# Projection operators, quantum field theory, and the pion-nucleon system

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(Received 16 July 1984)

An analysis is carried out of a $\pi$-N-$\Delta$ field theory using an extension of the Feshbach projection operator technique in conjunction with a set of identities which relate Green's functions acting in subspaces whose projection operators are related by the action of the meson creation and annihilation operators. Exact relations are derived for the propagators, self-energies, and vertex functions that arise in the field theory. A separation of the pion-nucleon elastic scattering amplitudes into their one-particle irreducible and one-particle reducible parts is obtained. An exact set of coupled integral equations for the pion-nucleon amplitudes is derived. When a certain approximation for the effective potentials in these equations is made, it is found that the one-particle irreducible amplitudes satisfy a set of three-particle equations that are exactly analogous to the quasiparticle equations of Alt, Grassberger, and Sandhas. Moreover, the one-particle reducible parts can be obtained from the solutions of the three-particle equations. These equations are nonlinear in that the kernels are determined by the solutions. An approximate set of linear equations is obtained whose solutions satisfy two-particle and three-particle unitarity and have reasonable analytic structure. An iteration scheme for improving on the linear approximation is presented.

## I. INTRODUCTION

Recently[1] the author has succeeded in deriving a set of three-particle equations for the coupled N$\pi$-N$\pi\pi$ system starting from the Hamiltonian of the Chew-Low theory.[2,3] This Hamiltonian describes the interaction of pions with a static nucleon through the process N$\rightleftharpoons$N+$\pi$. The equations were constructed so that the solutions satisfy two-particle and three-particle unitarity, as well as the discontinuity relations for the production amplitudes (N+$\pi$$\rightarrow$N+2$\pi$) in the subenergy variables, which are the final state energies of the pions.

The method used to obtain the three-particle equations is a dispersion relation technique based on unitarity and analyticity in the subenergy and total energy variables, and was developed by analyzing Lee[4] model-type field theories.[5,6] The technique is closely related to an approach used by other workers[7,8] to derive three-particle equations by imposing subenergy unitarity and analyticity on the isobar expansion for production amplitudes. The author's approach differs in that the isobar expansion is not assumed. An expansion similar to it emerges from the analysis. Also, the subenergy dispersion relations are supplemented by a total energy dispersion relation in order to determine the modifications of the standard Amado-Lovelace equations[9,10] that are necessary to account for single nucleon intermediate states.

One of the shortcomings of the field theory used by the author to derive three-particle equations for the N$\pi$-N$\pi\pi$ system is that it does not treat the $\Delta$ resonance on the same footing as the nucleon. According to the modern quark picture the $\Delta$ is just as "elementary" as the N. In recent years, effective field theories which treat the $\pi$, N, and $\Delta$ as elementary have been developed.[11-13] These theories have been obtained by coupling pions to quarks in the framework of either the MIT bag model[14] or the constituent quark model.[15] The so-called chiral bag models[11,12] are obtained by introducing a pion field into the original bag model Lagrangian so as to restore chiral symmetry.[16,17] The justification for including the pion field in the constituent quark model[13] is simply the fact that pions interact with baryons. In either approach, the three-quark core of a baryon serves essentially as an extended, static source for the pions. The model for the core determines the form factors or cutoff functions for the various vertices ($\pi$NN, $\pi$N$\Delta$, $\pi\Delta\Delta$, ...) that occur in the effective field theory.

Even though the bare N and $\Delta$ appear in these field theories on an equal footing, the physical or dressed particles are, of course, quite different in that the N is stable while the $\Delta$ is not. Unstable particles are somewhat awkward to handle in dispersion relations, because they deal with the physical states of a system, and unstable particles do not appear in the spectrum of these states. From the work of Luke[18] on unstable particles in the Lee model, it is clear that in principle the dispersion relation approach developed for a $\pi$-N field theory can be extended to a $\pi$-N-$\Delta$ field theory by assuming that the $\Delta$ is stable and then analytically continuing in the mass of the $\Delta$ to an unstable value. A careful justification for this procedure is fairly difficult, since the $\Delta$ sits quite close to the two-pion threshold. For this reason we will develop a different method for analyzing a $\pi$-N-$\Delta$ field theory. At the end we will return to the use of dispersion relations to justify an approximate form of the exact equations that we will derive for the pion-nucleon system.

The method that we will develop is based on the Feshbach projection operator formalism,[19] which was introduced to treat nuclear reactions with many channels present. It was originally formulated under the assumption that the number of "elementary" particles involved is conserved. Here, since we are dealing with a meson field theory, this is no longer true.

An extension of the Feshbach formalism[19] to allow for an indefinite number of mesons has been developed by Mizutani and Koltun for the pion-deuteron system[20] and for general pion-nucleus reactions.[21] Their techniques, which are discussed extensively in Ref. 22, bear some resemblance to Okubo's[23] use of projection operators in a discussion of the Tamm-Dancoff method in meson theory. Their approach is consistent with meson field theory, but is developed in terms of an effective Schrödinger equation so as to remain in close contact with conventional nuclear physics. The projection operators in their formalism deal with subspaces involving physical or dressed nucleons and pions.

Stingl and Stelbovics[24] have used the Feshbach formalism in conjunction with the Chew-Low Hamiltonian[2,3] to develop a model for the pion-deuteron system. In their approach, it is assumed that the Hamiltonian acts in the restricted space spanned by free $\mathrm{N}+\mathrm{N}$, $\mathrm{N}+\mathrm{N}+\pi$, and $\mathrm{N}+\mathrm{N}+2\pi$ states. The $\mathrm{N}+\mathrm{N}$ and $\mathrm{N}+\mathrm{N}+2\pi$ channels are eliminated so as to obtain an effective three-particle problem.

Projection operator techniques have also been used to analyze several sectors of the Lee model. The sectors studied correspond to simple models for the pion-nucleon,[25] pion-deuteron,[25] and three-nucleon systems,[26] as well as for nuclear matter.[27] In contrast to Mizutani and Koltun,[20,21] the projection operators used in these analyses deal with subspaces involving bare particles.

The projection operator relations that we will use here are developed in Sec. II. They are a fairly straightforward extension of the equations developed in the Green's function formalism of Fonda and Newton.[28,29] These relations make it possible to express a Green's function $G^{\alpha}$ defined in terms of a Hamiltonian projected onto a subspace $\alpha$ to its projection $P_{\beta}G^{\alpha}P_{\beta}$ onto a smaller subspace $\beta$, and to a Green's function $G^{\gamma}$ defined in terms of a Hamiltonian projected onto a subspace $\gamma$ obtained by removing subspace $\beta$ from subspace $\alpha$. The relations developed also make it possible to obtain the projected Green's function $P_{\beta}G^{\alpha}P_{\beta}$ from a pseudo-Hamiltonian that acts only in the subspace $\beta$. The formalism presented in Sec. II is of a very general nature in that the specific form of the Hamiltonian is irrelevant.

In Sec. III we will develop a set of relations that are peculiar to quantum field theory, in that they involve the meson annihilation and creation operators, $a(p)$ and $a^{\dagger}(p)$. The results of this section are based on a very simple observation. If, for example, $P_s$ is the projection operator for the space spanned by bare one-meson—one-fermion states, $|\pi\mathrm{N}\rangle$ and $|\pi\Delta\rangle$, and $P_a$ projects onto the space of bare one fermion states, $|\mathrm{N}\rangle$ and $|\Delta\rangle$, then $a(p)P_s=P_a a(p)$. Thus the meson operators take us from one subspace to another. The important consequence of this is that relationships exist between Green's functions defined on various subspaces, which are induced by the action of the meson creation and annihilation operators. These relations are extremely useful and greatly facilitate the analysis of a $\pi$-N-$\Delta$ field theory. Here we shall apply them to pion-nucleon scattering. In the future we will use them to analyze pion-nucleus reactions.

Section IV, which deals with the pion-nucleon system, is divided into a number of subsections. In Sec. IV A the underlying field theory is briefly discussed. We do not give the details of the various vertices involved ($\pi$NN, $\pi$N$\Delta$, and $\pi\Delta\Delta$) as they are irrelevant for the subsequent development. A specific version of the type of model we have in mind is given in Ref. 30. Expressions for the N and $\Delta$ propagators, the various vertex functions, and relationships among them are obtained in Secs. IV B and IV C.

In Sec. IV D we define Green's functions in the subspace of $|\pi\mathrm{N}\rangle$ and $|\pi\Delta\rangle$ states and show how they are related to the pion-nucleon scattering amplitudes. We also show how to separate out the one-particle reducible parts of these amplitudes, i.e., the parts that arise from single nucleon and single $\Delta$ intermediate states. These contributions are expressed in terms of the N and $\Delta$ propagators and the vertex functions. Exact integral equations are obtained for the Green's functions and pion-nucleon scattering amplitudes. The equations for the amplitudes are similar in structure to the Amado-Lovelace equations,[9,10] but with the modifications due to N and $\Delta$ intermediate states present. The propagators in these equations are simply related to the fermion propagators, while the effective, energy-dependent potentials that appear represent fairly sophisticated generalizations of the crossed Born term potentials in the original Amado-Lovelace equations.[9,10] These new potentials make possible the treatment of unstable particles, such as the $\Delta$, within the framework of three-particle theory. We also derive exact expressions for the vertex functions in terms of the one-particle irreducible amplitudes. Section IV D concludes with an analysis of the relationship between the equations obtained here and those of the well-known two-potential formalism of Gell-Mann and Goldberger.[31]

In Sec. IV E we analyze the effective potentials and the propagators that appear in the coupled, one-variable integral equations for the pion-nucleon scattering amplitudes. We develop a particular approximation for the potentials which lead to equations for the coupled N$\pi$-N$\pi\pi$ system, which are analogous to those of the quasiparticle method for three-particle systems developed by Alt, Grassberger, and Sandhas (AGS).[32] The analysis shows that with some modest modifications and extensions standard three-particle equations can be used to calculate the one-particle irreducible part of the pion-nucleon scattering amplitude. The rest of the amplitude can be obtained by carrying out integrations with the solutions of the three-particle equations. The three-particle equations are nonlinear in that their input is given by the solutions of the equations. Section IV E concludes with the development of approximate linear equations, whose solutions satisfy two-particle and three-particle unitarity and have reasonable analytic structure. Section V gives a discussion of the results, a comparison with other relevant work, and suggestions for the future.

