left.+f_M^a(q^2)\frac{i}{2m_N}\mathbf{q}\times\boldsymbol{\sigma}^a\right]\right|\psi_i\Big\rangle , \tag{35a}$$

$$\langle N_f(p')\,|\,J_0(0)\,|\,N_i(p)\rangle\,|_{\text{RCIA}}=\left\langle\psi_f\left|\sum_{a=1}^{A}e^{-i\mathbf{q}\cdot\mathbf{r}^a}\left[f_V^a(q^2)\left[1+\frac{\delta_V^a}{8m_N^2}\right]+f_M^a(q^2)\frac{\delta_m^a}{4m_N^2}\right]\right|\psi_i\right\rangle . \tag{35b}$$

Here the various entities entering Eqs. (35a) and (35b) are defined by[1]

$$f_V(q^2)=\tfrac{1}{2}e_S(q^2)+\frac{\tau_3}{2}e_V(q^2) , \tag{36a}$$

$$f_M(q^2)=\tfrac{1}{2}\mu_S(q^2)+\frac{\tau_3}{2}\mu_V(q^2) , \tag{36b}$$

$$e_S(0)=e_V(0)=1 , \tag{36c}$$

$$\mu_S(0)+\mu_V(0)=2\mu_p(0)=2\times 1.793 , \tag{36d}$$

$$\mu_S(0)-\mu_V(0)=2\mu_n(0)=2\times(-1.913) , \tag{36e}$$

$$F(\mathbf{r})=2m_N[2m_N+V(\mathbf{r})-U_0(\mathbf{r})]^{-1} , \tag{36f}$$

$$G(\mathbf{r})=[F(\mathbf{r})]^2 , \tag{36g}$$

$$\delta_V=+(i\mathbf{q}\cdot\mathbf{r})\mathbf{q}^2/A^2+\mathbf{q}^2/A^2-(2i\zeta/A)\boldsymbol{\sigma}\times\frac{\nabla}{i}\cdot\mathbf{q}$$
$$-G(\mathbf{r})\left[\mathbf{q}^2+2i\zeta\boldsymbol{\sigma}\cdot\mathbf{q}\times\frac{\nabla}{i}\right]$$
$$+\{-i[\nabla G(\mathbf{r})]\cdot\mathbf{q}+\boldsymbol{\sigma}\cdot[\nabla G(\mathbf{r})]\times\mathbf{q}\} , \tag{36h}$$

$$\delta_M=-F(\mathbf{r})\left[\mathbf{q}^2+2i\zeta\boldsymbol{\sigma}\cdot\mathbf{q}\times\frac{\nabla}{i}\right]$$
$$+\{-i\mathbf{q}\cdot[\nabla F(\mathbf{r})]+\boldsymbol{\sigma}\cdot[\nabla F(\mathbf{r})]\times\mathbf{q}\} , \tag{36i}$$

$$\zeta=1-\frac{1}{A} . \tag{36j}$$

Following the existing literature,[17] we assume a pure $1p_{1/2}$ configuration for the ^{15}N nucleus. We obtain, from Eqs. (31)–(36),

$$\mu=\mu^c+\mu^s+\mu^m , \tag{37a}$$

$$\mu^c=\tfrac{1}{3}2\tfrac{15}{16}\int_0^\infty dr\,r^2R^2(r)F(r) , \tag{36b}$$

$$\mu^s=-\tfrac{1}{3}\int_0^\infty dr\,r^2R^2(r)[F(r)+1.793], \tag{37c}$$

$$\mu^m=-\tfrac{1}{3}\int_0^\infty dr\,r^2R^2(r)rF'(r) . \tag{37d}$$

Here $R(r)$ is the radial part of the normalized $1p_{1/2}$ wave function, i.e.,

$$R(r)\sum_s\langle 1,m-s;\tfrac{1}{2},s\,|\,\tfrac{1}{2},m\rangle Y_{1,m-s}(\mathbf{r})\zeta_s .$$

μ^c, μ^s, and μ^m are due to, respectively, the convection current ($\propto \boldsymbol{\nabla}/i$), the spin current ($\propto i\mathbf{q}\times\boldsymbol{\sigma}$), and the medium-induced current $\{\propto i\boldsymbol{\sigma}\times[\boldsymbol{\nabla}F(r)]\}$. Numerically, we find, using the harmonic oscillator (HO) wave function with $b=1.64$ fm and V, U_0 as in Ref. 8,

$$\mu^c=0.859,\quad \mu^s=-1.056,\quad \mu^m=0.165\ ,$$
$$\mu=-0.032\ . \tag{38}$$

Accordingly, our small result for the magnetic moment μ is not very different from the value obtained by Miller[17] or Bawin *et al.*[17] This suggests that the RCIA is a good approximation to the original Dirac-phenomenology approach. This result is not unexpected in view of the other paper.[8] Unfortunately, the medium-induced current, which is important only in the surface region, yields a contribution which tends to upset the agreement between theory and experiment. Other effects, such as meson-exchange currents, are needed to assess whether the Dirac picture indeed fails.

