Statistical theory of multistep compound reactions

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We generalize a simple version of the optical background representation for the transition amplitude to the case of well-nested doorways. Using this simple generalization of the optical background representation an alternate statistical theory of multistep compound reactions is suggested, which uses the idea of nested averaging in the case of well-nested doorways and is valid under very general conditions. Simple connections are established between the present approach of multistep compound reactions and that of Feshbach, Kerman, and Koonin, of Agassi, Weidenmüller, and Mantzouranis, and of Hussein and McVoy. The present formulation exhibits the chaining condition of Feshbach, Kerman, and Koonin in the extreme low energy limit, is time reversal symmetric, and yields the Bohr description of compound nuclei in the appropriate limit.

I. INTRODUCTION

The evaporation model of Weisskopf[1,2] for compound reactions, which employs the independent hypothesis by Bohr[2] and uses complicated compound nuclear states, has been very successful in explaining the gross features of such reactions. Many of the fine features of such reactions were explained after the elegant mathematical work, based on the Bohr independent hypothesis and the Weisskopf evaporation model,[2] of Wolfenstein and of Hauser and Feshbach.[3] The characteristic time for such reactions is large and because of the large density of the complicated compound nuclear states employed statistical considerations are applicable and one obtains an isotropic angular distribution. On the other hand, for direct nuclear reactions, the characteristic time is short, wave functions employed are simple, and the angular distribution shows a forward peaking.[4] By changing energy as one moves from the domain of compound nuclear reactions towards the domain of direct nuclear reactions the characteristic time decreases; one needs less complicated nuclear wave functions and the angular distribution changes from full isotropy to a mere symmetry around 90° for multistep compound reactions and to an anisotropic distribution with forward peaking for multistep direct and direct reactions.[3,5] The aim of the present paper is to improve the understanding of the mechanism of multistep compound reactions using a statistical dynamical theory.[6]

There are several statistical theories already existing for multistep compound reactions.[5–9] The nested average approach[7,8] of multistep compound reactions of Hussein and McVoy (HM) is basically a generalization of the optical background representation of the fluctuation amplitude of Kawai, Kerman, and McVoy[10] (KKM) to the case of well-nested doorways. The nested average approach[7,8] is simple and exact and only makes some general assumptions about the widths and spacings of different classes of doorways. The model of the nested average approach assumes the presence of a hierarchy of doorway resonances to be grouped into classes according to their total widths. The average width Γ_n and spacing D_n in class n obey

$$\Gamma_n \gg \Gamma_{(n+1)} , \quad \Gamma_n \gg D_n , \tag{1.1}$$

where $n=1,2,\ldots,N$. Equations (1.1) define a well-nested sequence of classes of compound nucleus ($n=N$) and doorway ($n<N$) resonances.

The characteristic times corresponding to different classes of resonances in a multistep compound reaction are not expected to be measured directly in the near future. One can only hope to feel the presence of such characteristic times indirectly through the study of the autocorrelation function[8]

$$C(\epsilon)=\langle\sigma(E)\sigma(E+\epsilon)\rangle-\langle\sigma(E)\rangle\langle\sigma(E+\epsilon)\rangle , \tag{1.2}$$

which will contain more than one correlation width Γ. Equation (1.2) will yield a generalization of the Ericson expression[11] $C(\epsilon)\sim(\Gamma^2+\epsilon^2)^{-1}$ valid for the one class case. At this point one should recall that the idea of dividing the cross section into its multistep components was first used by Griffin[12] in his exciton model. The model of Hussein and McVoy[7] yields a simple generalization of the Ericson[11] expression for the autocorrelation function valid under the general condition (1.1) and contains the Hauser-Feshbach[3] result for conventional compound nuclear reactions.

Though the elegant formulation of Feshbach, Kerman, and Koonin[5] (FKK) yields useful results for multistep compound reactions, it possesses several unpleasant features[6,13] as has been pointed out previously. Firstly, the chaining condition that FKK introduce is merely a hypothesis and their dynamical formulation violates it. A part of this violation is a mathematical artifact of the special version of the optical background representation they use. As a consequence, while physically the chaining condition is expected to be obeyed at extreme low energies it is violated in the FKK approach. Secondly, unlike in the optical background representation of KKM and in its generalization by HM, the multistep fluctuation amplitudes of FKK do not average to zero without needing properties of statistical fluctuation of certain form factors. Thirdly, over and above the conditions defined by

Eq. (1.1) FKK impose the condition $\Gamma_n \gg D_{(n-1)}$ in order to achieve "self-averaging" of the multistep transition amplitude. Once the chaining condition is removed in the FKK approach one needs even a stronger condition: $\Gamma_n \gg D_j$ for all n and j.[13] (An attempt was made in Ref. 13 to eliminate the first two limitations of the FKK approach in the extreme low energy limit.) As a consequence of these unpleasant features the formulation of FKK is not time reversal symmetric.

The formulation of FKK is an attempted literal translation of the doorway state formulation of Feshbach, Kerman, and Lemmer[14,15] to the case of a hierarchy of doorways. Such a simple generalization maintaining the nested doorway structure appears to be useful for the analysis of experimental results. The FKK formulation sorts the levels associated with a model Hamiltonian into classes which then define the various stages of the reaction. For conventional compound nuclear reactions, though FKK write their final result in a way reminiscent of the Hauser-Feshbach form, any attempt to extract the symmetric Hauser-Feshbach[3] form from the manifestly asymmetric FKK formulation is doomed to failure.

Finally, there is the mathematically elegant and rigorous formulation of Agassi, Weidenmüller, and Mantzouranis[9] (AWM) which goes far beyond the approach of FKK in complexity and generality. As in the FKK approach the AWM formulation sorts out the levels of a model Hamiltonian into classes which then define the various stages of the reaction, and assumes that the many-body matrix elements of the Hamiltonian governing the reaction are random in phase. Though the AWM formulation does not have the unpleasant features of the FKK approach, the fluctuation cross section of AWM involves the inversion of a matrix in the class space, which makes the approach rather complicated to use.

It has been noted[8] that unlike the formulations of FKK and AWM that of HM does not employ a model Hamiltonian but is a generalization of the usual KKM optical background representation of the fluctuation amplitude to the case of well-nested doorways. Recently, we[13] have provided an alternative optical background representation of the fluctuation amplitude. In this work we generalize this alternate optical background representation of the transition amplitude to the case of well-nested doorways. It has been pointed out that the alternate optical background representation is a convenient rewriting of the KKM one; similarly the present generalization of the alternate optical background representation to the case of well-nested doorways is shown to be a convenient rewriting of the HM generalization. The present formulation yields multistep cross sections and correlation functions obeying the general condition (1.1) and having the following distinct advantages. Firstly, under appropriate conditions the present formulation exhibits chaining property. Secondly, the present formulation is explicitly time reversal symmetric and the multistep contribution corresponding to the most complicated compound nuclear states has the symmetric Hauser-Feshbach form.

The plan of the present paper is as follows. In Sec. II we present our optical background representation in a simple two-step doorway model. This model possesses all the interesting features of the multistep model but yet is very simple to treat algebraically. This section is divided into various subsections which deal with various aspects of the model. The generalization of this model to the case of multistep compound reactions is presented in Sec. III and finally in Sec. IV we present a brief discussion where we compare the present approach with some other approaches.

II. OPTICAL BACKGROUND REPRESENTATION

A. Formulation in a two-step model

In Ref. 13 we presented an alternate optical background representation for the transition matrix when the entire Hilbert space is divided into optical (P) and fluctuation (Q) parts representing the projections onto the open channel and closed channel spaces, respectively. In the present section we generalize the alternate optical background representation to the case where the entire Hilbert space is broken into three pieces; e.g., the optical (P) part, the doorway state (d) part, and the fluctuation (q) part. We introduce the usual orthogonal projection operators P, d, and q as in Ref. 14, such that

$$P+d+q=1 \tag{2.1}$$

and

$$d+q=Q \tag{2.2}$$

with the condition $Pd=dq=Pq=0$. The transition amplitude from the incident channel state ϕ_i to the final channel state ϕ_f is given by

$$T=\langle \phi_f \mid (V+VGV) \mid \phi_i \rangle \ , \tag{2.3}$$

where $G=(E-H+i0)^{-1}$ is the full resolvent operator, $H=(H_0+V)$ is the full Hamiltonian, V is the channel interaction, and H_0 is the kinetic energy operator of relative motion. Unless otherwise specified the initial (i) and the final (f) channel indices will not be explicitly shown as labels on transition amplitudes and cross sections in this paper. Also, we shall not explicitly show the $+i0$ part of the resolvent operators.

In this section we consider strongly overlapping resonances in the doorway (d) and compound nuclear (q) spaces and calculate fluctuation cross sections corresponding to these two classes. This two-step model of compound reactions is physically as rich as the multistep model we consider in Sec. III. The two classes of resonances are denoted by 1 (d space) and 2 (q space) in Eq. (1.1)—the corresponding average widths and spacings of compound nuclear states being denoted by Γ_1,Γ_2 and D_1,D_2. As in HM we introduce a nested sequence of energy averaging interval I_n ($n=1,2$) such that

$$I_1 \gg \Gamma_1 \gg I_2 \gg \Gamma_2 \ . \tag{2.4}$$

The scattering amplitude T given by Eq. (2.3) is then written as

$$T=T_{\rm op}+T_d+T_q \ , \tag{2.5}$$

where $T_{\rm op} \equiv \langle T \rangle_{I_1}$ is the optical amplitude;

$T_d \equiv (\langle T \rangle_{I_2} - \langle T \rangle_{I_1})$ and $T_q \equiv (T - \langle T \rangle_{I_2})$ are fluctuation amplitudes corresponding to d and q spaces, respectively. As usual $\langle\ \rangle_I$ denotes energy averaging with respect to a Lorentzian function of width I. The separation (2.5) of the transition amplitude into its components has the advantage that $\langle T_d \rangle_{I_1} \cong \langle T_q \rangle_{I_2} \cong 0$. Hence in the corresponding multistep cross sections interference terms can be avoided.