## II. PROJECTION OPERATOR FORMALISM

We begin by introducing a set of projection operators labeled with subscripts and superscripts. The projection operator $P_\rho$ includes the states denoted by the cover index $\rho$, while $P^\rho$ includes everything but the states labeled by $\rho$. Obviously, the two are related by

$$P_\rho + P^\rho = 1 \ . \tag{2.1}$$

These operators we combine with the Hamiltonian operator $H$ to define the operators

$$\begin{aligned} &H_{\lambda\rho} = P_\lambda H P_\rho \ , \quad H_\lambda{}^\rho = P_\lambda H P^\rho \ , \\ &H^\lambda{}_\rho = P^\lambda H P_\rho \ , \quad H^{\lambda\rho} = P^\lambda H P^\rho \ . \end{aligned} \tag{2.2}$$

We also define Green's functions according to

$$G(z) = \frac{1}{z-H} \ , \tag{2.3a}$$

$$G^\lambda(z) = \frac{P^\lambda}{z - H^{\lambda\lambda}} \ , \tag{2.3b}$$

$$G_\lambda(z) = \frac{P_\lambda}{z - H_{\lambda\lambda}} \ , \tag{2.3c}$$

where $z$ is a complex parameter.

We now derive relations between various Green's functions defined as in (2.3). We start with a subspace whose projection operator is $P^\alpha$ and decompose it into orthogonal projection operators $P_\beta$ and $P^\gamma$ according to

$$P_\beta + P^\gamma = P^\alpha \ , \tag{2.4a}$$

which by (2.1) implies that

$$P^\gamma = 1 - P_\alpha - P_\beta = P^{\alpha\beta} \ . \tag{2.4b}$$

From (2.2) and (2.4a), we can write

$$H^{\alpha\alpha} = H_{\beta\beta} + H^{\gamma\gamma} + H_\beta{}^\gamma + H^\gamma{}_\beta \ , \tag{2.5}$$

which when used in $G^\alpha(z)$ leads to the identity

$$G^\alpha(z) = \frac{P^\alpha}{z - H_{\beta\beta} - H^{\gamma\gamma}} [1 + (H_\beta{}^\gamma + H^\gamma{}_\beta) G^\alpha(z)] \ . \tag{2.6}$$

Using the fact that $P_\beta$ and $P^\gamma$ are orthogonal, it is straightforward to show that

$$(z - H_{\beta\beta} - H^{\gamma\gamma})[G_\beta(z) + G^\gamma(z)] = P^\alpha \ , \tag{2.7}$$

and therefore

$$\frac{P^\alpha}{z - H_{\beta\beta} - H^{\gamma\gamma}} = G_\beta(z) + G^\gamma(z) \ . \tag{2.8}$$

Putting this in (2.6), we obtain

$$P_\beta G^\alpha(z) = G_\beta(z) + G_\beta(z) H_\beta{}^\gamma P^\gamma G^\alpha(z) \ , \tag{2.9a}$$

$$P^\gamma G^\alpha(z) = G^\gamma(z) + G^\gamma(z) H^\gamma{}_\beta P_\beta G^\alpha(z) \ , \tag{2.9b}$$

which, with a little algebra, can be combined to give

$$P_\beta G^\alpha(z) = g_\beta^\alpha(z)[1 + H_\beta{}^\gamma G^\gamma(z)] \ , \tag{2.10}$$

where

$$g_\beta^\alpha(z) = \frac{P_\beta}{z - H_\beta^\alpha(z)} \ , \tag{2.11}$$

with

$$H_\beta^\alpha(z) = H_{\beta\beta} + H_\beta{}^\gamma G^\gamma(z) H^\gamma{}_\beta \ . \tag{2.12}$$

At first sight the notation $H_\beta^\alpha$ appears unnatural, but it should be kept in mind that according to (2.4b) the index $\gamma$ is determined by $\beta$ and $\alpha$. The operator $H_\beta^\alpha(z)$ is a pseudo-Hamiltonian that acts in the subspace $\beta$.

Putting (2.10) into (2.9b), and using (2.4a), we arrive at the important identity

$$G^\alpha(z) = G^\gamma(z) + [1 + G^\gamma(z) H^\gamma{}_\beta] g_\beta^\alpha(z) [1 + H_\beta{}^\gamma G^\gamma(z)] \ . \tag{2.13}$$

Using the fact that $P_\beta$ and $P^\gamma$ are orthogonal, we find from (2.13) that

$$g_\beta^\alpha(z) = P_\beta G^\alpha(z) P_\beta \ . \tag{2.14}$$

Equations (2.13) and (2.14) relate the Green's function $G^\alpha(z)$, which acts in the subspace of $P^\alpha$, to its projection, $g_\beta^\alpha(z)$, onto the smaller subspace $\beta$, and to a Green's function $G^\gamma(z)$ which acts in the subspace of $P^\gamma$, obtained by removing $P_\beta$ from $P^\alpha$. According to (2.11), $g_\beta^\alpha(z)$ is the Green's function for the pseudo-Hamiltonian given by (2.12). The first term in this pseudo-Hamiltonian is simply the true Hamiltonian $H$ projected onto the subspace $\beta$, while the second term describes a transition from this subspace to the subspace of $P^\gamma = P^\alpha - P_\beta$, propagation there according to the Hamiltonian $H^{\gamma\gamma}$, and return to subspace $\beta$.

As is well known,[19] the pseudo-Hamiltonian (2.12) with $P^\alpha = 1$ arises when the Schrödinger equation

$$(E - H) \mid \psi \rangle = 0 \tag{2.15}$$

is replaced by an effective equation in the subspace $\beta$. If we let

$$H_\beta(z) = H_\beta^\alpha(z) \tag{2.16a}$$

with

$$P^\alpha = 1 \ , \tag{2.16b}$$

the effective Schrödinger equation is

$$[E - H_\beta(E \pm i\epsilon)] P_\beta \mid \psi \rangle_\pm = 0 \ . \tag{2.17}$$

The complete state vector is related to its projection onto the subspace $\beta$ by

$$\mid \psi \rangle_\pm = [1 + G^\beta(E \pm i\epsilon) H^\beta{}_\beta] P_\beta \mid \psi \rangle_\pm \ , \tag{2.18a}$$

with

$$P^\beta = 1 - P_\beta \ . \tag{2.18b}$$

All of the relations in the projection operator formalism are of a very general nature in that the structure of the Hamiltonian has played no role whatsoever. In the next section we shall derive a set of identities which greatly facilitate the application of this formalism to quantum field theory.

## III. USEFUL IDENTITIES

We assume that the system of interest consists of bosons and fermions whose dynamics is determined by a quantum field theory Hamiltonian

$$H = H_0 + H_1 , \tag{3.1}$$

where $H_0$ is the free Hamiltonian and $H_1$ contains the interactions. We introduce a set of operators, $a^\dagger(p)$ and $a(p)$, which create and annihilate bosons labeled with the cover index $p$. These operators satisfy the commutation rule

$$[a(p), a^\dagger(q)] = \delta(p,q) , \tag{3.2}$$

with their other commutators zero. Here $\delta(p,q)$ is a product of Dirac delta functions and Kronecker delta symbols. We assume that

$$[H_0, a^\dagger(p)] = \omega_p a^\dagger(p) , \tag{3.3a}$$

$$[H_1, a^\dagger(p)] = J(p) , \tag{3.3b}$$

where $\omega_p$ is the energy of meson $p$, and $J(p)$ is an operator which depends only on fermion creation and annihilation operators, and hence commutes with $a(p)$ and $a^\dagger(p)$.

Our purpose here is to derive identities which describe the action of $a(p)$ and $a^\dagger(p)$ on the projection operators and Green's functions of Sec. II. We begin by assuming $P_\rho$ and $P_\sigma$ are two projection operators related by

$$a(p) P_\rho = P_\sigma a(p) . \tag{3.4}$$

We shall see several examples of this in Sec. IV. A simple example is where $P_\rho$ projects onto a one-boson—one-fermion subspace and $P_\sigma$ projects onto the one-fermion subspace. Using (3.3), (3.4), and the definitions (2.2), it is straightforward to carry out the following manipulations.

$$\begin{aligned} a(p) H_{\rho\rho} &= P_\sigma a(p) H P_\rho , \\ &= P_\sigma [J^\dagger(p) + (\omega_p + H) a(p)] P_\rho , \\ &= P_\sigma J^\dagger(p) P_\rho + (\omega_p + H_{\sigma\sigma}) a(p) P_\rho , \\ P_\sigma a(p)(H_{\rho\rho} - z) &= P_\sigma J^\dagger(p) P_\rho + (\omega_p + H_{\sigma\sigma} - z) a(p) P_\rho . \end{aligned}$$

Multiplying the last line from the left by $(z - \omega_p - H_{\sigma\sigma})^{-1}$, from the right by $(z - H_{\rho\rho})^{-1}$, and using (2.3), we obtain

$$a(p) G_\rho(z) = G_\sigma(z - \omega_p)[a(p) + J^\dagger(p) G_\rho(z)] , \tag{3.5a}$$

$$G_\rho(z) a^\dagger(p) = [a^\dagger(p) + G_\rho(z) J(p)] G_\sigma(z - \omega_p) , \tag{3.5b}$$

where (3.5b) follows from (3.5a) by replacing $z$ with $z^*$ and taking the adjoint. Using (3.3b) and (3.5) it is easy to show that

$$a(p)[1 + H_1 G_\rho(z)] = [1 + H_1 G_\sigma(z - \omega_p)] \times [a(p) + J^\dagger(p) G_\rho(z)] , \tag{3.6a}$$

$$[1 + G_\rho(z) H_1] a^\dagger(p) = [a^\dagger(p) + G_\rho(z) J(p)] \times [1 + G_\sigma(z - \omega_p) H_1] . \tag{3.6b}$$

We assume that there is a third projection operator $P_\tau$ related to $P_\sigma$ by

$$a(p) P_\sigma = P_\tau a(p) . \tag{3.7}$$

Using this, (3.3b) and (3.5), we find

$$\begin{aligned} a(p) G_\rho(z) a^\dagger(q) = {} & G_\sigma(z - \omega_p) \delta(p,q) + [a^\dagger(q) + G_\sigma(z - \omega_p) J(q)] G_\tau(z - \omega_p - \omega_q) \\ & \times [a(p) + J^\dagger(p) G_\sigma(z - \omega_q)] + G_\sigma(z - \omega_p) J^\dagger(p) G_\rho(z) J(q) G_\sigma(z - \omega_q) . \end{aligned} \tag{3.8}$$

Finally, with the help of (3.3b), (3.5), and (3.8), we arrive at the fairly complicated, but very useful identity

$$\begin{aligned} a(p) H_1 G_\rho(z) H_1 a^\dagger(q) = {} & H_1 G_\sigma(z - \omega_p) J(q) a(p) + a^\dagger(q) J^\dagger(p) G_\sigma(z - \omega_q) H_1 \\ & + a^\dagger(q) H_1 G_\tau(z - \omega_p - \omega_q) H_1 a(p) + a^\dagger(q) H_1 G_\tau(z - \omega_p - \omega_q) J^\dagger(p)[1 + G_\sigma(z - \omega_q) H_1] \\ & + [1 + H_1 G_\sigma(z - \omega_p)] J(q) G_\tau(z - \omega_p - \omega_q) H_1 a(p) + [1 + H_1 G_\sigma(z - \omega_p)] \\ & \times [J(q) G_\tau(z - \omega_p - \omega_q) J^\dagger(p) + J^\dagger(p) G_\rho(z) J(q)][1 + G_\sigma(z - \omega_q) H_1] \\ & + H_1 G_\sigma(z - \omega_p) \delta(p,q) H_1 . \end{aligned} \tag{3.9}$$

At this point there appears to be little motivation for having developed the identities of this section; their usefulness will be amply demonstrated in Sec. IV.