Analogously, the RCIA expression for the charge density, Eq. (35b), yields, for a pure $1p_{1/2}$ configuration and to order $q^2/m_{\rm N}^2$,

$$\begin{aligned}\langle {}^{15}{\rm N}(p')\,|\,J_0(0)\,|\,{}^{15}{\rm N}(p)\rangle = {} & e_p(q^2)\int_0^\infty dr\, r^2R^2(r)(1-\tfrac{1}{6}q^2r^2)\\ & -\frac{q^2}{8m_{\rm N}^2}\int_0^\infty dr\, r^2R^2(r)\left[-\frac{1}{A^2}+G(r)+2\times 1.793F(r)\right]\\ & +\frac{q^2}{8m_{\rm N}^2}\frac{4}{3}\frac{15}{16}\int_0^\infty dr\, r^2R^2(r)\left[-\frac{1}{A}+G(r)+2\times 1.793F(r)\right]\\ & -\frac{q^2}{8m_{\rm N}^2}\frac{1}{3}\int_0^\infty dr\, r^2R^2(r)r[G'(r)+2\times 1.793F'(r)]\ .\end{aligned} \tag{39}$$

Here the second, third, and fourth terms arise, respectively, from the q^2, $\boldsymbol{\sigma}\times\mathbf{p}\cdot\mathbf{q}$, and $\mathbf{q}\cdot[\boldsymbol{\nabla}F(r)]$ terms in δ_V and δ_M [Eqs. (36h) and (36i)]. It is clear from Eq. (39) that recoil effects are negligible except for light nuclei. Numerically, Eq. (39) yields, with the HO wave function as given above,

$$\langle r_E^2\rangle = \langle r_E^2\rangle_{\rm p} + 6.64\ {\rm fm}^2\ , \tag{40}$$

which is to be compared with the experimental value in Eq. (28). The proton charge radius $\langle r_E^2\rangle_{\rm p}$ describes the q^2 dependence of $e_p(q^2)$ which appears in the first term of Eq. (39). We note that the oscillator parameter b can be adjusted to make a prediction in agreement with the experimental value [Eq. (28)] without affecting appreciably the prediction on the magnetic moment. We also note that the medium-induced charge density {which is proportional to $[\boldsymbol{\nabla}F(r)]$ or $[\boldsymbol{\nabla}G(r)]$} is again sensitive to the surface region but is of negligible importance (only -0.5% of the overall contribution).

In closing our numerical example, it is useful to note that Eqs. (17)—(22) establish the link (or, in fact, the equivalence) between the nondiagonalized (original) Dirac Hamiltonian and the unitarily-diagonalized Hamiltonian, with the latter being assumed for the RCIA used in the present section. In particular, the smallness of recoil effects in the RCIA treatment of the above numerical example *cannot* be taken as to imply that recoil effects in the relativistic mean-field calculation for the same problem are also negligible. As a matter of fact, Eqs. (17)—(22) indicate that the boost operator, as described by Eqs. (15d) and (16d) [with the single-particle boost operator (16d) appearing exactly in Eq. (21)], must be used in such relativistic mean-field or Dirac-phenomenology calculations. This boost operator [Eq. (16d)] contains both a term linear in $\boldsymbol{\alpha}$ [as already given in Eq. (10b)] and terms involving the potentials. For the illustrative example discussed above, numerical importance of these corrections are currently under detailed investigation. However, any such effects are suppressed by a factor $1/A$ [as implied by Eq. (19) with M the *nuclear* mass], and so, are in general negligible except for very light nuclei or for some exceptional cases where delicate cancellation occurs or measurements are very accurate. It is clearly of great interest to pin down cases where inclusion of such relativistic recoil effects becomes essential. Although this task is well beyond the main focus of the present paper, it is gratifying to note that the equivalence between the nondiagonalized Dirac picture (as used in relativistic mean-field calculations) and the unitarily-diagonalized picture (as used in connection with RCIA) is established reasonably well by Eqs. (17)—(22) and, in both cases, recoil effects are understood at least to the leading order in v/c.

IV. SUMMARY

We have invoked the formalism developed by Krajcik and Foldy to treat recoil effects in the relativistically corrected impulse approximation in the presence of a local central potential. Whereas recoil effects in such impulse approximations remain negligible, recoil corrections in Dirac phenomenology may be of numerical significance for very light nuclei or for exceptional cases where delicate cancellation occurs or accurate measurements have been made. As an illustrative numerical example, we have considered the magnetic moment and charge radius of the ^{15}N nucleus.

ACKNOWLEDGMENTS

We acknowledge Prof. G. E. Walker for stimulating discussions. Two of us (Gattone and Hwang) are grateful to Prof. M. H. Macfarlane for bringing to their attention Refs. 4 and 5. This work was supported in part by the National Science Foundation and the U. S. Department of Energy.

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