Next let us introduce the operators V_n and G_n through the recursion relations

$$V_n = V_{(n-1)} + V_{(n-1)} G_{(n-1)} V_{(n-1)} \tag{2.6a}$$

and

$$G_n = d_n (E - d_n H_n d_n)^{-1} d_n \ , \tag{2.6b}$$

with $n = 0,1,2$. In Eq. (2.6) the projectors d_n are defined by $d_0 = P$, $d_1 = d$, and $d_2 = q$, and $V_0 = V$ is the channel interaction and $H_n = (H_0 + V_n)$. This notation is introduced here for the sake of future convenience. In terms of these operators the transition amplitude T of Eq. (2.3) can be written as[16]

$$T = \langle \phi_f \mid (V_2 + V_2 G_2 V_2) \mid \phi_i \rangle \ . \tag{2.7}$$

In explicit notation Eq. (2.7) reads

$$\begin{aligned} T = {} & \langle \phi_f \mid (V + VG_0V) \mid \phi_i \rangle + \langle \phi_f \mid (V + VG_0V) G_1 (V + VG_0V) \mid \phi_i \rangle \\ & + \langle \phi_f \mid [(V + VG_0V) + (V + VG_0V) G_1 (V + VG_0V)] G_2 [(V + VG_0V) + (V + VG_0V) G_1 (V + VG_0V)] \mid \phi_i \rangle \ . \end{aligned} \tag{2.8}$$

Equation (2.8) is essentially Eq. (3.9) of de Toledo Piza and Kerman[16] and is easy to derive. In Eq. (2.8) the first term on the right-hand side containing G_0 represents fluctuations in the optical space. The second term containing G_1 represents doorway space fluctuations governed by resonances of average width and spacing Γ_1 and D_1. Finally, the last term containing G_2 represents rapid q space fluctuations governed by resonances of average width and spacing Γ_2 and D_2. The operators G_0, G_1, and G_2 are responsible for these fluctuations.

Now it is easy to separate T_q of Eq. (2.5) from the expression of T given by Eq. (2.7) for $n = 2$. By definition $T_q \equiv (T - \langle T \rangle_{I_2})$ is given by

$$T_q = \langle \phi_f \mid V_2 (G_2 - \langle G_2 \rangle_{I_2}) V_2 \mid \phi_i \rangle \ , \tag{2.9}$$

such that $\langle T_q \rangle_{I_2} \cong 0$ insofar as $\langle G_2 \rangle_{I_2}$ is unchanged by reaveraging. In writing (2.9) we have noted that V_2 is slowly varying over the averaging interval I_2 and hence $\langle V_2 \rangle_{I_2} \cong V_2$. Now the difference $T - T_q$ given by

$$\langle T \rangle_{I_2} \equiv (T - T_q) = \langle \phi_f \mid (V_2 + V_2 \langle G_2 \rangle_{I_2} V_2) \mid \phi_i \rangle \ , \tag{2.10}$$

will contain the components T_d and $T_{\rm op}$. By interchanging the role of d_2 and d_1 Eq. (2.10) can be written as

$$\langle T \rangle_{I_2} = \langle \phi_f \mid (V_1^{(1)} + V_1^{(1)} G_1^{(1)} V_1^{(1)}) \mid \phi_i \rangle \ , \tag{2.11}$$

with

$$V_1^{(1)} = V_1 + V_1 \langle G_{22} \rangle_{I_2} V_1 \ , \tag{2.12a}$$

$$= V_0^{(1)} + V_0^{(1)} G_0^{(1)} V_0^{(1)} \ , \tag{2.12b}$$

where

$$V_0^{(1)} = V + V \langle G_{qq} \rangle_{I_2} V \ . \tag{2.12c}$$

Here

$$\begin{aligned} G_{22} &= d_2 (E - d_2 H_1 d_2)^{-1} d_2 \ , \\ G_{qq} &= d_2 (E - d_2 H d_2)^{-1} d_2 \ , \\ G_0^{(1)} &= d_0 (E - d_0 H_0^{(1)} d_0)^{-1} d_0 \ , \\ G_1^{(1)} &= d_1 (E - d_1 H_1^{(1)} d_1)^{-1} d_1 \ , \end{aligned}$$

where $H_0^{(1)} = (H_0 + V_0^{(1)})$ and $H_1^{(1)} = (H_0 + V_1^{(1)})$. Now in order to calculate T_d or $T_{\rm op}$ we need to evaluate $\langle T \rangle_{I_1}$. For this purpose we rewrite Eq. (2.3) or Eq. (2.7) as

$$T = \langle \phi_f \mid (V_1 + V_1 \bar{G}_{QQ} V_1) \mid \phi_i \rangle \ , \tag{2.13}$$

with $Q = (d + q) = (d_1 + d_2)$, and $\bar{G}_{QQ} = Q(E - QH_1Q)^{-1}Q$. Then to find the energy average $\langle T \rangle_{I_1}$ of Eq. (2.13) we note that V_1 is a slowly varying function over the averaging interval I_1 and that the energy averaging should be done only on $\bar{G}_{QQ}$ in Eq. (2.13). After performing this energy average we have

$$\langle T \rangle_{I_1} = \langle \phi_f \mid (V_1 + V_1 \langle \bar{G}_{QQ} \rangle_{I_1} V_1) \mid \phi_i \rangle \ . \tag{2.14}$$

Now making use of the identity

$$\begin{aligned} V_1 + V_1 \langle \bar{G}_{QQ} \rangle_{I_1} V_1 = {} & (V_1 + V_1 \langle G_{22} \rangle_{I_2} V_1) \\ & + (V_1 + V_1 \langle G_{22} \rangle_{I_2} V_1) \langle G_1^{(1)} \rangle_{I_1} \\ & \times (V_1 + V_1 \langle G_{22} \rangle_{I_2} V_1) \ , \end{aligned} \tag{2.15}$$

one can immediately find out T_d and $T_{\rm op}$. Equation (2.15) is basically Eq. (3.34a) of Ref. 8. In Eq. (2.15) the average over I_1 guarantees that the states in d_1 are treated on the average on the right-hand side and since in $(V_1 + V_1 \langle \bar{G}_{QQ} \rangle_{I_1} V_1)$ the states in $q = d_2$ are averaged out, all states in Q appear only on the average in this equation. Substituting Eq. (2.15) in (2.14) and using Eq. (2.12) one has

$$\langle T \rangle_{I_1} = \langle \phi_f \mid (V_1^{(1)} + V_1^{(1)} \langle G_1^{(1)} \rangle_{I_1} V_1^{(1)}) \mid \phi_i \rangle \ . \tag{2.16}$$

From Eqs. (2.11) and (2.16) one immediately has

$$T_d \equiv \langle T \rangle_{I_2} - \langle T \rangle_{I_1}$$
$$= \langle \phi_f \mid V_1^{(1)} (\langle G_1^{(1)} \rangle_{I_2} - \langle G_1^{(1)} \rangle_{I_1}) V_1^{(1)} \mid \phi_i \rangle \ . \quad (2.17)$$

Finally, changing the role of $d_0 = P$ and Q in Eq. (2.14) we get

$$T_{\rm op} \equiv \langle T \rangle_{I_1} = \langle \phi_f \mid (U + U g_{\rm op} U) \mid \phi_i \rangle \ , \quad (2.18)$$

where

$$U = V + V \langle G_{QQ} \rangle_{I_1} V \quad (2.19)$$

and

$$g_{\rm op} = P[E - P(H_0 + U)P]^{-1} P \ , \quad (2.20)$$

where $G_{QQ} = Q(E - QHQ)^{-1} Q$. Equation (2.18) is the usual expression for $T_{\rm op}$ and is identical with Eq. (2.10) of Ref. 13. Equations (2.9), (2.17), and (2.18) for T_q, T_d, and $T_{\rm op}$ constitute the generalization of the optical background representation of Ref. 13 when two types of fluctuations are present in the transition amplitude. Substituting $d_1 \equiv d = 0$ in our model one easily obtains the optical background representation of Ref. 13, which has been shown to be equivalent to that of KKM. The present definition of T_d, though based on the same definition of HM, appears to be different from that obtained in Ref. 8. In the following subsection we show the equivalence between these two expressions for T_d before calculating the multistep cross sections.

B. Equivalence with the HM representation

The doorway space contribution $T_d \equiv (\langle T \rangle_{I_2} - \langle T \rangle_{I_1})$ of Hussein and McVoy in the same model is given by[7,8]

$$T_d = \langle \phi_f \mid (1 + U g_{\rm op}) V_{Pd} \, g_{dd}^{\rm HM} V_{dP} (1 + g_{\rm op} U) \mid \phi_i \rangle \ , \quad (2.21)$$

where

$$g_{dd}^{\rm HM} = d(E - d H_0^{(1)} d - V_{dP} \, g_{\rm op} \, V_{Pd})^{-1} d \ , \quad (2.22)$$

$$V_{Pd} = P V_0^{(1)} d_1 [i(I_1/2) \langle G_d^{(1)} \rangle_{I_1}]^{1/2} d_1 \ , \quad (2.23a)$$

$$V_{dP} = d_1 [i(I_1/2) \langle G_d^{(1)} \rangle_{I_1}]^{1/2} d_1 V_0^{(1)} P \ , \quad (2.23b)$$

where

$$G_d^{(1)} = d_1 (E - d_1 H_0^{(1)} d_1)^{-1} d_1 \ .$$

The equivalence between T_d of Eq. (2.17) and of Eq. (2.21) can be established by using the following easily verified identities:

$$B \equiv V_{Pd} \, g_{dd}^{\rm HM} V_{dP}$$
$$= P V_0^{(1)} [i(I_1/2) \langle G_d^{(1)} \rangle_{I_1}]^{1/2} (G_d^{(1)} + g_{dd}^{\rm HM} V_{dP} \, g_{\rm op} V_{Pd} G_d^{(1)}) [i(I_1/2) \langle G_d^{(1)} \rangle_{I_1}]^{1/2} V_0^{(1)} P \ , \quad (2.24a)$$
$$= P V_0^{(1)} i(I_1/2) \langle G_d^{(1)} \rangle_{I_1} \{ G_d^{(1)} + G_d^{(1)} V_0^{(1)} P [g_{\rm op}^{-1} - P V_0^{(1)} \langle G_d^{(1)} \rangle_{I_1} i(I_1/2) G_d^{(1)} V_0^{(1)} P]^{-1} P V_0^{(1)} i(I_1/2) \langle G_d^{(1)} \rangle_{I_1} G_d^{(1)} \} V_0^{(1)} P \ , \quad (2.24b)$$
$$= P V_0^{(1)} i(I_1/2) \langle G_d^{(1)} \rangle_{I_1} [G_d^{(1)} + G_d^{(1)} V_0^{(1)} \, g_{PP} V_0^{(1)} (G_d^{(1)} - \langle G_d^{(1)} \rangle_{I_1})] V_0^{(1)} P \ , \quad (2.24c)$$
$$= P V_0^{(1)} i(I_1/2) \langle G_d^{(1)} \rangle_{I_1} G_1^{(1)} (V_0^{(1)} - V_0^{(1)} G_0^{(1)} V_0^{(1)} \langle G_d^{(1)} \rangle_{I_1} V_0^{(1)}) P \ , \quad (2.24d)$$

where

$$g_{PP} = P(E - P H_0^{(1)} P - P V_0^{(1)} G_d^{(1)} V_0^{(1)} P)^{-1} P \ . \quad (2.24e)$$

Equation (2.24b) follows from Eq. (2.24a) by considering the Neumann series expansion of relevant resolvent operators of both sides. Equation (2.24c) follows by using the following identity:

$$g_{\rm op}^{-1} = (E - PHP - PV \langle G_{QQ} \rangle_{I_1} VP)$$
$$= (E - P H_0 P - P V_0^{(1)} P - P V_0^{(1)} \langle G_d^{(1)} \rangle_{I_1} V_0^{(1)} P) \ . \quad (2.25)$$

Finally, Eq. (2.24d) follows by using the definition of $G_1^{(1)}$ given after Eq. (2.12). Using Eqs. (2.19) and (2.20), Eq. (2.24d) can be rewritten as

$$B \equiv g_{\rm op}^{-1} \, g_{\rm op} V_0^{(1)} \langle G_d^{(1)} \rangle_{I_1} i(I_1/2) G_1^{(1)} V_0^{(1)} G_0^{(1)} \, g_{\rm op}^{-1} \ , \quad (2.26a)$$
$$= g_{\rm op}^{-1} G_0^{(1)} V_0^{(1)} \langle G_1^{(1)} \rangle_{I_1} i(I_1/2) G_1^{(1)} V_0^{(1)} G_0^{(1)} \, g_{\rm op}^{-1} \ , \quad (2.26b)$$
$$= g_{\rm op}^{-1} G_0^{(1)} V_0^{(1)} (G_1^{(1)} - \langle G_1^{(1)} \rangle_{I_1}) V_0^{(1)} G_0^{(1)} \, g_{\rm op}^{-1} \ , \quad (2.26c)$$

where again Eq. (2.26b) follows by considering the Neumann series expansion of the relevant resolvent operators of both sides. Substituting Eq. (2.26c) in Eq. (2.21) one arrives at Eq. (2.17) with the use of the identities

$$(1 + U g_{\rm op}) g_{\rm op}^{-1} G_0^{(1)} = V_1^{(1)} \ , \quad (2.27a)$$

$$G_0^{(1)} g_{\rm op}^{-1} (1 + g_{\rm op} U) = V_1^{(1)} \ . \quad (2.27b)$$

Hence, the equivalence of the present optical background representation with that of Ref. 8 is established. Though these two optical background representations are equivalent, the present one has some distinct advantages over that of Ref. 8 in calculating multistep cross sections as we see in the following.

C. Multistep fluctuation cross sections

In this subsection we calculate the multistep contributions to the fluctuation cross section. The amplitude T_q given by Eq. (2.9) can be written as

$$T_q = \langle \phi_f \mid V_2 [i(I_2/2) \langle G_2 \rangle_{I_2}]^{1/2}$$
$$\times G_2 [i(I_2/2) \langle G_2 \rangle_{I_2}]^{1/2} V_2 \mid \phi_i \rangle \ . \quad (2.28)$$

Next we make an eigenfunction expansion of G_2 in Eq. (2.28) in terms of the biorthogonal basis states $|\psi_{2\mu}\rangle$ and $\langle\tilde{\psi}_{2\mu}|$ of the Hamiltonian $d_2 H_2 d_2$ corresponding to the complex eigenvalue $\epsilon_{2\mu}\equiv(E_{2\mu}-i\Gamma_{2\mu}/2)$, and satisfying $\langle\tilde{\psi}_{2\mu}|\psi_{2\nu}\rangle=\delta_{\mu\nu}$. Then one has

$$T_q=\sum_\mu \frac{g_{2f\mu}\, g_{2i\mu}}{E-\epsilon_{2\mu}} , \tag{2.29}$$

where

$$g_{2c\mu}\equiv\langle\phi_c|V_2 f_{2\mu}(E)|\psi_{2\mu}\rangle , \tag{2.30a}$$

$$=\langle\tilde{\psi}_{2\mu}|f_{2\mu}(E)V_2|\phi_c\rangle , \tag{2.30b}$$

where c represents either the initial or the final channel level i or f. The function

$$f_{2\mu}(E)=\{i(I_2/2)[E-\epsilon_{2\mu}+i(I_2/2)]^{-1}\}^{1/2} , \tag{2.31}$$

is a slowly varying function of energy E and because of this weak dependence on E of $g_{2c\mu}$ one has $\langle T_q\rangle_{I_2}\cong 0$ and hence $\langle g_{2f\mu}\, g_{2i\mu}\rangle_{\mu\epsilon I_2}\cong 0$. It is because of this condition that one can employ average unitarity as suggested by Kerman and Sevgen[17] and as a consequence neglect the level-level correlation term, and obtain[17,18] the following multistep cross section σ_q:

$$\sigma_q\cong\frac{2\pi}{D_2}\left\langle\frac{|g_{2f\mu}|^2|g_{2i\mu}|^2}{\Gamma_{2\mu}}\right\rangle_{\mu\epsilon I_1} , \tag{2.32a}$$

$$\cong\frac{2\pi}{D_2\Gamma_2}\langle|g_{2f\mu}|^2|g_{2i\mu}|^2\rangle_{\mu\epsilon I_1} , \tag{2.32b}$$

where $\langle 1/\Gamma_{2\mu}\rangle_\mu\equiv 1/\Gamma_2$. Next following KKM we define the X matrix by

$$X_{2cc'}=\langle g_{2c\mu}\, g^*_{2c'\mu}\rangle_{\mu\epsilon I_2} , \tag{2.33}$$

when Eq. (2.32) reduces to[10,19]

$$\sigma_q\cong\frac{2\pi}{D_2\Gamma_2}\langle(X_{2ff}X_{2ii}+X_{2fi}X_{2if})\rangle_{I_1} . \tag{2.34}$$

Here Γ_2 is the average width and D_2 is the average spacing of q space states. In the absence of direct channel coupling Eq. (2.34) yields the following simplified relation in the limit of many open channels:

$$\sigma_q\cong\frac{2\pi}{D_2\Gamma_2}\langle X_{2ff}\rangle_{I_1}\langle X_{2ii}\rangle_{I_1}(1+\delta_{fi}) . \tag{2.35}$$

Before presenting a detailed analysis of the result (2.35) we calculate the fluctuation cross section σ_d corresponding to the doorway space d_1. Again T_d can be written as

$$T_d\cong\sum_\mu \frac{g_{1f\mu}\, g_{1i\mu}}{E-\epsilon_{1\mu}} , \tag{2.36}$$

where

$$g_{1c\mu}\equiv\langle\phi_c|V_1^{(1)}[i(I_1/2)(E-\epsilon_{1\mu}+iI_1/2)^{-1}]^{1/2}|\psi_{1\mu}\rangle , \tag{2.37a}$$

$$=\langle\tilde{\psi}_{1\mu}|[i(I_1/2)(E-\epsilon_{1\mu}+iI_1/2)^{-1}]^{1/2}V_1^{(1)}|\phi_c\rangle , \tag{2.37b}$$

where $|\psi_{1\mu}\rangle$ and $\langle\tilde{\psi}_{1\mu}|$ are again the biorthogonal set of eigenstates of the operator $d_1 H_1^{(1)} d_1$ appearing in $G_1^{(1)}$ corresponding to the complex eigenvalue $\epsilon_{1\mu}\equiv(E_{1\mu}-i\Gamma_{1\mu}/2)$. Again using the arguments of average unitarity and neglecting level-level correlation the d space cross section σ_d becomes

$$\sigma_d\cong\frac{2\pi}{D_1\Gamma_1}\langle|g_{1f\mu}|^2|g_{1i\mu}|^2\rangle_{\mu\epsilon I_1} , \tag{2.38}$$

where $\langle 1/\Gamma_{1\mu}\rangle_\mu=1/\Gamma_1$ and D_1 is the average spacing of states. In terms of the X matrix defined in this case by

$$X_{1cc'}=\langle g_{1c\mu}\, g^*_{1c'\mu}\rangle_{\mu\epsilon I_1} , \tag{2.39}$$

the expression (2.38) for cross section becomes

$$\sigma_d\cong\frac{2\pi}{D_1\Gamma_1}\langle(X_{1ff}X_{1ii}+X_{1fi}X_{1if})\rangle_{I_1} , \tag{2.40}$$

which in the absence of direct reaction channel coupling becomes (for many open channels)

$$\sigma_d\cong\frac{2\pi}{D_1\Gamma_1}\langle X_{1ff}\rangle_{I_1}\langle X_{1ii}\rangle_{I_1}(1+\delta_{fi}) . \tag{2.41}$$

Finally, defining $\langle X_{ncc}\rangle_{I_1}\equiv\Gamma_{nc}$, $n=1,2$, the full fluctuation cross section $\sigma^{\rm fl}=(\sigma_d+\sigma_q)$ can be written as

$$\sigma^{\rm fl}=\sum_{n=1}^{2}\frac{2\pi\Gamma_{ni}}{D_n}\frac{D_n}{2\pi\Gamma_n}\frac{2\pi\Gamma_{nf}}{D_n}(1+\delta_{fi}) . \tag{2.42}$$

In Eq. (2.42) the factor $(2\pi\Gamma_{nf}/D_n)$ or $(2\pi\Gamma_{ni}/D_n)$ represents the probability of reaching the stage n starting from the final channel f or the initial channel i.