## IV. THE PION-NUCLEON SYSTEM

### A. The model

We assume that our field theory Hamiltonian describes the interaction of $P$-wave pions with a static nucleon (N) and a static delta ($\Delta$) through the virtual processes

$$\begin{aligned} &\mathrm{N} \rightleftarrows \mathrm{N} + \pi , \\ &\mathrm{N} \rightleftarrows \Delta + \pi , \\ &\Delta \rightleftarrows \Delta + \pi . \end{aligned} \tag{4.1}$$

Conservation of angular momentum and parity imply that only $P$- and $F$-wave mesons can interact with a static N and $\Delta$ by means of the preceding processes. Thus, aside from not allowing $F$-wave mesons, the model we are using is essentially determined by the assumption that the static fermions interact with the pions only through the virtual processes (4.1).

We take the pion creation operators $a^\dagger_{mn}(q)$ to be the standard components of an irreducible tensor operator of order one[33] in configuration and isospin space. Here $m$ and $n$ are the $z$ components of the meson's angular momentum and isospin, respectively. We note that $(-)^{m+n} a_{-m,-n}(q)$ is also a component of a spherical tensor of order one, with indices $m$ and $n$. In order to compress the notation we shall frequently use $\mu$ and $\nu$ as cover indices for the pair $(m,n)$. The nonzero commutation relation for the meson operators is taken to be

$$[a_\mu(p),a_\nu^\dagger(q)]=\frac{\delta(p-q)}{p^2}\delta_{\mu\nu}\ . \tag{4.2}$$

Keeping in mind (3.3), we see that our Hamiltonian must be given by (3.1) with

$$H_0=H_0^F+\sum_\nu\int_0^\infty dq\, q^2 a_\nu^\dagger(q)a_\nu(q)\omega_q\ , \tag{4.3a}$$

$$H_1=\sum_\nu\int_0^\infty dq\, q^2[a_\nu(q)J_\nu(q)+a_\nu^\dagger(q)J_\nu^\dagger(q)]\ , \tag{4.3b}$$

where $H_0^F$ is the free Hamiltonian for the fermions, N and $\Delta$, and $J_\nu(q)$ contains only fermion creation and annihilation operators. According to (4.2) and (4.3b), we have

$$[H_1,a_\mu^\dagger(p)]=J_\mu(p)\ . \tag{4.4}$$

Since $H_1$ is invariant under rotations in configuration and isospin space, (4.4) implies that $J_\mu(p)=J_{mn}(p)$ is the component of a rank one spherical tensor and so is $(-)^{m+n}J^\dagger_{-m,-n}(p)$. As we go along, it will become clear that this is all we need to know about the $J_\mu$'s in order to derive the results of this section.

### B. The fermion propagators

We define the fermion propagators by

$$\Gamma_f(z)\delta_{ff'}\delta_{rr'}=\langle fr\mid G(z)\mid f'r'\rangle\ ,\quad f=\mathrm{N},\Delta\ , \tag{4.5}$$

where $G(z)$ is given by (2.3a) and $\mid fr\rangle$ is a bare fermion state which satisfies

$$H_0\mid fr\rangle=M_f^{(0)}\mid fr\rangle\ ,\quad f=\mathrm{N},\Delta\ , \tag{4.6}$$

with $M_f^{(0)}$ the bare fermion mass. Here $r$ is a cover index for the pair $(M,N)$, where $M$ and $N$ are the $z$ components of the fermions angular momentum and isospin, respectively.

We introduce the projection operators

$$P_f=\sum_r\mid fr\rangle\langle fr\mid\ ,\quad f=\mathrm{N},\Delta\ . \tag{4.7}$$

If in Sec. II we take $P^\alpha=1$ and $P_\beta=P_f$, then in (2.14) $G^\alpha=G$, and we can express $\Gamma_f$ as a matrix element of $g_f^\alpha(z)$. Using (2.11) and (2.12), it is then easy to show that

$$\Gamma_f(z)=\frac{1}{z-\langle fr\mid H_f(z)\mid fr\rangle}\ . \tag{4.8}$$

Here the pseudo-Hamiltonian $H_f(z)$ is defined by the relations

$$P^f=1-P_f\ , \tag{4.9}$$

$$H_f(z)=P_f[H+HG^f(z)H]P_f\ , \tag{4.10}$$

where $G^f(z)$ is of the form (2.3b). Using the fact that $H_0$ conserves the number of mesons, while $H_1$ changes the number by one, we can rewrite (4.8) as

$$\Gamma_f(z)=\frac{1}{z-M_f^{(0)}-\Sigma_f(z)}\ , \tag{4.11}$$

where the self-energy function is given by

$$\Sigma_f(z)=\langle fr\mid H_1G^f(z)H_1\mid fr\rangle\ . \tag{4.12}$$

We expect the denominator of $\Gamma_\mathrm{N}(z)$ to have a simple zero on the physical sheet at $z=M_\mathrm{N}$, the nucleon mass, while the denominator of $\Gamma_\Delta(z)$ should have a pair of conjugate zeroes on an unphysical sheet, corresponding to the complex mass of the $\Delta$ resonance. The complex zeroes occur in pairs because $\Sigma_f(z^*)=\Sigma_f^*(z)$, i.e., the self-energy, is a real, analytic function of $z$.

A useful decomposition of the complete Green's function $G(z)$ can be obtained from (2.13) by making the identifications $P^\alpha=1$, $P_\beta=P_f$, and $P^\gamma=P^f$. Using (2.14), (4.7), and (4.5), we find

$$G(z)=G^f(z)+\sum_r\mid F_r(z)\rangle\widetilde{\Gamma}_f(z)\langle F_r(z^*)\mid\ , \tag{4.13}$$

with

$$\mid F_r(z)\rangle=[1+G^f(z)H_1]\mid fr\rangle Z_f^{1/2} \tag{4.14}$$

and

$$\widetilde{\Gamma}_f(z)=\Gamma_f(z)/Z_f\ . \tag{4.15}$$

In order for (4.13) to be true, $Z_f$ can be any real number different from zero. It is convenient to choose $Z_\mathrm{N}$ to be the wave function renormalization constant for the nucleon, i.e.,

$$Z_\mathrm{N}^{1/2}=\langle \mathrm{N}r\mid\mathrm{N}r\rangle_\pm\ , \tag{4.16}$$

where $\mid\mathrm{N}r\rangle_\pm$ is the physical nucleon state, then $\widetilde{\Gamma}_\mathrm{N}(z)$ has a unit residue at the nucleon pole. Also with this choice we find from (2.18) that

$$\mid\mathrm{N}_r(M_\mathrm{N}\pm i\epsilon)\rangle=\mid\mathrm{N}r\rangle_\pm\ . \tag{4.17}$$

Using (4.13) and (4.14), it is easy to show that

$$\mid F_r(z)\rangle=G(z)\mid fr\rangle\widetilde{\Gamma}_f^{-1}(z)Z_f^{-1/2}\ . \tag{4.18}$$

Starting with this, we find with the help of (3.5a) $(P_\rho=P_\sigma=1)$ that

$$a_\mu(p)\mid F_r(z)\rangle=G(z-\omega_p)J_\mu^\dagger(p)\mid F_r(z)\rangle\ . \tag{4.19}$$

This result will be useful in subsequent developments.

An alternative expression for the self-energy functions can be obtained by using (4.14) in (4.12). The result is

$$\Sigma_f(z)=\langle fr\mid H_1\mid F_r(z)\rangle Z_f^{-1/2}\ . \tag{4.20}$$

Besides (4.13), there is another useful decomposition of the complete Green's function. We again use (2.13), but now make the identifications $P^\alpha=1$, $P_\beta=P_a$, and $P^\gamma=P^a$ where

$$P_a=P_\mathrm{N}+P_\Delta\ ,\quad P^a=1-P_a\ . \tag{4.21}$$

With the help of (2.14), (4.7), and (4.5) we find

$$G(z)=G^a(z)+[1+G^a(z)H_1]\times\sum_{fr}\mid fr\rangle\Gamma_f(z)\langle fr\mid[1+H_1G^a(z)]\ . \tag{4.22}$$

Because of the action of the projection operators, putting

(4.22) into (4.18) and using (4.15) leads to

$$|F_r(z)\rangle=[1+G^a(z)H_1]\,|fr\rangle Z_f^{1/2}\,, \tag{4.23}$$

which in turn allows us to rewrite (4.22) as

$$G(z)=G^a(z)+\sum_{fr}|F_r(z)\rangle\tilde{\Gamma}_f(z)\langle F_r(z^*)|\ . \tag{4.24}$$

This decomposition is particularly useful since the second term contains all of the contributions of single fermion states to the complete Green's function.

### C. The vertex functions

We begin this subsection by considering matrix elements of the complete Green's function between one-fermion and one-meson—one-fermion states. If in Sec. III we identify $P_\rho$ and $P_\sigma$ with 1 we find that (3.5a) and (4.18) allow us to write these matrix elements in the form

$$\langle fr\,|a_\mu(p)G(z)\,|f'r'\rangle=Z_f^{1/2}\tilde{\Gamma}_f(z-\omega_p)\langle F_r(z^*-\omega_p)\,|J_\mu^\dagger(p)\,|F'_{r'}(z)\rangle\tilde{\Gamma}_{f'}(z)Z_{f'}^{1/2}\ . \tag{4.25}$$

From (4.18) it follows that $|F_r(z)\rangle$ is an eigenstate of angular momentum and isospin whose eigenvalues we label $(J,M)$ and $(T,N)$, respectively. For N and $\Delta$ we have $(J,T)=(\frac{1}{2},\frac{1}{2})$ and $(J,T)=(\frac{3}{2},\frac{3}{2})$, respectively. Recall that $r$ is a cover index for $(M,N)$, while $\mu$ is a cover index for $m$ and $n$, the $z$ components of the pions angular momentum and isospin, respectively. Since $J_{mn}(p)$ is a component of a spherical tensor in configuration and isospin space, the Wigner-Eckart theorem[33] allows us to write

$$\langle F_r(z_1^*)\,|J_\mu^\dagger(p)\,|F'_{r'}(z_2)\rangle=\gamma_{ff'}(p,z_1,z_2)\langle fr\mu\,|f'r'\rangle\ , \tag{4.26a}$$

$$\langle F_r(z_1^*)\,|J_{\mu'}(p)\,|F'_{r'}(z_2)\rangle=\bar{\gamma}_{ff'}(p,z_1,z_2)\langle f'r'\mu'\,|fr\rangle\ , \tag{4.26b}$$

with

$$\gamma_{ff'}(p,z_1,z_2)=\bar{\gamma}^{\,*}_{f'f}(p,z_2^*,z_1^*)\ . \tag{4.27}$$

Here $\langle f'r'\mu'|fr\rangle$ is a product of Clebsch-Gordan (CG) coefficients defined by

$$\begin{aligned}&\langle f'r'\mu'\,|fr\rangle=\langle J'1M'm'\,|JM\rangle\langle T'1N'n'\,|TN\rangle\ ,\\ &f'=(J',T')\ ;\ r'=(M',N')\ ;\ \mu'=(m',n')\ ;\\ &f=(J,T)\ ;\ r=(M,N)\ ,\end{aligned} \tag{4.28}$$

where we are using Messiah's notation for the CG coefficients.[33] We shall refer to the functions $\gamma_{ff'}(p,z_1,z_2)$ as the vertex functions.