The main advantage of the present approach is its simplicity and the special chaining property that $\langle X_{2cc}\rangle_{I_1}$ exhibits. We shall see that it contains the direct transition from channel c to the q space and a route via the d space. In order to see this property of chaining we evaluate $\langle X_{2cc}\rangle_{I_1}$ using the present optical background representation. For this purpose we rewrite the g as

$$g_{2c\mu}\equiv\langle\phi_c|(V_1+V_1G_1V_1)f_{2\mu}(E)|\psi_{2\mu}\rangle$$
$$=\langle\phi_c|V_1' f_{2\mu}(E)|\psi_{2\mu}\rangle+\langle\phi_c|V_1[i(I_1/2)\langle G_1\rangle_{I_1}]^{1/2}G_1[i(I_1/2)\langle G_1\rangle_{I_1}]^{1/2}V_1 f_{2\mu}(E)|\psi_{2\mu}\rangle , \tag{2.43}$$

where

$$V_1'=V_1+V_1\langle G_1\rangle_{I_1}V_1 . \tag{2.44}$$

In the first term in Eq. (2.43) the d_1 space has been averaged out and it provides a direct transition from channel c to the d_2 space. The second term in this equation represents a transition from the channel c to the d_2 space via the doorway space d_1. Moreover, the second term in Eq. (2.43) averages to zero by construction. Hence, in X the interference term will not contribute and one has

$$\frac{2\pi}{D_2}\langle X_{2cc}\rangle_{I_1}=\frac{2\pi}{D_2}\langle\,|\,g_{2c\mu}\,|^2\rangle_{\mu\epsilon I_1}$$
$$=\frac{2\pi\Gamma_{2c}^{(2)}}{D_2}+\frac{2\pi\Gamma_{1c}^{(21)}}{D_1^{(21)}}\frac{\Gamma_{1\to 2}^{\downarrow(2)}}{\Gamma_1^{(2)}}\,, \tag{2.45}$$

where

$$\Gamma_{2c}^{(2)}=\langle\,|\,\langle\phi_c\,|\,V_1'\,f_{2\mu}(E)\,|\,\psi_{2\mu}\rangle\,|^2\rangle_{\mu\epsilon I_1}\,, \tag{2.46a}$$

$$\Gamma_{1c}^{(21)}=\langle\,|\,\langle\phi_c\,|\,V_1\,f_{1\nu}^{(2)}(E)\,|\,\psi_{1\nu}^{(2)}\rangle\,|^2\rangle_{\nu\epsilon I_1}\,, \tag{2.46b}$$

$$\Gamma_{1\to 2}^{(2)}=\frac{2\pi}{D_2}\langle\,|\,\langle\tilde{\psi}_{1\nu}^{(2)}\,|\,f_{1\nu}^{(2)}(E)V_1\,f_{2\mu}(E)\,|\,\psi_{2\mu}\rangle\,|^2\rangle_{\substack{\nu\epsilon I_1\\ \mu\epsilon I_1}}\,. \tag{2.46c}$$

In order to derive Eq. (2.45) we have employed eigenfunction expansion of the operator G_1 appearing in Eq. (2.43). The biorthogonal set of eigenfunctions of the Hamiltonian $d_1H_1d_1$ are denoted $|\psi_{1\nu}^{(2)}\rangle$ and $\langle\tilde{\psi}_{1\nu}^{(2)}|$ with the corresponding eigenvalue $\epsilon_{1\nu}^{(2)}\equiv(E_{1\nu}^{(2)}-i\Gamma_{1\nu}^{(2)}/2)$. Again $f_{1\nu}^{(2)}$ represent the slow energy dependence corresponding to the square root factor in Eq. (2.43); $D_1^{(21)}$ is the average spacing of states in this space and $\Gamma_1^{(2)}\equiv 1/\langle 1/\Gamma_{1\nu}^{(2)}\rangle_\nu$.

In Eqs. (2.45) and (2.46) the index 2 on various functions refer to the fact that these functions originated from the optical background representation of the g's appearing in T_q. For example, $D_1^{(21)}$ of Eq. (2.45) is in general expected to be different from D_1 of Eq. (2.38) and the same is true for the other variables with different superfixes. It should be noted that none of these variables are appropriate for the actual physical system but they represent a convenient parametrization of the fluctuation cross section in terms of these quantities. Though it is expected that the variables with different superfixes will yield quantities of the same order of magnitude, they are not expected to yield identical results as the eigenfunctions employed in the calculation of g is different from that employed in the calculation of multistep cross sections.

Using Eqs. (2.45) and (2.46), Eq. (2.42) for the fluctuation cross section becomes ($i\neq f$)

$$\sigma^{\mathrm{fl}}=\frac{2\pi\Gamma_{1i}^{(1)}}{D_1^{(1)}}\frac{D_1^{(1)}}{2\pi\Gamma_1}\frac{2\pi\Gamma_{1f}^{(1)}}{D_1^{(1)}}+\left[\frac{2\pi\Gamma_{2i}^{(2)}}{D_2^{(2)}}+\frac{2\pi\Gamma_{1i}^{(21)}}{D_1^{(21)}}\frac{\Gamma_{1\to 2}^{(2)}}{\Gamma_1^{(2)}}\right]\frac{D_2^{(2)}}{2\pi\Gamma_2^{(2)}}\left[\frac{2\pi\Gamma_{2f}^{(2)}}{D_2^{(2)}}+\frac{2\pi\Gamma_{1f}^{(21)}}{D_1^{(21)}}\frac{\Gamma_{1\to 2}^{(2)}}{\Gamma_1^{(2)}}\right], \tag{2.47a}$$

$$=\frac{2\pi\Gamma_{1i}^{(1)}}{D_1^{(2)}}\frac{\Gamma_{1f}^{(1)}}{\Gamma_1}+\left[\frac{2\pi\Gamma_{2i}^{(2)}}{D_2^{(2)}}+\frac{2\pi\Gamma_{1i}^{(21)}}{D_1^{(21)}}\frac{\Gamma_{1\to 2}^{(2)}}{\Gamma_1^{(2)}}\right]\left[\frac{\Gamma_{2f}^{(2)}}{\Gamma_2^{(2)}}+\frac{\Gamma_{2\to 1}^{(2)}}{\Gamma_2^{(2)}}\frac{\Gamma_{1f}^{(21)}}{\Gamma_1^{(21)}}\right], \tag{2.47b}$$

where

$$\Gamma_{2\to 1}^{(2)}=\Gamma_{1\to 2}^{(2)}\frac{D_2^{(2)}}{D_1^{(21)}}\,. \tag{2.48}$$

Here $D_n^{(n)}\equiv D_n$ and $\Gamma_n^{(n)}\equiv\Gamma_n$, $n=1,2$. The superfix is introduced to remind that these variables belong to the chain that end on class n and is added for future convenience in notation. The first expression given by Eq. (2.47a) is manifestly time reversal symmetric whereas the second expression given by (2.47b) is more appropriate for exhibiting the correct time evolution of the system. The first term on the right-hand side of Eq. (2.47) represents the multistep cross section σ_d and shows the time evolution of the system through the d space which is the only space existing as in this term the q space has been averaged out. The last term in Eq. (2.47b) represents the four possible time developments of the system symbolically denoted by $i\to q\to f$, $i\to d\to q\to d\to f$, $i\to q\to d\to f$, and $i\to d\to q\to f$. The HM formulation for multistep cross section has denied this simple physical structure of time evolution because of the complicated square root factors inherent in their approach.

The fluctuation cross section in this model given by Eq. (2.47) is the principal result of this section. At extreme low energies when only the elastic channel is open the second term in Eq. (2.47b) obeys chaining properties if the Hamiltonian H of the system obeys chaining, i.e., $d_0Hd_2\equiv PHq=0$, $d_0Vd_2\equiv PVq=0$. In order to establish this claim one should note that under these conditions $\Gamma_{2c}^{(2)}$ defined by Eq. (2.46a) reduces to zero as the d space in V_1' defined by Eq. (2.44) has been averaged out, and consequently V_1' cannot connect the P and q spaces, i.e., $PV_1'q=0$.

Another advantage of the present formulation is that in the absence of direct reaction channel coupling the q space contribution to cross section σ_q has a symmetric Hauser-Feshbach form and hence the doorway component and the compound nuclear component of the cross section comes out of a unified whole. To prove this as in KKM we introduce the Satchler's generalized transmission matrix P by[19]

$$P_{2cc'}=\frac{2\pi}{D_2\Gamma_2}\sum_{c''}(X_{2cc'}X_{2c''c''}+X_{2cc''}X_{2c''c'})\,. \tag{2.49}$$

Again in the absence of direct reaction channel coupling and in the presence of many open channels

$$P_{2cc'}=\frac{2\pi}{D_2\Gamma_2}X_{2cc'}\,\mathrm{Tr}X_2\,, \tag{2.50}$$

which has the following approximate solution in the strong absorption limit:[8,10]

$$X_2\cong\left[\frac{D_2\Gamma_2}{2\pi}\right]^{1/2}P_2(\mathrm{Tr}P_2)^{-1/2}\,. \tag{2.51}$$

Substituting Eq. (2.51) in Eq. (2.35) one obtains the usual Hauser-Feshbach expression[3]

$$\sigma_q = (1+\delta_{fi})\frac{T_f^q T_i^q}{\sum_c T_c^q} , \tag{2.52}$$

where $T_c^q = P_{2cc}$. Hence the present approach includes the standard Bohr description of the compound nucleus for the most complicated stage of resonances.