Combining (4.25), (4.26a), and (4.18), we can write

$$\langle fr\,|a_\mu(p)\,|F'_{r'}(z)\rangle=Z_f^{1/2}\tilde{\Gamma}_f(z-\omega_p)\gamma_{ff'}(p,z-\omega_p,z)\times\langle fr\mu\,|f'r'\rangle\ , \tag{4.29}$$

which shows that the projections of the $|F_r(z)\rangle$ onto the one-meson—one-fermion states are determined by the fermion propagators and the vertex functions.

It is convenient to introduce bare or undressed vertex functions by

$$\langle fr\,|J_\mu^\dagger(p)\,|f'r'\rangle=\gamma_{ff'}^{(0)}(p)\langle fr\mu\,|f'r'\rangle\ , \tag{4.30a}$$

$$\langle fr\,|J_{\mu'}(p)\,|f'r'\rangle=\bar{\gamma}_{ff'}^{(0)}(p)\langle f'r'\mu'\,|fr\rangle\ , \tag{4.30b}$$

with

$$\gamma_{ff'}^{(0)}(p)=\bar{\gamma}_{f'f}^{(0)*}(p)\ . \tag{4.31}$$

As the interaction is turned off the $\gamma$'s approach the $\gamma^{(0)}$'s. The renormalization of the coupling constants can be expressed in terms of the dressed and undressed vertex functions. If, for example, $f_0$ and $f$ are the bare and renormalized coupling constants for $\mathrm{N}\rightleftarrows\mathrm{N}+\pi$, they are related by[3]

$$\gamma_{\mathrm{NN}}(p,M_{\mathrm{N}},M_{\mathrm{N}})=f\gamma_{\mathrm{NN}}^{(0)}(p)/f_0\ . \tag{4.32}$$

It is well known[34] that in quantum field theories with interactions of the type given by (4.1) there exists an integral expression for the self-energy functions in terms of the vertex functions and propagators. It is easy to derive this relation with the results we have obtained so far. If we put (4.3) into (4.20) we encounter the matrix element

$$\langle fr\,|J_\nu(q)a_\nu(q)\,|F_r\rangle\ .$$

Since $J_\nu(q)$ contains only fermion creation and annihilation operators we can evaluate this matrix element by inserting a set of bare fermion states to the right of $J_\nu$. Using (4.29), (4.30b), and the orthogonality relation for the Clebsch-Gordan coefficients that occur in (4.28), we arrive at

$$Z_f^{1/2}\Sigma_f(z)=\sum_{f'}\int_0^\infty dq\,q^2\bar{\gamma}_{ff'}^{(0)}(q)Z_{f'}^{1/2}\tilde{\Gamma}_{f'}(z-\omega_q)\gamma_{f'f}(q,z-\omega_q,z)\ . \tag{4.33}$$

### D. Pion-nucleon scattering

In this subsection essentially all of the results we have obtained so far will be used to derive integral equations for the pion-nucleon scattering amplitudes. For notational purposes we define

$$\begin{aligned}&P_0=0\ ,\\ &P^0=1-P_0=1\ .\end{aligned} \tag{4.34}$$

We introduce a Green's function in the space spanned by the one-meson—one-fermion states $a_\mu^\dagger(p)\,|\,fr\,\rangle$ by the definition

$$G^e_{fr,f'r'}(p\mu,q\nu;z)=Z_f^{-1/2}\langle fr\,|\,a_\mu(p)G^e(z)a_\nu^\dagger(q)\,|\,f'r'\rangle Z_{f'}^{-1/2}\quad(e=0,a)\,, \tag{4.35}$$

where $G^e(z)$ is given by (2.3b) with $P^\lambda=P^e$. If $e=0$, $G^e(z)$ is the complete Green's function, while if $e=a$ the N and $\Delta$ states are taken out of play [see (4.21)]. We are going to use (3.8) to obtain an alternate expression for (4.35). We therefore identify $P_\rho$ in Sec. III with $P^e$. Since

$$a_\mu(p)P^e=1a_\mu(p)\quad(e=0,a)\,, \tag{4.36}$$

(3.4) and (3.7) tell us that we must identify $P_\sigma$ and $P_\tau$ with the unit operator. With the help of (3.2), (4.2), (4.5), (4.15), and (4.18), we find

$$G^e_{fr,f'r'}(p\mu,q\nu;z)=\widetilde{\Gamma}_f(z-\omega_p)\delta_{ff'}\delta_{rr'}\frac{\delta(p-q)}{p^2}\delta_{\mu\nu}+\widetilde{\Gamma}_f(z-\omega_p)X^e_{fr,f'r'}(p\mu,q\nu;z)\widetilde{\Gamma}_{f'}(z-\omega_q)\quad(e=0,a)\,, \tag{4.37}$$

where

$$X^e_{fr,f'r'}(p\mu,q\nu;z)=\langle F_r(z^*-\omega_p)\,|\,J_\nu(q)G(z-\omega_p-\omega_q)J_\mu^\dagger(p)+J_\mu^\dagger(p)G^e(z)J_\nu(q)\,|\,F'_{r'}(z-\omega_q)\rangle\quad(e=0,a)\,. \tag{4.38}$$

Using (4.17) and (4.19), it is straightforward to verify that

$$X^0_{\mathrm{N}r,\mathrm{N}r'}(k\mu,k\nu;M_\mathrm{N}+\omega_k+i\epsilon)={}_+\langle \mathrm{N}r\,|\,J_\mu^\dagger(k)\,|\,k\nu\mathrm{N}r'\rangle_+\,, \tag{4.39a}$$

where

$$|\,k\nu\mathrm{N}r'\rangle_+=[a_\nu^\dagger(k)+G(M_\mathrm{N}+\omega_k+i\epsilon)J_\nu(k)]\,|\,\mathrm{N}r'\rangle\,, \tag{4.39b}$$

which, according to Eqs. (7) and (12) of Ref. 5, shows that

$$X^0_{\mathrm{N}r,\mathrm{N}r'}(k\mu,k\nu;M_\mathrm{N}+\omega_k+i\epsilon)$$

is the amplitude for the elastic scattering from the initial state $(k\nu\mathrm{N}r')$ to the final state $(k\mu\mathrm{N}r)$. Amplitudes defined by (4.38) with one or both of the $f$'s equal to $\Delta$ are unphysical, even on shell, since the $\Delta$ is an unstable particle.

If we put (4.24) into (4.38) when $e=0$ and use (4.26), we find

$$X^0_{fr,f'r'}(p\mu,q\nu;z)=X^a_{fr,f'r'}(p\mu,q\nu;z)+\sum_{f''r''}\gamma_{ff''}(p,z-\omega_p,z)\langle fr\mu\,|\,f''r''\rangle\widetilde{\Gamma}_{f''}(z)\langle f'r'\nu\,|\,f''r''\rangle\overline{\gamma}_{f''f'}(q,z,z-\omega_q)\,. \tag{4.40}$$

Because the operator $G^a(z)$ does not allow for any single fermion intermediate states [see (2.3b) and (4.21)], the amplitude $X^a_{fr,f'r'}$ [see (4.35) and (4.37)] describes those processes in which pion absorption is missing, i.e., it is one-particle irreducible. These processes are contained in the second term on the right-hand side of (4.40). It is not surprising that this term involves the fermion propagators. In most models of the $\pi$N amplitude[20,35,36] the absorption term does not appear in the $\Delta$ channel, but we have seen here that it must be present when the underlying field theory contains an elementary $\Delta$.

Our job now is to derive integral equations for the Green's functions and scattering amplitudes. To this end, we introduce a projection operator for the one-meson—one-fermion subspace, i.e.,

$$P_s=\sum_{fr\mu}\int_0^\infty a_\mu^\dagger(p)\,|\,fr\,\rangle p^2dp\,\langle fr\,|\,a_\mu(p)\,. \tag{4.41}$$

According to (4.35) we need $G^e(z)$ in the subspace of $P_s$, so in (2.14) we identify $P^\alpha$ with $P^e$ and $P_\beta$ with $P_s$ and let

$$g_s^e(z)=P_sG^e(z)P_s\,. \tag{4.42}$$

From (2.4) we see that we must identify $P^\gamma$ with

$$P^{es}=1-P_e-P_s\,, \tag{4.43}$$

and then, using (2.11) and (2.12), we obtain

$$[z-H_s^e(z)]g_s^e(z)=P_s\,, \tag{4.44}$$

with

$$\begin{aligned}H_s^e(z)&=P_s[H+HG^{es}(z)H]P_s\,,\\&=P_s[H_0+H_1G^{es}(z)H_1]P_s\,.\end{aligned} \tag{4.45}$$

If $e=0$, $G^{es}$ excludes only the one-meson—one-fermion states, while if $e=a$ it also excludes the one-fermion states [see (2.3b), (4.43), (4.34), (4.21), and (4.41)].