Finally, we conclude this subsection by calculating the autocorrelation function defined by Eq. (1.2). The t matrix autocorrelation function C^T is given by[8]

$$C^T(\epsilon) \equiv \langle [T_d(E)+T_q(E)][T_d^*(E+\epsilon)+T_q^*(E+\epsilon)] \rangle_{I_1} , \tag{2.53a}$$

$$\cong \sum_{n-1}^{2} \frac{2\pi}{D_n} \left\langle \frac{|g_{nf\mu}|^2 |g_{ni\mu}|^2}{\Gamma_{n\mu}+i\epsilon} \right\rangle_{\mu\epsilon I_1} , \tag{2.53b}$$

$$\cong \sum_{n=1}^{2} \frac{2\pi}{D_n} \frac{\langle |g_{nf\mu}|^2 |g_{ni\mu}|^2 \rangle_{\mu\epsilon I_1}}{\Gamma_n + i\epsilon} , \tag{2.53c}$$

$$= \sum_{n=1}^{2} \sigma_n \frac{\Gamma_n}{\Gamma_n + i\epsilon} , \tag{2.53d}$$

where we again suppress the exit and entrance channel labels on σ and T and where we again employ the usual approximations used in the calculation of the cross section.

Following Ref. 8 now we can easily evaluate the cross section autocorrelation function given by Eq. (1.2). This will give the useful generalization of Ericson's formula and is given by

$$C(\epsilon) = \langle \sigma(E)\sigma(E+\epsilon) \rangle_{I_1} - \langle \sigma(E) \rangle_{I_1} \langle \sigma(E+\epsilon) \rangle_{I_1} , \tag{2.54a}$$

$$= |C^T(\epsilon)|^2 + 2\sigma^{\rm dir} {\rm Re} C^T(\epsilon) , \tag{2.54b}$$

where C^T is given by Eq. (2.53) and $\sigma^{\rm dir} \equiv |\langle T \rangle_{I_1}|^2$ is the "direct" cross section. The study of the autocorrelation function gives a useful way of finding out the presence of various classes of resonances in a reaction.[8,20]

D. Summary

In this section we have derived the present optical background representation in a simple doorway model and mentioned its advantages. Firstly, the present formulation is time reversal symmetric and includes the Bohr description of compound nuclear reactions. In quantitative language it means that the compound nuclear cross section σ_q has a symmetric Hauser-Feshbach form. Secondly, the present formulation exhibits chaining property. This means that the system evolves first through a succession of classes of nuclear states of increasing complexities, and then escapes to the exit channel from one of these classes through a succession of classes of nuclear states of decreasing complexities. In the next section we shall generalize the present formulation to the nested doorway model where a hierarchy of doorways are present.

III. GENERALIZATION TO THE NESTED DOORWAY MODEL

The model of nested average approach assumes the presence of a hierarchy of doorway resonances to be grouped into classes according to their total widths. The average width Γ_n and spacing D_n in class n obey

$$\Gamma_n \gg \Gamma_{(n+1)} , \quad \Gamma_n \gg D_n , \tag{3.1}$$

where $n = 1,2,\ldots,N$. Equation (3.1) defines a well-nested sequence of classes of compound nuclear ($n=N$) and doorway ($n<N$) resonances.[6]

As in HM we introduce a nested sequence of energy averaging intervals I_n such that

$$\Gamma_{(n-1)} \gg I_n \gg \Gamma_n . \tag{3.2}$$

The scattering amplitude T is written as

$$T = T_0 + \sum_{n=1}^{N} T_n^{\rm fl} , \tag{3.3}$$

where $T_0 = \langle T \rangle_{I_1}$ is the optical amplitude and

$$T_n^{\rm fl} \equiv \langle T \rangle_{I_{(n+1)}} - \langle T \rangle_{I_n} \tag{3.4}$$

are the multistep fluctuation amplitudes with $I_{(N+1)} = 0$. Let d_i's represent the projection operators for successive doorway spaces i where increasing i denotes increasing complexity, and d_0 represent the projection operators onto the open channel space. We consider the following decomposition of the full Hilbert space:

$$1 = P_{(n-1)} + d_n + Q_{(n+1)} , \tag{3.5}$$

where

$$P_{(n-1)} = \sum_{i=0}^{(n-1)} d_i , \quad Q_{(n+1)} = \sum_{i=(n+1)}^{N} d_i .$$

This separation is similar to the separation (2.1) of the last section. Here d_0 is the projector onto the open channel space and Q_1 is that onto the closed channel space.

Let us now calculate $T_n^{\rm fl}$. We introduce the operators V_i through the recursion relation

$$V_i = V_{(i-1)} + V_{(i-1)} G_{(i-1)} V_{(i-1)} , \quad i = 1,2,3,\ldots,(n+2) , \tag{3.6}$$

with $V_0 = V$ the full channel interaction. In Eq. (3.6) G_i's are defined by

$$G_i = d_i (E - d_i H_i d_i)^{-1} d_i , \quad i = 0,1,2,\ldots,n \tag{3.7}$$

and

$$G_{(n+1)} = Q_{(n+1)} (E - Q_{(n+1)} H_{(n+1)} Q_{(n+1)})^{-1} Q_{(n+1)} , \tag{3.8}$$

where

$$H_i = H_0 + V_i , \quad i = 1,2,\ldots,(n+1) . \tag{3.9}$$

In Eq. (3.9) H_0 is again the total relative kinetic energy operator. In terms of these operators the transition amplitude T can be written as

$$T=\langle\phi_f\,|\,(V_{(n+1)}+V_{(n+1)}G_{(n+1)}V_{(n+1)})\,|\,\phi_i\rangle\ . \tag{3.10}$$

Equation (3.10) is a generalization of Eq. (2.7) of Sec. II to the case of nested doorways. Now we introduce the fluctuation amplitude corresponding to $Q_{(n+1)}$ space by

$$T_Q^{\rm fl}=\langle\phi_f\,|\,V_{(n+1)}(G_{(n+1)}-\langle G_{(n+1)}\rangle_{I_{(n+1)}})V_{(n+1)}\,|\,\phi_i\rangle\ , \tag{3.11}$$

such that $\langle T_Q^{\rm fl}\rangle_{I_{(n+1)}}\cong 0$ insofar as $\langle G_{(n+1)}\rangle_{I_{(n+1)}}$ is unchanged by reaveraging. Now the difference

$$T-T_Q^{\rm fl}=\langle\phi_f\,|\,(V_{(n+1)}+V_{(n+1)}\times\langle G_{(n+1)}\rangle_{I_{(n+1)}}V_{(n+1)})\,|\,\phi_i\rangle \tag{3.12}$$

will contain all multistep components $T_i^{\rm fl}$, $i=1,2,\ldots,n$. Using the definition (3.6) of V_i we rewrite Eq. (3.12) as

$$T-T_Q^{\rm fl}=\langle\phi_f\,|\,(V_n+V_nG_nV_n)+(V_n+V_nG_nV_n)\times\langle G_{(n+1)}\rangle_{I_{(n+1)}}(V_n+V_nG_nV_n)\,|\,\phi_i\rangle\ , \tag{3.13a}$$

$$=\langle\phi_f\,|\,(V_n^{(n)}+V_n^{(n)}G_n^{(n)}V_n^{(n)})\,|\,\phi_i\rangle\ , \tag{3.13b}$$

where

$$V_i^{(n)}=V_i+V_i\langle G_{(i+1)(i+1)}\rangle_{I_{(n+1)}}V_i\ , \tag{3.14a}$$

$$G_{(i+1)(i+1)}=Q_{(n+1)}(E-Q_{(n+1)}H_iQ_{(n+1)})^{-1}Q_{(n+1)}\ , \tag{3.14b}$$

$$G_i^{(n)}=d_i(E-d_iH_i^{(n)}d_i)^{-1}d_i\ , \tag{3.14c}$$

with $H_i^{(n)}=H_0+V_i^{(n)}$ and $i=0,1,2,\ldots,n$. Equations (3.13) and (3.14) are generalizations of Eqs. (2.11) and (2.12) to this case. The resolvent operator $G_n^{(n)}$ in Eq. (3.13b) contributes to poles appearing at a rate D_n as the fluctuations of the $Q_{(n+1)}$ space have been averaged out, and this is the most rapidly varying part in this equation. Using the same arguments that led to Eq. (2.17) one arrives at the following expression for the nth multistep amplitude:

$$T_n^{\rm fl}=\langle\phi_f\,|\,V_n^{(n)}(G_n^{(n)}-\langle G_n^{(n)}\rangle_{I_n})V_n^{(n)}\,|\,\phi_i\rangle\ , \tag{3.15}$$

such that $\langle T_n^{\rm fl}\rangle_{I_n}\cong 0$. Equation (3.15) provides the representation of the nth fluctuation amplitude we desire. The present definition of $T_n^{\rm fl}$ is identical with that of HM and the equivalence between the present representation and that of Ref. 8 can be established by following the analysis of Sec. II B.