With the help of (4.45), (4.2), (4.11), and (4.15), it is straightforward to show that we can write

$$\begin{aligned}&Z_f^{1/2}\langle fr\,|\,a_\mu(p)[z-H_s^e(z)]a_\nu^\dagger(q)\,|\,f'r'\rangle Z_{f'}^{1/2}\\&\quad=\widetilde{\Gamma}_f^{-1}(z-\omega_p)\delta_{ff'}\delta_{rr'}\frac{\delta(p-q)}{p^2}\delta_{\mu\nu}-V^e_{fr,f'r'}(p\mu,q\nu;z)\,,\end{aligned} \tag{4.46}$$

where the effective potential is given by

$$V^e_{fr,f'r'}(p\mu,q\nu;z)=Z_f^{1/2}\langle fr\,|\,a_\mu(p)[H_1G^{es}(z)H_1-\Sigma_f(z-\omega_p)]a_\nu^\dagger(q)\,|\,f'r'\rangle Z_{f'}^{1/2}\,. \tag{4.47}$$

Using (4.12) and conservation of angular momentum and isospin, we can rewrite (4.47) as

$$V^e_{fr,f'r'}(p\mu,q\nu;z)=Z_f^{1/2}\langle fr\,|\,a_\mu(p)H_1G^{es}(z)H_1a^\dagger_\nu(q)-H_1G^f(z-\omega_p)H_1\frac{\delta(p-q)}{p^2}\delta_{\mu\nu}\,|\,f'r'\rangle Z_{f'}^{1/2}\ . \tag{4.48}$$

In order to put this into a more useful form, we will use (3.9) with $P_p=P^{es}$. With the help of (4.43), (4.21), (4.34), (4.41), (4.2), and (4.7), we find

$$a_\mu(p)P^{es}=P^a a_\mu(p)\ , \tag{4.49}$$

which, according to (3.4), (3.7), and (4.21), implies that we must identify $P_\sigma$ with $P^a$ and $P_\tau$ with the unit operator. Using (3.9) with these identifications, as well as (4.23) and (4.14), we obtain

$$V^e_{fr,f',r'}(p\mu,q\nu;z)=\langle F_r(z^*-\omega_p)\,|\,J_\nu(q)G(z-\omega_p-\omega_q)J^\dagger_\mu(p)+J^\dagger_\mu(p)G^{es}(z)J_\nu(q)\,|\,F'_{r'}(z-\omega_q)\rangle\ . \tag{4.50}$$

If we write out (4.44) in detail, and use (4.2), (4.46), (4.42), (4.41), and (4.35), we are led to the following integral equation for the Green's functions:

$$G^e_{fr,f'r'}(p\mu,q\nu;z)=\widetilde{\Gamma}_f(z-\omega_p)\delta_{ff'}\delta_{rr'}\frac{\delta(p-q)}{p^2}\delta_{\mu\nu}+\widetilde{\Gamma}_f(z-\omega_p)\sum_{f''r''\lambda}\int_0^\infty V^e_{fr,f''r''}(p\mu,x\lambda;z)x^2dx\,G^e_{f''r'',f'r'}(x\lambda,q\nu;z)\ . \tag{4.51}$$

It is now convenient to introduce a set of operators and a set of states in the one-meson—one-fermion space. These are not to be thought of as Fock space operators and states, but rather as the sort of entities that are encountered in potential scattering. We label the operators in this space with the subscript $s$ and write the completeness relation for the states as

$$\sum_{fr\mu}\int_0^\infty |p\mu fr\rangle p^2dp\langle p\mu fr\,|=1_s\ . \tag{4.52}$$

The operators are defined by the relations

$$\langle p\mu fr\,|\,G^e_s(z)\,|\,q\nu f'r'\rangle=G^e_{fr,f'r'}(p\mu,q\nu;z)\ , \tag{4.53a}$$

$$\langle p\mu fr\,|\,t_s(z)\,|\,q\nu f'r'\rangle=\widetilde{\Gamma}_f(z-\omega_p)\delta_{ff'}\delta_{rr'}\frac{\delta(p-q)}{p^2}\delta_{\mu\nu}\ , \tag{4.53b}$$

$$\langle p\mu fr\,|\,X^e_s(z)\,|\,q\nu f'r'\rangle=X^e_{fr,f',r'}(p\mu,q\nu;z)\ , \tag{4.53c}$$

$$\langle p\mu fr\,|\,V^e_s(z)\,|\,q\nu f'r'\rangle=V^e_{fr,f'r'}(p\mu,q\nu;z)\ . \tag{4.53d}$$

In this notation (4.37) and (4.51) become

$$G^e_s(z)=t_s(z)+t_s(z)X^e_s(z)t_s(z)\ , \tag{4.54}$$

$$G^e_s(z)=t_s(z)+t_s(z)V^e_s(z)G^e_s(z)\ , \tag{4.55a}$$

$$G^e_s(z)=t_s(z)+G^e_s(z)V^e_s(z)t_s(z)\ , \tag{4.55b}$$

where (4.55b) can be seen to be equivalent to (4.55a) by comparing iterations of the two equations. By comparing (4.37) and (4.51) or (4.54) and (4.55), we find that

$$X^e_s(z)t_s(z)=V^e_s(z)G^e_s(z)\ , \tag{4.56a}$$

$$t_s(z)X^e_s(z)=G^e_s(z)V^e_s(z)\ , \tag{4.56b}$$

and if we put (4.55b) in (4.56a) we are led to

$$X^e_s(z)=V^e_s(z)+V^e_s(z)G^e_s(z)V^e_s(z)\ , \tag{4.57}$$

which when combined with (4.56) gives

$$X^e_s(z)=V^e_s(z)+V^e_s(z)t_s(z)X^e_s(z)\ , \tag{4.58a}$$

$$X^e_s(z)=V^e_s(z)+X^e_s(z)t_s(z)V^e_s(z)\ . \tag{4.58b}$$

In order to take advantage of conservation of angular momentum and isospin, it is convenient to introduce eigenstates of the total angular momentum and isospin according to

$$|pf\alpha a\rangle=\sum_{r\mu}|p\mu fr\rangle\langle fr\mu\,|\,\alpha a\rangle\ , \tag{4.59a}$$

with the completeness relation

$$\sum_{f\alpha a}\int_0^\infty |pf\alpha a\rangle p^2dp\langle pf\alpha a\,|=1_s\ . \tag{4.59b}$$

Here $\alpha$ is a cover index for the total angular momentum and isospin setup according to

$$\alpha=(J_\alpha,T_\alpha)\ ,\quad \alpha=1,2,3,4\ ,$$
$$T_1=T_2=J_1=J_3=\tfrac{1}{2}\ ;\quad T_3=T_4=J_2=J_4=\tfrac{3}{2}\ , \tag{4.60}$$

and $a$ is a cover index for the $z$ components of the total angular momentum and isospin. The coefficients in (4.59a) are defined by (4.28). In this new basis we have

$$\langle pf\alpha a\,|\,G^e_s(z)\,|\,qf'\alpha'a'\rangle=G^{e\alpha}_{ff'}(p,q;z)\delta_{\alpha\alpha'}\delta_{aa'}\ , \tag{4.61a}$$

$$\langle pf\alpha a\,|\,X^e_s(z)\,|\,qf'\alpha'a'\rangle=X^{e\alpha}_{ff'}(p,q;z)\delta_{\alpha\alpha'}\delta_{aa'}\ , \tag{4.61b}$$

$$\langle pf\alpha a\,|\,V^e_s(z)\,|\,qf'\alpha'a'\rangle=V^{e\alpha}_{ff'}(p,q;z)\delta_{\alpha\alpha'}\delta_{aa'}\ , \tag{4.61c}$$

$$\langle pf\alpha a\,|\,t_s(z)\,|\,qf'\alpha'a'\rangle=\widetilde{\Gamma}_f(z-\omega_p)\delta_{ff'}\frac{\delta(p-q)}{p^2}\delta_{\alpha\alpha'}\delta_{aa'}\ , \tag{4.61d}$$

where (4.61a)—(4.61c) follow from rotational invariance in configuration and isospin space, while in obtaining (4.61d) we have used (4.59a) and (4.53b).

From (4.58a), (4.59b), and (4.61b)—(4.61d), it follows that

$$X^{e\alpha}_{ff'}(p,q;z)=V^{e\alpha}_{ff'}(p,q;z)+\sum_{f''}\int_0^\infty V^{e\alpha}_{ff''}(p,x;z)x^2dx\,\widetilde{\Gamma}_{f''}(z-\omega_x)X^{e\alpha}_{f''f'}(x,q;z)\ , \tag{4.62}$$

which for fixed $\alpha$ and $f'$ is a pair of coupled integral equations with the $\pi$N elastic scattering amplitude in channel $\alpha$ given by $X^{0\alpha}_{\rm NN}(k,k;M_{\rm N}+\omega_k+i\epsilon)$. Using (4.61b), (4.59a), (4.53c), and (4.40), it is straightforward to show that

$$X^{0\alpha}_{ff'}(p,q;z)=X^{a\alpha}_{ff'}(p,q;z)+(\delta_{\alpha 1}+\delta_{\alpha 4})\gamma_{f\alpha}(p,z-\omega_p,z)\widetilde{\Gamma}_{\alpha}(z)\overline{\gamma}_{\alpha f'}(q,z,z-\omega_q)\ . \tag{4.63}$$

In this expression the pion absorption terms have been separated out explicitly, and it is seen that they contribute only to the $P_{11}$ and $P_{33}$ channels. There is no distinction between $X^{0\alpha}_{ff'}$ and $X^{a\alpha}_{ff'}$ in the $P_{13}$ and $P_{31}$ channels.

We shall now derive some useful relations among the vertex functions, the fermion propagators, and the one-particle irreducible amplitudes $X^{a\alpha}_{ff'}(p,q;z)$. We start with (4.23) and replace $G^a(z)$ with the expression

$$G^a(z)=G^{as}(z)+[1+G^{as}(z)H_1]g^a_s(z)[1+H_1G^{as}(z)]\ , \tag{4.64}$$

which is obtained from (2.13) and (2.14) by making the identifications $P^\alpha=P^a$, $P_\beta=P_s$, and $P^\gamma=P^{as}$. Since $H_1\,|\,fr\rangle$ is in the subspace of $P_s$, we find that (4.23) becomes

$$|\,F_r(z)\rangle=|\,fr\rangle Z_f^{1/2}+[1+G^{as}(z)H_1]P_s\,|\,F_r(z)\rangle\ . \tag{4.65}$$

If we put (4.41) in this, use (4.29) and (3.6b) with $P_\rho=P^{as}$ and $P_\sigma=P^a$ [see (4.49) and (3.4)], and (4.23), we obtain

$$|\,F_r(z)\rangle=|\,fr\rangle Z_f^{1/2}+\sum_{f'r'\nu}\int_0^\infty dq\,q^2[a^\dagger_\nu(q)+G^{as}(z)J_\nu(q)]\,|\,F'_{r'}(z-\omega_q)\rangle\widetilde{\Gamma}_{f'}(z-\omega_q)\gamma_{f'f}(q,z-\omega_q,z)\langle f'r'\nu\,|\,fr\rangle\ . \tag{4.66}$$

When we use this expression for the ket on the left-hand side of (4.26a), we obtain an integral equation for the vertex function. The inhomogeneous term in the equation can be worked out by observing that $J^\dagger_\mu(p)\,|\,f'r'\rangle$ is in the subspace of $P_a$, and by using (4.23) and (4.30a), while the kernel can be simplified by using (4.19), (4.50), the orthogonality relation for Clebsch-Gordan coefficients, (4.53d), (4.59a), and (4.61c). The resulting integral equation for the vertex function is

$$\gamma_{ff'}(p,z-\omega_p,z)=Z_f^{1/2}\gamma^{(0)}_{ff'}(p)Z_{f'}^{1/2}+\sum_{f''}\int_0^\infty V^{af'}_{ff''}(p,x;z)x^2dx\,\widetilde{\Gamma}_{f''}(z-\omega_x)\gamma_{f''f'}(x,z-\omega_x,z)\ , \tag{4.67}$$

where we see that the kernel is the same as that of (4.62) with $e=a$ and $\alpha=f'$.