Now using the definitions of V_i and $V_i^{(n)}$ given, respectively, by Eqs. (3.6) and (3.14), one has the following identity:

$$V_i^{(n)}=V_{(i-1)}^{(n)}+V_{(i-1)}^{(n)}G_{(i-1)}^{(n)}V_{(i-1)}^{(n)}\ . \tag{3.16}$$

Equation (3.16) can be verified by interchanging the roles of $Q_{(n+1)}$ contained in $G_{(i+1)(i+1)}$ and $d_{(i-1)}$ contained in V_i. In Eq. (3.16) $G_{(i-1)}^{(n)}$ contributes to poles appearing at a rate given by $D_{(i-1)}$ and as the fluctuations of the $Q_{(n+1)}$ space have been averaged out $V_{(i-1)}^{(n)}$ is slowly varying on this scale. As in Sec. II Eq. (3.15) can be rewritten as

$$T_n^{\rm fl}=\langle\phi_f\,|\,V_n^{(n)}[i(I_n/2)\langle G_n^{(n)}\rangle_{I_n}]^{1/2}\times G_n^{(n)}[i(I_n/2)\langle G_n^{(n)}\rangle_{I_n}]^{1/2}V_n^{(n)}\,|\,\phi_i\rangle\ . \tag{3.17}$$

We shall use representation (3.17) for the purpose of calculating the nth multistep cross section and exhibit the property of chaining. As the multistep component of the transition amplitude given by Eq. (3.17) averages to zero by construction, crossed terms are avoided. Next we use Eqs. (3.16) and (3.17) in order to calculate multistep cross section and exhibit the property of chaining. As in Sec. II next we make an eigenfunction expansion of $G_n^{(n)}$ of Eq. (3.17) and rewrite $T_n^{\rm fl}$ as

$$T_n^{\rm fl}=\sum_\mu\frac{g_{nf\mu}g_{ni\mu}}{E-\epsilon_{n\mu}}\ , \tag{3.18}$$

where

$$g_{nc\mu}\equiv\langle\phi_c\,|\,V_n^{(n)}f_{n\mu}(E)\,|\,\psi_{n\mu}\rangle\ , \tag{3.19a}$$

$$=\langle\tilde{\psi}_{n\mu}\,|\,f_{n\mu}(E)V_n^{(n)}\,|\,\phi_c\rangle\ , \tag{3.19b}$$

and where $|\,\psi_{n\mu}\rangle$ and $\langle\tilde{\psi}_{n\mu}|$ are the biorthogonal set of eigenstates of the Hamiltonian $d_nH_n^{(n)}d_n$ appearing in $G_n^{(n)}$ corresponding to the complex eigenvalue $\epsilon_{n\mu}$ and $f_{n\mu}$ is defined by

$$f_{n\mu}\equiv[i(I_n/2)(E-\epsilon_{n\mu}+iI_n/2)^{-1}]^{1/2}\ . \tag{3.20}$$

The function f is a slowly varying function of energy E and because of this weak dependence on E of $g_{nc\mu}$ one has $\langle T_n^{\rm fl}\rangle_{I_n}\cong 0$ and hence

$$\langle g_{nf\mu}g_{ni\mu}\rangle_{\mu\epsilon I_n}\cong 0\ .$$

It is because of this condition that one can apply average unitarity as suggested by Kerman and Sevgen[17] and as a consequence neglect the level-level correlation term, and obtain

$$\sigma_n^{\rm fl}\cong\frac{2\pi}{D_n\Gamma_n}\langle\,|\,g_{nf\mu}\,|^2\,|\,g_{ni\mu}\,|^2\rangle_{\mu\epsilon I_n}\ , \tag{3.21}$$

where Γ_n is the average width and D_n the mean spacing of resonances in this space.

Next as in Sec. II we define the X matrix by

$$X_{ncc'}=\langle g_{nc\mu}g_{nc'\mu}^*\rangle_{\mu\epsilon I_n}\ , \tag{3.22}$$

in terms of which $\sigma_n^{\rm fl}$ reduces to

$$\sigma_n^{\rm fl}\cong\frac{2\pi}{D_n\Gamma_n}\langle(X_{nff}X_{nii}+X_{nif}X_{nfi})\rangle_{I_1}\ . \tag{3.23}$$

In the absence of direct reaction channel coupling Eq. (3.23) yields the simplified relation

$$\sigma_n^{\rm fl}\cong\frac{2\pi}{D_n\Gamma_n}\langle X_{nff}\rangle_{I_1}\langle X_{nii}\rangle_{I_1}(1+\delta_{fi})\ . \tag{3.24}$$

Now we would like to prove the chaining property of the X's in Eq. (3.24), and in order to establish this, using Eq. (3.16) recursively, we arrive at

$$V_n^{(n)}=V_1^{(n)}\left[1+\sum_{0<i<n}G_i^{(n)}V_i^{(n)}+\sum_{0<i<j<n}\sum G_i^{(n)}V_i^{(n)}G_j^{(n)}V_j^{(n)}+\cdots+G_1^{(n)}V_1^{(n)}G_2^{(n)}V_2^{(n)}\cdots G_{(n-1)}^{(n)}V_{(n-1)}^{(n)}\right], \tag{3.25a}$$

$$=\left[1+\sum_{0<i<n}V_i^{(n)}G_i^{(n)}+\sum_{0<i<j<n}\sum V_j^{(n)}G_j^{(n)}V_i^{(n)}G_i^{(n)}+\cdots+V_{(n-1)}^{(n)}G_{(n-1)}^{(n)}V_{(n-2)}^{(n)}G_{(n-2)}^{(n)}\cdots V_2^{(n)}G_2^{(n)}V_1^{(n)}G_1^{(n)}\right]V_1^{(n)} . \tag{3.25b}$$

Then we substitute expression (3.25a) in X_{nff} and expression (3.25b) in X_{nii} appearing in (3.24). In Eq. (3.25) the most rapidly varying function is $G_{(n-1)}^{(n)}$ which contributes to poles appearing at a rate $D_{(n-1)}$. Then comes the functions $G_{(n-2)}^{(n)}$ and $V_{(n-1)}^{(n)}$ which fluctuate at a rate slower than $G_{(n-1)}^{(n)}$, and so on. Equation (3.25) when substituted into Eqs. (3.19) and (3.22) provides all possible time development of the system from the entrance channel to the stage of complexity n. First, one can have a direct transition from the open channel space to states of complexity n; then, one has the transition via m stages with m varying from 1 to $(n-1)$. For example, for $n=4$ one has the following possible routes: $i\to4$, $i\to1\to4$, $i\to2\to4$, $i\to3\to4$, $i\to1\to2\to4$, $i\to1\to3\to4$, $i\to2\to3\to4$, and $i\to1\to2\to3\to4$. But before evaluating X_{ncc} using Eqs. (3.19) and (3.22) it is necessary to introduce the optical background representation of Sec. II A to the operator $V_n^{(n)}$ so that interference terms between various pieces of $V_n^{(n)}$ in the calculation of X_{ncc} can be eliminated. Now it is obvious that this can be achieved by applying successively the optical background representation to $V_n^{(n)}$, though in practice the algebra could be tedious for a general n. Also it is intuitively obvious that this will not change the possible time development of the system from the open channel space to the stage of complexity n.

Let us illustrate these aspects for a particular n and show how Eq. (3.25) gets modified after introducing the optical background representation. Let us consider the case $n=3$ which is sufficiently complex but still algebraically simple, so that we can write the result for a general n. In this case $T_3^{\rm fl}$ is given by Eq. (3.17) with $n=3$, where $V_3^{(3)}$ satisfies the following hierarchy of equations:

$$V_3^{(3)}=V_2^{(3)}+V_2^{(3)}G_2^{(3)}V_2^{(3)} , \tag{3.26a}$$

$$V_2^{(3)}=V_1^{(3)}+V_1^{(3)}G_1^{(3)}V_1^{(3)} , \tag{3.26b}$$

$$V_1^{3}=V_0^{(3)}+V_0^{(3)}G_0^{(3)}V_0^{(3)} , \tag{3.26c}$$

and

$$V_0^{(3)}=V+VQ_4\langle(E-Q_4HQ_4)^{-1}\rangle_{I_4}Q_4V . \tag{3.26d}$$

We can easily rewrite Eq. (3.26a) as

$$V_3^{(3)}=(V_2^{(3)}+V_2^{(3)}\langle G_2^{(3)}\rangle_{I_2}V_2^{(3)}) + V_2^{(3)}(G_2^{(3)}-\langle G_2^{(3)}\rangle_{I_2})V_2^{(3)} , \tag{3.27}$$

where we have separated the rapidly and slowly varying parts so that the rapidly varying part given by the last term in Eq. (3.27) averages to zero. Using this method of separating the rapidly and slowly varying parts repeatedly we have

$$V_3^{(3)}=(V_1^{(23)}+V_1^{(23)}G_1^{(23)}V_1^{(23)})+[(V_1^{(3)}+V_1^{(3)}\langle G_1^{(3)}\rangle_{I_1}V_1^{(3)})+V_1^{(3)}(G_1^{(3)}-\langle G_1^{(3)}\rangle_{I_1})V_1^{(3)}](G_2^{(3)}-\langle G_2^{(3)}\rangle_{I_2})V_2^{(3)} , \tag{3.28}$$

where $V_i^{(23)}$ satisfies

$$V_1^{(23)}=V_0^{(23)}+V_0^{(23)}G_0^{(23)}V_0^{(23)} , \tag{3.29a}$$

$$V_0^{23}=V+V(Q_4+d_2)\langle[E-(Q_4+d_2)H(Q_4+d_2)]^{-1}\rangle_{I_2}(Q_4+d_2)V , \tag{3.29b}$$

and

$$G_1^{(23)}=d_1(E-d_1H_1^{(23)}d_1)^{-1}d_1 , \tag{3.29c}$$

where $H_1^{(23)}=H_0+V_1^{(23)}$. In writing Eq. (3.28) we note that the first terms on the right-hand sides (rhs) of Eqs. (3.27) and (3.28) are identical. Now separating the rapidly and slowly varying parts in this term again Eq. (3.28) can be written as

$$V_3^{(3)}=(V_1^{(23)}+V_1^{(23)}\langle G_1^{(23)}\rangle_{I_1}V_1^{(23)})+[V_1^{(23)}(G_1^{(23)}-\langle G_1^{(23)}\rangle_{I_1})V_1^{(23)}]+[(V_1^{(3)}+V_1^{(3)}\langle G_1^{(3)}\rangle_{I_1}V_1^{(3)})(G_2^{(3)}-\langle G_2^{(3)}\rangle_{I_2})V_2^{(3)}] +[V_1^{(3)}(G_1^{(3)}-\langle G_1^{(3)}\rangle_{I_1})V_1^{(3)}(G_2^{(3)}-\langle G_2^{(3)}\rangle_{I_2})V_2^{(3)}] . \tag{3.30}$$

This equation should be compared with Eq. (3.25a):

$$V_3^{(3)}=(V_1^{(3)})+(V_1^{(3)}G_1^{(3)}V_1^{(3)})+(V_1^{(3)}G_2^{(3)}V_2^{(3)}) +(V_1^{(3)}G_1^{(3)}V_1^{(3)}G_2^{(3)}V_2^{(3)}) . \tag{3.31}$$

This $V_3^{(3)}$ should be used to calculate $g_{nc\mu}$ of Eq. (3.19a). The first term on the rhs of Eqs. (3.30) or (3.31) represent the direct transition to the $n=3$ space from the open channel space c. The second and the third terms represent the transition from the open channel space to

the $n=3$ space via the $n=1$ and $n=2$ spaces, respectively. Finally, the last term in Eqs. (3.30) or (3.31) represent such a transition via both the $n=1$ and $n=2$ spaces. By construction, the last three terms of Eq. (3.30) average to zero when averaged over appropriate averaging intervals and hence in calculating $g_{nc\mu}$ of Eq. (3.19a) or X of Eq. (3.22) the interference terms between various pieces can be avoided.

Before using Eq. (3.30) to construct X we note that the first term on the rhs is slowly varying and is the optical background term. The second term contains fluctuations of the d_1 space contained in $G_1^{(23)}$; the third term contains fluctuations of the d_2 space contained in $G_2^{(3)}$. Finally, the last term contains a chain of fluctuations of the d_1 and d_2 spaces contained, respectively, in $G_1^{(3)}$ and $G_2^{(3)}$. Now it is straightforward to calculate X_{nff} using (3.30)

$$\frac{X_{3ff}}{\Gamma_3}=\frac{\Gamma_{3f}^{(3)}}{\Gamma_3^{(3)}}+\frac{\Gamma_{3\to 2}^{\uparrow(3)}}{\Gamma_3^{(3)}}\left[\frac{\Gamma_{2f}^{(32)}}{\Gamma_2^{(32)}}+\frac{\Gamma_{2\to 1}^{\uparrow(32)}}{\Gamma_2^{(32)}}\frac{\Gamma_{1f}^{\uparrow(321)}}{\Gamma_1^{(321)}}\right]+\frac{\Gamma_{3\to 1}^{\uparrow(3)}}{\Gamma_3^{(3)}}\frac{\Gamma_{1f}^{(31)}}{\Gamma_1^{(31)}}\,, \tag{3.32}$$

where

$$\Gamma_{3f}^{(3)}=\langle\,|\,\langle\phi_f\,|\,(V_1^{(23)}+V_1^{(23)}\langle G_1^{(23)}\rangle_{I_1}V_1^{(23)})f_{3\mu}(E)\,|\,\psi_{3\mu}\rangle\,|^2\rangle_{\mu\epsilon I_1}\,, \tag{3.33a}$$

$$\Gamma_{3\to 2}^{\uparrow(3)}=\frac{2\pi}{D_2^{(3)}}\langle\,|\,\langle\tilde{\psi}_{2\nu}^{(3)}\,|\,f_{2\nu}^{(3)}(E)V_2^{(3)}f_{3\mu}(E)\,|\,\psi_{3\mu}\rangle\,|^2\rangle_{\mu,\nu\epsilon I_1}\,, \tag{3.33b}$$

$$\Gamma_{2\to 1}^{\uparrow(32)}=\frac{2\pi}{D_1^{(3)}}\langle\,|\,\langle\tilde{\psi}_{1\nu}^{(3)}\,|\,f_{1\nu}^{(3)}(E)V_1^{(3)}f_{2\mu}^{(3)}(E)\,|\,\psi_{2\mu}^{(3)}\rangle\,|^2\rangle_{\mu,\nu\epsilon I_1}\,, \tag{3.33c}$$

$$\Gamma_{2f}^{(32)}=\langle\,|\,\langle\phi_f\,|\,[V_1^{(3)}+V_1^{(3)}\langle G_1^{(3)}\rangle_{I_1}V_1^{(3)}]f_{2\nu}^{(3)}(E)\,|\,\psi_{2\nu}^{(3)}\rangle\,|^2\rangle_{\nu\epsilon I_1}\,, \tag{3.33d}$$

$$\Gamma_{3\to 1}^{\uparrow(3)}=\frac{2\pi}{D_1^{(32)}}\langle\,|\,\langle\tilde{\psi}_{1\nu}^{(23)}\,|\,f_{1\nu}^{(23)}(E)V_1^{(23)}f_{3\mu}(E)\,|\,\psi_{3\mu}\rangle\,|^2\rangle_{\mu,\nu\epsilon I_1}\,, \tag{3.33e}$$

$$\Gamma_{1f}^{(31)}=\langle\,|\,\langle\phi_f\,|\,V_1^{(23)}f_{1\nu}^{(23)}(E)\,|\,\psi_{1\nu}^{(23)}\rangle\,|^2\rangle_{\nu\epsilon I_1}\,, \tag{3.33f}$$

$$\Gamma_{1f}^{(321)}=\langle\,|\,\langle\phi_f\,|\,V_1^{(3)}f_{1\nu}^{(3)}(E)\,|\,\psi_{1\nu}^{(3)}\rangle\,|^2\rangle_{\nu\epsilon I_1}\,, \tag{3.33g}$$

and $\Gamma_3^{(3)}=\Gamma_3$. Equations (3.32) and (3.33) follow by the repeated application of the approximations needed to arrive at Eq. (3.24) in the absence of direct reaction channel coupling. In Eq. (3.33) $D_1^{(32)}$, $D_1^{(3)}$, and $D_2^{(3)}$ are the average spacing of resonances contained in $G_1^{(23)}$, $G_1^{(3)}$, and $G_2^{(3)}$; and $\Gamma_1^{(321)}$, $\Gamma_1^{(31)}$, and $\Gamma_2^{(32)}$ are the corresponding average widths, respectively, of these resonances. The functions $|\,\psi_{2\mu}^{(3)}\rangle$ and $\langle\tilde{\psi}_{2\mu}^{(3)}\,|$ are the biorthogonal set of states used to expand $G_2^{(3)}$. The functions $|\,\psi_{1\nu}^{(23)}\rangle$ and $\langle\tilde{\psi}_{1\nu}^{(23)}\,|$ are the biorthogonal set of states used to expand $G_1^{(23)}$ and finally, $|\,\psi_{1\nu}^{(3)}\rangle$ and $\langle\tilde{\psi}_{1\nu}^{(3)}\,|$ are the biorthogonal set used to expand $G_1^{(3)}$. The functions $f(E)$ in Eq. (3.33) are the square root factors arising from the present optical background representation.

After having calculated X_{3ff} we would like to calculate X_{3ii} in order to demonstrate the property of chaining. In this case we should use the time reversed expression for $V_3^{(3)}$, e.g.,

$$V_3^{(3)}=(V_1^{(23)}+V_1^{(23)}\langle G_1^{(23)}\rangle_{I_1}V_1^{(23)})+[V_1^{(23)}(G_1^{(23)}-\langle G_1^{(23)}\rangle_{I_1})V_1^{(23)}]+[V_2^{(3)}(G_2^{(3)}-\langle G_2^{(3)}\rangle_{I_2})(V_1^{(3)}+V_1^{(3)}\langle G_1^{(3)}\rangle_{I_1}V_1^{(3)})]$$
$$+[V_2^{(3)}(G_2^{(3)}-\langle G_2^{(3)}\rangle_{I_2})V_1^{(3)}(G_1^{(3)}-\langle G_1^{(3)}\rangle_{I_1})V_1^{(3)}]\,. \tag{3.34}$$

Now we use $g_{nc\mu}$ of Eq. (3.19b) in order to calculate X of Eq. (3.22) and using similar approximations we arrive at

$$\frac{2\pi X_{3ii}}{D_3}=\frac{2\pi\Gamma_{3i}^{(3)}}{D_3^{(3)}}+\left[\frac{2\pi\Gamma_{2i}^{(32)}}{D_2^{(32)}}+\frac{2\pi\Gamma_{1i}^{(321)}}{D_1^{(321)}}\frac{\Gamma_{1\to 2}^{\downarrow(32)}}{\Gamma_1^{(32)}}\right]\frac{\Gamma_{2\to 3}^{\downarrow(3)}}{\Gamma_2^{(3)}}+\frac{2\pi\Gamma_{1i}^{(31)}}{D_1^{(31)}}\frac{\Gamma_{1\to 3}^{\downarrow(3)}}{\Gamma_1^{(3)}}\,, \tag{3.35}$$

where $\Gamma_{3i}^{(3)}$, $\Gamma_{2i}^{(32)}$, $\Gamma_{1i}^{(31)}$, and $\Gamma_{1i}^{(321)}$ are defined by Eq. (3.33) with i replaced by f; $D_3^{(3)}\equiv D_3$, and $D_2^{(32)}$, $D_1^{(31)}$, and $D_1^{(321)}$ are the average spacing of resonances contained in $G_2^{(3)}$, $G_1^{(23)}$, and $G_1^{(3)}$, respectively; $\Gamma_{1\to 2}^{\downarrow(32)}=\Gamma_{2\to 1}^{\uparrow(32)}D_1^{(3)}/D_2^{(32)}$; $\Gamma_{2\to 3}^{\downarrow(3)}=\Gamma_{3\to 2}^{\uparrow(3)}D_2^{(3)}/D_3^{(3)}$; $\Gamma_{1\to 3}^{\downarrow(3)}=\Gamma_{3\to 1}^{\uparrow(3)}D_1^{(32)}/D_3^{(3)}$. Again we emphasize that the various Γ's appearing in Eqs. (3.32) and (3.35) are not the physically relevant widths but should be considered as parameters for expressing the cross section.