Extending our abstract notation, we write

$$\langle pf\alpha a\,|\,\gamma_{f'r'}(z)\rangle=\gamma_{ff'}(p,z-\omega_p,z)\delta_{\alpha f'}\delta_{ar'}\ , \tag{4.68a}$$

$$\langle pf\alpha a\,|\,u_{f'r'}\rangle=Z_f^{1/2}\gamma^{(0)}_{ff'}(p)Z_{f'}^{1/2}\delta_{\alpha f'}\delta_{ar'}\ , \tag{4.68b}$$

which when used in conjunction with the notation given by (4.61c) and (4.61d), allows us to replace (4.67) with

$$|\,\gamma_{fr}(z)\rangle=|\,u_{fr}\rangle+V^a_s(z)t_s(z)\,|\,\gamma_{fr}(z)\rangle\ . \tag{4.69}$$

From (4.58b) it follows that

$$[1+X^e_s(z)t_s(z)][1-V^e_s(z)t_s(z)]=1\ , \tag{4.70}$$

which leads to the solution of (4.69), i.e.,

$$|\,\gamma_{fr}(z)\rangle=[1+X^a_s(z)t_s(z)]\,|\,u_{fr}\rangle\ . \tag{4.71a}$$

Using (4.68), (4.59b), (4.61b), and (4.61d), we find

$$\gamma_{ff'}(p,z-\omega_p,z)=Z_f^{1/2}\gamma^{(0)}_{ff'}(p)Z_{f'}^{1/2}+\sum_{f''}\int_0^\infty X^{af'}_{ff''}(p,x;z)x^2dx\,\widetilde{\Gamma}_{f''}(z-\omega_x)Z_{f''}^{1/2}\gamma^{(0)}_{f''f'}(x)Z_{f'}^{1/2}\ . \tag{4.71b}$$

Using (4.68), (4.59b), (4.61d), (4.31), (4.71a), and (4.54), we find that we can rewrite (4.33) as

$$Z_f\Sigma_f(z)=\langle u_{fr}\,|\,t_s(z)\,|\,\gamma_{fr}(z)\rangle\ , \tag{4.72a}$$

$$=\langle u_{fr}\,|\,G^a_s(z)\,|\,u_{fr}\rangle\ , \tag{4.72b}$$

while (4.63), (4.61b), (4.68a), and (4.27) lead to

$$X^0_s(z)=X^a_s(z)+\sum_{fr}|\,\gamma_{fr}(z)\rangle\widetilde{\Gamma}_f(z)\langle\gamma_{fr}(z^*)\,|\ . \tag{4.73}$$

If we put (4.71) into (4.73) we obtain

$$X^0_s(z)=X^a_s(z)+[1+X^a_s(z)t_s(z)]Y_s(z)[1+t_s(z)X^a_s(z)]\ , \tag{4.74}$$

where

$$Y_s(z)=\sum_{fr} |u_{fr}\rangle \widetilde{\Gamma}_f(z)\langle u_{fr}| \; . \tag{4.75}$$

Equation (4.74) is very similar to the separation of the $T$ matrix obtained in the two-potential formalism.[31] It is worth pursuing this analogy a little further. According to (4.50) and (4.34), we can write

$$V^0_{fr,f'r'}(p\mu,q\nu;z)=V^a_{fr,f'r'}(p\mu,q\nu;z)+W_{fr,f'r'}(p\mu,q\nu;z) \; , \tag{4.76}$$

where

$$W_{fr,f'r'}(p\mu,q\nu;z)=\langle F_r(z^*-\omega_p)|J^\dagger_\mu(p)[G^s(z)-G^{as}(z)]J_\nu(q)|F'_{r'}(z-\omega_q)\rangle \; . \tag{4.77}$$

If in (2.11)–(2.14) we choose $P^\alpha=P^s$, $P^\gamma=P^{as}$, and $P_\beta=P_a$, and use the fact that $P^{as}HP_a=0$, we find

$$G^s(z)=G^{as}(z)+\sum_{fr}|fr\rangle\frac{1}{z-M_f^{(0)}}\langle fr| \; . \tag{4.78}$$

Putting this in (4.77), and using (4.23) and (4.30), we obtain

$$W_{fr,f'r'}(p\mu,q\nu;z)=\sum_{f''r''}Z_f^{1/2}\gamma^{(0)}_{ff''}(p)\frac{\langle fr\mu|f''r''\rangle\langle f'r'\nu|f''r''\rangle}{z-M^{(0)}_{f''}}\overline{\gamma}^{(0)}_{f''f'}(q)Z^{1/2}_{f'} \; , \tag{4.79}$$

which with the help of (4.76), (4.61c), (4.59a), and (4.53d) leads to

$$V^{0\alpha}_{ff'}(p,q;z)=V^{a\alpha}_{ff'}(p,q;z)+Z_f^{1/2}\gamma^{(0)}_{f\alpha}(p)\times\frac{(\delta_{\alpha1}+\delta_{\alpha4})}{z-M^{(0)}_\alpha}\overline{\gamma}^{(0)}_{\alpha f'}(q)Z^{1/2}_{f'} \; . \tag{4.80}$$

Using (4.68b), (4.31), and (4.61c), we obtain

$$V^0_s(z)=V^a_s(z)+W_s(z) \; , \tag{4.81}$$

with

$$W_s(z)=\sum_{fr}\frac{|u_{fr}\rangle Z_f^{-1}\langle u_{fr}|}{z-M_f^{(0)}} \; . \tag{4.82}$$

It is now straightforward to show that $Y_s(z)$ is the solution of the equation

$$Y_s(z)=W_s(z)+W_s(z)G^a_s(z)Y_s(z) \; , \tag{4.83}$$

where we have used (4.72b), (4.11), and (4.15). Thus the separation of the full scattering operator in (4.74) is exactly analogous to that given in the two-potential formalism.[31]

If the potentials $V^a_s(z)$ and the propagators were known, we would now have a complete set of equations. The scattering operator $X^a_s(z)$ could be obtained by solving (4.58) and the result could be used in (4.71) to calculate the vertex functions. The complete scattering operator would then be given by (4.73). Of course, we do not know the potentials and the propagators.

In Sec. IV E, we shall show that a particular truncation of the formalism leads to a closed system of equations for all the quantities of interest. The scheme we shall develop has the attractive feature that it leads to amplitudes that satisfy the constraints of unitarity below the three-pion threshold.

### E. The coupled N$\pi$-N$\pi\pi$ system

We begin by deriving an alternative expression for the potentials. If we use (4.19) in (4.50), we find

$$V^a_{fr,f'r'}(p\mu,q\nu;z)=\langle F_r(z^*-\omega_p)|J^\dagger_\mu(p)|\phi(f'r',q\nu;z)\rangle \; , \tag{4.84}$$

with

$$|\phi(fr,q\nu;z)\rangle=[a^\dagger_\nu(q)+G^{as}(z)J_\nu(q)]|F_r(z-\omega_q)\rangle \; . \tag{4.85}$$

Putting (4.66) in (4.84), using (4.27) and (4.85) and the fact that $\langle fr|J^\dagger_\mu(p)$ is in the subspace of $P_a$, we are led to

$$V^a_{fr,f'r'}(p\mu,q\nu;z)=\sum_{f''r''\lambda}\langle f''r''\lambda|fr\rangle\int_0^\infty dx\,x^2\overline{\gamma}_{ff''}(x,z-\omega_p,z-\omega_p-\omega_x)\widetilde{\Gamma}_{f''}(z-\omega_p-\omega_x)A_{f''r'',f'r'}(p\mu,x\lambda;q\nu;z) \; , \tag{4.86}$$

where

$$A_{fr,f'r'}(p\mu,x\lambda;q\nu;z)=\langle\phi(fr,x\lambda;z^*-\omega_p)|J^\dagger_\mu(p)|\phi(f'r',q\nu;z)\rangle \; . \tag{4.87}$$

This expression can be put in a more convenient form by using the identity

$$a_\mu(p)|\phi(fr,q\nu;z)\rangle=\frac{\delta(p-q)}{p^2}\delta_{\mu\nu}|F_r(z-\omega_q)\rangle+G^a(z-\omega_p)J^\dagger_\mu(p)|\phi(fr,q\nu;z)\rangle \; , \tag{4.88}$$

which can be obtained by following the procedure used to derive Eq. (10) of Ref. 5. The relevant equations here are (3.5a), (4.49), (4.19), (3.5b), (4.36), and (4.2). Using this identity, (4.87), (4.85), and (4.26a), we find

$$A_{fr,f'r'}(p\mu,x\lambda;q\nu;z)=\frac{\delta(x-q)}{x^2}\delta_{\lambda\nu}\gamma_{ff'}(p,z-\omega_p-\omega_x,z-\omega_x)\langle fr\mu\,|\,f'r'\rangle$$
$$+\langle F_r(z^*-\omega_p-\omega_x)\,|\,J^\dagger_\mu(p)G^a(z-\omega_x)J^\dagger_\lambda(x)+J^\dagger_\lambda(x)G^{as}(z-\omega_p)J^\dagger_\mu(p)\,|\,\phi(f'r',q\nu;z)\rangle \ . \quad (4.89)$$

From (4.42), (4.41), (4.35), (2.13), (3.6), (3.4), (4.49), (4.23), and (4.85), it follows that

$$G^a(z)=G^{as}(z)+\sum_{\substack{fr\mu\\ f'r'\nu}}\int_0^\infty |\phi(fr,p\mu;z)\rangle p^2dp\, G^a_{fr,f'r'}(p\mu,q\nu;z)q^2dq\langle\phi(f'r',q\nu;z^*)| \ . \quad (4.90)$$

Putting this into (4.89) and using (4.84), (4.87), (4.53), (4.52), and (4.56a) leads to

$$A_{fr,f'r'}(p\mu,x\lambda;q\nu;z)=\frac{\delta(x-q)}{x^2}\delta_{\lambda\nu}\gamma_{ff'}(p,z-\omega_p-\omega_x,z-\omega_x)\langle fr\mu\,|\,f'r'\rangle$$
$$+\sum_{f''r''\rho}\int_0^\infty X^a_{fr,f''r''}(p\mu,y\rho;z-\omega_x)\widetilde{\Gamma}_{f''}(z-\omega_x-\omega_y)y^2dy$$
$$\times A_{f''r'',f'r'}(x\lambda,y\rho;q\nu;z)+R_{fr,f'r'}(p\mu,x\lambda;q\nu;z) \ , \quad (4.91a)$$

with

$$R_{fr,f'r'}(p\mu,x\lambda;q\nu;z)=\langle F_r(z^*-\omega_p-\omega_x)\,|\,J^\dagger_\mu(p)G^{as}(z-\omega_x)J^\dagger_\lambda(x)+J^\dagger_\lambda(x)G^{as}(z-\omega_p)J^\dagger_\mu(p)\,|\,\phi(f'r',q\nu;z)\rangle \ . \quad (4.91b)$$

Our basic approximation is to neglect $R_{fr,f'r'}$, for by so doing we are led to a closed set of equations for all of the quantities of interest. This closed set of equations is of the same structure as those that occur in the quasiparticle method for solving nonrelativistic three particle problems.[32] The starting point of the quasiparticle method is a set of coupled integral equations, the so-called AGS equations,[32] which are exact within the framework of the nonrelativistic potential theory. In the quasiparticle method, the off-shell $T$ matrices that appear in the kernels of the equations are split into separable terms and a weak remainder. Using this splitting, which is just like (4.73), the AGS equations can be put in the same form as (4.62) with $e=a$. The quasiparticle effective potentials are determined by equations which are analogous to (4.86) and (4.91a) with $R_{fr,fr'}=0$.

There are some differences between the AGS quasiparticle equations[32] and those obtained here. First of all, the equations developed here allow for the inclusion of an unstable particle, i.e., the $\Delta$ resonance. Second, solving the equations developed here does not lead immediately to the physical scattering amplitudes, but rather to the one-particle irreducible amplitudes $X^{aa}_{ff'}$. The physical amplitudes are obtained by adding the one particle reducible contributions as in (4.63) or (4.73). Finally, the input to the equations developed here is also the output, i.e., in contrast to a potential model, these equations are nonlinear. The N$\pi$ and N$\pi\pi$ sectors are coupled and must be treated self-consistently, which can be done by iteration.

A natural first approximation for solving the equations iteratively is obtained by assuming that the nucleon and $\Delta$ pole terms in (4.63) or (4.73) are dominant, which means that we begin by setting $X^a(z)=0$. Then (4.71) implies that the vertex functions are given approximately by

$$\gamma_{ff'}(p,z-\omega_p,z)\simeq Z_f^{1/2}\gamma^{(0)}_{ff'}(p)Z_{f'}^{1/2} \ . \quad (4.92)$$

Since the bare vertex functions are in general imaginary,[37] it is convenient to define real functions by

$$Z_f^{1/2}\gamma^{(0)}_{ff'}(p)Z_{f'}^{1/2}=-ig_{ff'}(p) \ , \quad (4.93a)$$

$$Z_f^{1/2}\overline{\gamma}^{(0)}_{ff'}(p)Z_{f'}^{1/2}=i\overline{g}_{ff'}(p) \ . \quad (4.93b)$$

If we put this approximation for the vertex functions into (4.33) we get an approximation for the self-energy, and from (4.11) and (4.15) an approximation for the propagators. However, in order to do the integrals in (4.33) we need to know the propagators.

Since the equations obtained by dropping the last term in (4.91a) are three-particle equations, it only makes sense to solve them for energies below the three-pion threshold, otherwise we violate unitarity. So we write $z=E+i\epsilon$ and assume that $E<M_{\rm N}+3\mu$. In (4.62) and (4.86) the real parts of the arguments of the propagators are less than $M_{\rm N}+2\mu$ and $M_{\rm N}+\mu$, respectively, which implies that in (4.33) $\mathrm{Re}(z)<M_{\rm N}+2\mu$. Accordingly, we only need an approximation for the propagators in the integrals of (4.33) which is good when $\mathrm{Re}(z-\omega_q)\lesssim M_{\rm N}+\mu$. A reasonable approximation is to replace $\widetilde{\Gamma}_{f'}(z-\omega_q)$ by $(z-\omega_q-M_{f'})^{-1}$. We will assume that $M_\Delta$ is real and shall shortly see a way of determining it. Using (4.33), (4.31), (4.92), and (4.93), we now have

$$Z_f\Sigma_f(z)\simeq\sum_{f'}\int_0^\infty dq\, q^2\frac{g^2_{f'f}(q)}{z-\omega_q-M_{f'}} \ , \quad (4.94)$$

and from (4.11) an approximation for the propagators, which leads to the correct analytic structure in the energy range of interest.

It is convenient to carry out a mass renormalization in the propagators so as to eliminate the bare masses. The procedure is given in Appendix A of Ref. 30. The result is

$$\widetilde{\Gamma}_f(z)=\frac{1}{z-M_f-Z_fR_f(z)} \ , \quad (4.95a)$$

where the physical masses are obtained by solving

$$M_f = M_f^{(0)} + \Sigma_f^{\mathrm{P}}(M_f)\ , \tag{4.95b}$$

the wave function renormalization constants are given by

$$Z_f^{-1} = 1 - \Sigma_f^{\mathrm{P}'}(M_f) \tag{4.95c}$$

and

$$R_f(z) = \Sigma_f(z) - \Sigma_f^{\mathrm{P}}(M_f) - (z - M_f)\Sigma_f^{\mathrm{P}'}(M_f)\ . \tag{4.95d}$$

The superscript P indicates that a principal value integral is to be used when necessary. The nucleon propagator has a pole at $z = M_{\mathrm{N}}$ with a residue of one. The real part of the denominator of the $\Delta$ propagator vanishes at $z = M_\Delta$. This propagator will have a pair of conjugate poles on the second Riemann sheet of the analytic function defined by (4.94).

To facilitate comparison with the author's previous work on the coupled N$\pi$-N$\pi\pi$ system,[1] and to avoid confusion between approximate and exacts results, we shall write the approximate propagators as

$$\tilde{\Gamma}_f(z) \simeq d_f^{-1}(z - M_{\mathrm{N}}) = \left[ Z_f(z - M_f^{(0)}) - \sum_{f'} \int_0^\infty dq\, q^2 \frac{g_{f'f}^2(q)}{z - \omega_q - M_{f'}} \right]^{-1} . \tag{4.96}$$

The first approximation for the potentials is obtained by putting the first term on the right-hand side of (4.91a) into (4.86) with the vertex functions given by (4.92). With the help of (4.61c), (4.59a), and (4.53d), we find

$$V_{ff'}^{a\alpha}(p,q;z) \simeq B_{ff'}^{\alpha}(p,q;z - M_{\mathrm{N}}) = \sum_{f''} \frac{\bar{g}_{ff''}(q) C_{ff'}^{\alpha f''} g_{f''f'}(p)}{d_{f''}(z - M_{\mathrm{N}} - \omega_p - \omega_q)}\ , \tag{4.97}$$

where the coefficients are given in terms of Wigner $6j$ symbols by

$$C_{\beta\gamma}^{\alpha\delta} = (-)^{T_\beta + J_\beta + T_\gamma + J_\gamma} [(2T_\beta + 1)(2J_\beta + 1)(2T_\gamma + 1)(2J_\gamma + 1)]^{1/2} \begin{Bmatrix} T_\beta & 1 & T_\alpha \\ T_\gamma & 1 & T_\delta \end{Bmatrix} \begin{Bmatrix} J_\beta & 1 & J_\alpha \\ J_\gamma & 1 & J_\delta \end{Bmatrix} . \tag{4.98}$$

In obtaining this result we have used

$$\sum_{\substack{b\nu \\ d\mu \\ g}} \langle \beta b \nu \mid \alpha a \rangle \langle \delta d \mu \mid \beta b \rangle \langle \delta d \nu \mid \gamma g \rangle \langle \gamma g \mu \mid \alpha' a' \rangle = \delta_{\alpha\alpha'} \delta_{aa'} C_{\beta\gamma}^{\alpha\delta}\ , \tag{4.99}$$

which follows from the standard expression for the $6j$ symbols[38] in terms of a sum of products of four Clebsch-Gordan coefficients [see (4.28) and (4.60)].

There is a constraint on the form factor $g_{\mathrm{NN}}(k)$ that is worth imposing. Using (4.50), (4.24), (4.26), (4.61c), (4.59a), (4.53d), (4.99), and (4.32), it is straightforward to show that

$$V_{\mathrm{NN}}^{e\alpha}(k,k;M_{\mathrm{N}} + \omega_k) \underset{\omega_k \to 0}{\longrightarrow} \frac{C_{11}^{\alpha 1}(f/f_0)^2 \mid \gamma_{\mathrm{NN}}^{(0)}(k) \mid^2}{(-\omega_k)}\ . \tag{4.100}$$

From (4.30), Eq. (2) of Ref. 1, and the Wigner-Eckart theorem,[33] it follows that

$$\gamma_{\mathrm{NN}}^{(0)}(k) = -i \left[ \frac{3}{\pi} \right]^{1/2} \left[ \frac{f_0}{\mu} \right] \frac{k v(k)}{\omega_k^{1/2}}\ , \tag{4.101}$$

where $v(k)$ is a cutoff function which we normalize to 1 at $\omega_k = 0$. Putting this into (4.100), we have

$$V_{\mathrm{NN}}^{e\alpha}(k,k;M_{\mathrm{N}} + \omega_k) \underset{\omega_k \to 0}{\longrightarrow} -\frac{3k^2}{\pi\omega_k} \left[ \frac{f}{\mu} \right]^2 \frac{C_{11}^{\alpha 1}}{\omega_k}\ , \tag{4.102}$$

which shows [see Eqs. (78) and (54) of Ref. 1] that the potential contains the crossed nucleon pole. By comparing (4.102) with (4.97), we see that we have the constraint

$$\lim_{\omega_k \to 0} \frac{\omega_k}{k^2} g_{\mathrm{NN}}^2(k) = \frac{3}{\pi} \left[ \frac{f}{\mu} \right]^2 , \tag{4.103}$$

which guarantees that our scattering amplitudes correctly incorporate the crossed nucleon pole even in first approximation.

The one-particle irreducible amplitudes $X_{ff'}^{a\alpha}$ are obtained in first approximation by solving (4.62) with the potentials and propagators given by (4.97) and (4.96). In obtaining the complete amplitudes, the second term on the right-hand side of (4.63) should be constructed by using (4.71b), (4.11), and (4.33). In (4.33) and (4.71b), the propagators should be replaced by (4.96). By so doing a better approximation for the vertex functions and propagators is obtained, and three-particle unitarity is satisfied for energies below the three-pion threshold. The results obtained at this stage can be improved upon by using (4.91a) and (4.86) to obtain a better approximation for the potentials, and then solving (4.62) once again. Hopefully, this procedure converges rapidly.

In order to see that the first approximation is a reasonable one, we will now show that it satisfies two-particle and three-particle unitarity and leads to production amplitudes with the correct analytic structure in the subenergy variables. For equations of the form (4.58), it can be shown (see Sec. VI of Ref. 1) that

$$DX_s^e = X_s^e(\pm)(Dt_s)X_s^e(\mp) + [1 + X_s^e(\pm)t_s(\pm)] \times [DV_s^e][1 + t_s(\mp)X_s^e(\mp)]\ , \tag{4.104}$$

where $(\pm) = (M_{\mathrm{N}} + \omega_k \pm i\epsilon)$ with $\omega_k \geq \mu$ and, e.g.,

$$DX_s^e = X_s^e(+) - X_s^e(-)\ . \tag{4.105}$$

Note that according to (4.81) and (4.82)

$$DV_s^0 = DV_s^a \; , \tag{4.106}$$

so that $X_s^0(z)$ and $X_s^a(z)$ satisfy the same discontinuity relation. Here we have assumed that $M_f^{(0)} < M_N + \mu$ so that $W_s(z)$ has no discontinuity in the range of interest. Even if it does, it can still be shown that $X_s^0(z)$ and $X_s^a(z)$ satisfy the same discontinuity relation.[6]

For our approximate propagator (4.96) we have

$$\frac{1}{d_f(\omega_k + i\epsilon)} - \frac{1}{d_f(\omega_k - i\epsilon)} = -2\pi i \left[ \delta_{fN}\delta(\omega_k) + \frac{k\omega_k g_{Nf}^2(k)}{|d_f(\omega_k + i\epsilon)|^2}\theta(\omega_k - \mu) \right] , \quad \omega_k \le 2\mu \; , \tag{4.107}$$

where we have assumed that $M_\Delta - M_N > \mu$. Actually $M_\Delta - M_N \gtrsim 2\mu$, so this is a good assumption. Using (4.107) in conjunction with (4.97), we find

$$B_{ff'}^a(p,q;\omega_k + i\epsilon) - B_{ff'}^a(p,q;\omega_k - i\epsilon) = -2\pi i \bar{g}_{fN}(q) C_{ff'}^{a1} \delta(\omega_k - \omega_p - \omega_q) g_{Nf'}(p) \; , \quad \omega_k \le 3\mu \; , \tag{4.108}$$

and we note that this discontinuity vanishes if $\omega_k < 2\mu$ or when either $\omega_p = \omega_k$ or $\omega_q = \omega_k$. With the help of (4.104), (4.106), (4.61b), (4.59b), (4.61d), (4.96), and (4.107), it is straightforward to show that

$$X_{ff'}^{e\alpha}(p,q;+) - X_{ff'}^{e\alpha}(p,q;-) = -2\pi i X_{fN}^{e\alpha}(p,k;\pm) k\omega_k X_{Nf'}^{e\alpha}(k,q;\mp) \; , \quad \mu < \omega_k < 2\mu \; , \tag{4.109}$$

which is an off-shell unitarity relation, valid in the elastic range. This relation will always be valid as long as the nucleon propagator $\tilde{\Gamma}_N(z)$ has a simple pole at $z = M_N$ with unit residue. Above the inelastic threshold, the off-shell unitarity relation becomes quite complicated as all the terms in (4.104) contribute. Following Ref. 1, it is straightforward to show that the on-shell unitarity relations for our approximate amplitudes are

$$\begin{aligned} X_{NN}^{e\alpha}(k,k;M_N + \omega_k + i\epsilon) - X_{NN}^{e\alpha}(k,k;M_N + \omega_k - i\epsilon) = -2\pi i \Bigg[ & k\omega_k \, |X_{NN}^{e\alpha}(k,k;M_N + \omega_k + i\epsilon)|^2 \\ & + \int p\omega_p d\omega_p q\omega_q d\omega_q \delta(\omega_k - \omega_p - \omega_q) \tfrac{1}{2} \\ & \times \sum_\beta \Big| F_\beta^{e\alpha}(p,\omega_q + i\epsilon,k) + \sum_\gamma C_{\beta\gamma}^{\alpha 1} F_\gamma^{e\alpha}(q,\omega_p + i\epsilon,k) \Big|^2 \Bigg] , \\ & \mu \le \omega_k \le 3\mu \; , \end{aligned} \tag{4.110}$$

where

$$\begin{aligned} F_\beta^{e\alpha}(p,\omega_q + i\epsilon,k) &= g_{N\beta}(q) \frac{X_{\beta N}^{e\alpha}(p,k;M_N + \omega_k + i\epsilon)}{d_\beta(\omega_q + i\epsilon)} \; , \quad \beta = N \; , \quad \Delta = 1,4 \; , \quad (\omega_p + \omega_q = \omega_k) \\ &= 0 \; , \quad \beta = 2,3 \; . \end{aligned} \tag{4.111}$$

Equation (4.110) is a three-particle unitarity relation, and it holds for the full amplitude ($e = 0$), as well as the one-particle irreducible amplitude ($e = a$). It is shown in Ref. 1 that the amplitude for $N + \pi \to N + \pi + \pi$ can be constructed from the functions $F_\beta^{0\alpha}(p,z,k)$ where $\beta$ stands for the quantum numbers of one of the $N\pi$ subsystems in the final state. According to (4.111) only the $P_{11}$ and $P_{33}$ subsystems ($\beta = 1,4$) contribute to the production amplitude in our first approximation. The $P_{13}$ and $P_{31}$ subsystems ($\beta = 2,3$) will contribute in the second approximation obtained by including the effect of the second term on the right-hand side of (4.91a) on the potentials.

It is shown in Ref. 1 that in general the functions $F_\beta^{0\alpha}(p,z,k)$ will have a right-hand cut in the subenergy variable $z$. Using the techniques developed there it is possible to show that with our first approximation

$$\begin{aligned} F_f^{e\alpha}(p,\omega_q + i\epsilon,k) - F_f^{e\alpha}(p,\omega_q - i\epsilon,k) = -(e^{-2i\theta_f(\omega_q)} - 1) & \\ \times \left[ F_f^{e\alpha}(p,\omega_q + i\epsilon,k) + \sum_{f'} C_{ff'}^{\alpha 1} F_{f'}^{e\alpha}(q,\omega_q + i\epsilon,k) \right] , & \\ \mu \le \omega_q \le 2\mu \; , \quad \omega_p + \omega_q = \omega_k \; , & \end{aligned} \tag{4.112}$$

where the phase $\theta_f$ is given by

$$d_f(\omega_q - i\epsilon)/d_f(\omega_q + i\epsilon) = e^{2i\theta_f(\omega_q)} \; , \quad \mu \le \omega_q \le 2\mu \; . \tag{4.113}$$

Equation (4.112) agrees in form with the exact discontinuity relation [see Eqs. (29), (30), and (34) of Ref. 1].

We conclude that our first approximation for the scattering amplitudes is about as good as one could hope for as it has the correct analytic structure, satisfies unitarity, and has the correct behavior at the direct and crossed nucleon poles. Moreover what led us to this approximation was the very reasonable assumption that the dominant contributions to the amplitudes are the N and $\Delta$ direct pole terms. Numerical calculations are now in progress to test this assumption.

## V. DISCUSSION

We have seen that a quantum field theory involving $\pi$'s, N's, and $\Delta$'s can be readily analyzed by using the extension of the Feshbach projection operator formalism[19] presented in Sec. II in conjunction with the identities derived in Sec. III. These identities are relationships between Green's functions which act in various subspaces whose projection operators are related by the action of the meson creation and annihilation operators. Although we have only applied these identities to the pion-nucleon system, it is clear that they will be of value in analyzing pion-nucleus reactions.

It is important to stress that except for Sec. IV E, all of the results of Sec. IV are exact within the framework of the $\pi$-N-$\Delta$ field theory that we have assumed. It is very useful to have exact relationships between the various entities that arise within the context of field theory (propagators, self-energies, vertex functions, ...) as these serve as a guide in the development of realistic approximation schemes.

Of particular value is the decomposition of the pion-nucleon amplitude given by Eqs. (4.40), (4.63), and (4.73). As we have seen in Sec. IV E, removing the one-particle reducible contributions from the full amplitude leaves an amplitude which approximately satisfies three-particle equations similar to these obtained in potential theory, i.e., the AGS equations.[32] Moreover the one-particle reducible contributions can be constructed from the solutions of the three-particle equations. It is interesting to note that the kernels of the three-particle integral equations, whose solutions are the one-particle irreducible amplitudes, involve the complete pion-nucleon amplitudes, not just the one-particle irreducible parts.

In contrast to potential models, our three-particle equations are nonlinear in that the input is also the output. We have developed approximate linear equations whose solutions have been shown to have a number of desirable features. They satisfy two-particle and three-particle unitarity, the production amplitudes have the correct analytic structure in the subenergy variables, and the elastic amplitude can be made to have the correct residues at the direct and crossed nucleon poles. These linear equations generalize those obtained in Ref. 1 in that they allow for intermediate states containing a $\Delta$. As we have seen in Sec. IV E corrections to the linear approximation can be obtained by a well-defined iteration scheme.

The approach advocated here for treating the pion-nucleon system in the framework of a $\pi$-N-$\Delta$ field theory goes somewhat beyond previous work on the subject[11,39–41] in that it involves a more careful treatment of the coupling between the N$\pi$ and N$\pi\pi$ channels. This is desirable because of the large inelasticity in the $P_{11}$ channel. Also, it seems reasonable to use three-particle equations to describe the low-energy pion-nucleon system in that the crossed Born terms are three particle in nature.

A very different, but more traditional, approach has been pursued by Cheung[30] who has used perturbation theory to calculate $\pi$-N scattering. He finds quite rapid convergence. An interesting possibility is to use perturbation theory to calculate the potentials and propagators in Eq. (4.62) and then solve this equation exactly. By so doing it should be possible to satisfy two-particle and three-particle unitarity.

An important extension of the equations developed here will be the inclusion of recoil effects. As pointed out previously,[1] the situation with regard to $\pi$-N scattering with nonstatic nucleons is somewhat confusing. The results of Aaron, Amado, and Young[42] suggest that the recoil effect makes the "force" in the $P_{33}$ channel more attractive. However, there are other calculations[43,44] based on relativistic generalizations of the Chew-Low theory that indicate that the inclusion of recoil effects makes the force in the $P_{33}$ channel less attractive. In fact, it is claimed in Ref. 43 that once recoil effects are taken into account it is not possible to generate the $\Delta$ resonance with the process $N \rightleftarrows N+\pi$. This suggests that it is necessary to have the $\Delta$ occur as an "elementary" particle in the underlying field theory. Clearly, it is of some importance to resolve this issue.

Finally, we note that it should be possible to apply the techniques developed here to relativistic quantum field theories. There is nothing inherently nonrelativistic about the relations obtained in Secs. II and III. Relativistic few particle equations derived using projection operator techniques would not be explicitly covariant, but they would have the advantage of involving fewer variables than covariant approaches.


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