Now the nth multistep cross section in the limit of many open channels and no direct reaction channel coupling is given by the product of the quantities appearing in Eqs. (3.32) and (3.35) and is given by

$$\sigma_3^{\mathrm{fl}}=\frac{2\pi}{D_3\Gamma_3}X_{3ii}X_{3ff} \tag{3.36}$$

$$=\left[\frac{2\pi\Gamma_{3i}^{(3)}}{D_3^{(3)}}+\left[\frac{2\pi\Gamma_{2i}^{(32)}}{D_2^{(32)}}+\frac{2\pi\Gamma_{1i}^{(321)}}{D_1^{(321)}}\frac{\Gamma_{1\to 2}^{\downarrow(32)}}{\Gamma_1^{(32)}}\right]\frac{\Gamma_{2\to 3}^{\downarrow(3)}}{\Gamma_2^{(3)}}+\frac{2\pi\Gamma_{1i}^{(31)}}{D_1^{(31)}}\frac{\Gamma_{1\to 3}^{\downarrow(3)}}{\Gamma_1^{(3)}}\right]$$
$$\times\left[\frac{\Gamma_{3f}^{(3)}}{\Gamma_3^{(3)}}+\frac{\Gamma_{3\to 2}^{\uparrow(3)}}{\Gamma_3^{(3)}}\left[\frac{\Gamma_{2f}^{(32)}}{\Gamma_2^{(32)}}+\frac{\Gamma_{2\to 1}^{\uparrow(32)}}{\Gamma_2^{(32)}}\frac{\Gamma_{1f}^{(321)}}{\Gamma_1^{(321)}}\right]+\frac{\Gamma_{3\to 1}^{\uparrow(3)}}{\Gamma_3^{(3)}}\frac{\Gamma_{1f}^{(31)}}{\Gamma_1^{(31)}}\right]\,. \tag{3.37}$$

In Eq. (3.37) the chaining property of the system is explicitly exhibited. The possible chains to the stage $n=3$ are given by $i\to 3$, $i\to 1\to 3$, $i\to 2\to 3$, and $i\to 1\to 2\to 3$. After reaching the stage 3 the system decays via one of the following routes: $3\to f$, $3\to 1\to f$, $3\to 2\to f$, and $3\to 2\to 1\to f$. At extreme low energies when the elastic channel is the only open channel the only route that survives in Eq. (3.37) is $i\to 1\to 2\to 3\to 2\to 1\to f$ ($i=f$). At higher energies, physically the $P=d_0$ space is very large and contains more complicated states and the system can bypass some stages in the doorway-hallway space d_n, $n>0$, through the P space. Then one has the possibilities $i\to 3$, $i\to 1\to 3$, $3\to 1\to f$, etc. Let us try to understand how $i\to 1\to 3$ becomes physically allowed at higher energies. The system after coming to the d_1 space may pass to the open channel $P=d_0$ space which contains states more complex than states in d_1 and less complex than states in d_3. In the usual shell model language d_n is an $(n+1)$-particle–n-hole state in the closed channel space. At higher energies $P=d_0$ will contain three-particle–two-hole states in the open channel space. Hence the system can pass from d_1 to d_3 via these states in the P space. As a result at higher energies strict chaining hypothesis of FKK is supposed to be violated as was pointed out[21] in Refs. 6 and 13 and one has what we call the "broken chaining hypothesis" as in Eq. (3.37).

The superfix on the Γ's and the D's in Eq. (3.37) refer to the chain to which this particular Γ or D belong. For example, $\Gamma^{\downarrow(32)}_{1\to 2}$ belongs to the chain $i\to 1\to 2\to 3$ whereas $\Gamma^{\downarrow(3)}_{1\to 3}$ belongs to the chain $i\to 1\to 3$. Similarly, $\Gamma^{\downarrow(43)}_{1\to 3}$ belongs to $i\to 1\to 3\to 4$. The functions $\Gamma^{\downarrow(3)}_{1\to 3}$ and $\Gamma^{\downarrow(43)}_{1\to 3}$ are both partial widths for decay from stage 1 to stage 3, the former one appearing in σ_3 and the latter one appearing in σ_4. The physical partial width for decay from stage 1 to stage 3 $\Gamma_{1\to 3}$ is expected to be different from both of these. All these three widths identified by the superfixes are, however, supposed to have the same order of magnitude. The same comment is true for $\Gamma^{\downarrow(32)}_{1\to 2}$ and $\Gamma^{\downarrow(2)}_{1\to 2}$ or for $D^{(321)}_1$, etc.

Now it is not difficult to generalize the present result to the case of multistep cross section in the limit of many open channels and no direct reaction channel coupling as

$$\sigma^{\rm fl}_n=\mathscr{P}^{(n)}(i\to n)\mathscr{Q}^{(n)}(n\to f)\ , \tag{3.38a}$$

$$\equiv\frac{2\pi}{D^{(n)}_n\Gamma^{(n)}_n}\langle X_{nii}\rangle_{I_1}\langle X_{nff}\rangle_{I_1}\ , \tag{3.38b}$$

where

$$\mathscr{P}^{(n)}(i\to n)=2\pi\langle X_{nii}\rangle_{I_1}/D^{(n)}_n$$

and

$$\mathscr{Q}^{(n)}(n\to f)=\langle X_{nff}\rangle_{I_1}/\Gamma^{(n)}_n\ ,$$

where i and f refer to the initial and the final channels, respectively. In Eq. (3.38) $\mathscr{P}^{(n)}(i\to n)$ and $\mathscr{Q}^{(n)}(n\to f)$ are constructed through the following recursion relations:

$$\mathscr{P}^{(n)}(i\to n)=\frac{2\pi\Gamma^{(n)}_{ni}}{D^{(n)}_n}+\sum_{0<m<n}\mathscr{P}^{(n)}(i\to m)\frac{\Gamma^{\downarrow(n)}_{m\to n}}{\Gamma^{(n)}_m}\ , \tag{3.39a}$$

$$\mathscr{P}^{(n)}(i\to m)=\frac{2\pi\Gamma^{(nm)}_{mi}}{D^{(nm)}_m}+\sum_{0<k<m}\mathscr{P}^{(nm)}(i\to k)\frac{\Gamma^{\downarrow(nm)}_{k\to m}}{\Gamma^{(nm)}_k}\ , \tag{3.39b}$$

$$\mathscr{P}^{(nm)}(i\to k)=\frac{2\pi\Gamma^{(nmk)}_{ki}}{D^{(nmk)}_k}+\sum_{0<n<k}\mathscr{P}^{(nmk)}(i\to r)\frac{\Gamma^{\downarrow(nmk)}_{r\to k}}{\Gamma^{(nmk)}_r}\ , \tag{3.39c}$$

$$\cdots$$

$$\mathscr{P}^{(nmkl\cdots p)}(i\to 1)=2\pi\Gamma^{(nmkl\cdots p)}_{1i}/D^{(nmkl\cdots p)}_1\ , \tag{3.39d}$$

and

$$\mathscr{Q}^{(n)}(n\to f)=\frac{\Gamma^{(n)}_{nf}}{\Gamma^{(n)}_n}+\sum_{0<m<n}\frac{\Gamma^{\uparrow(n)}_{n\to m}}{\Gamma^{(n)}_n}\mathscr{Q}^{(n)}(m\to f)\ , \tag{3.40a}$$

$$\mathscr{Q}^{(n)}(m\to f)=\frac{\Gamma^{(nm)}_{mf}}{\Gamma^{(nm)}_m}+\sum_{0<k<m}\frac{\Gamma^{\uparrow(nm)}_{m\to k}}{\Gamma^{(nm)}_m}\mathscr{Q}^{(n)}(k\to f)\ ,$$

$$\cdots \tag{3.40b}$$

$$\mathscr{Q}^{(nmkl\cdots p)}(1\to f)=\Gamma^{(nmkl\cdots p)}_{1f}/\Gamma^{(nmkl\cdots p)}_1\ , \tag{3.40c}$$

etc. Equations (3.38)–(3.40) illustrate all possible paths in the broken chaining hypothesis, which connect the initial channel i to the final channel f via various doorway-hallway stages. The total multistep cross section is given by

$$\sigma^{\rm fl}\equiv\sum_n\sigma^{\rm fl}_n=\sum_n\mathscr{P}^{(n)}(i\to n)\mathscr{Q}^{(n)}(n\to f)\ . \tag{3.41}$$

All the Γ's and D's appearing in Eqs. (3.39)–(3.41) can be defined as in Eqs. (3.32). But as this gives no new physical insight we do not write them explicitly here. The main result of this section is that in the limit of many open channels and no direct reaction channel coupling the multistep cross section can be expressed in terms of several widths Γ and spacings D which depend on the chain (or the multistep component) to which they belong. This dependence of Γ and D on the width is expected, at best, to be a weak dependence arising out of the process of using the present optical background representation. Once this weak dependence of the Γ's and D's on the chain is neglected, which may not be a poor approximation in view of our ignorance of these widths and spacings at the present time, one has the following simple result:

$$\sigma^{\rm fl}_n=\mathscr{P}(i\to n)\mathscr{Q}(n\to f)\ . \tag{3.42}$$

In Eq. (3.42) $\mathscr{P}$ and $\mathscr{Q}$ obey the recursion relations

$$\mathscr{P}(i\to n)=\frac{2\pi\Gamma_{ni}}{D_n}+\sum_{0<m<n}\mathscr{P}(i\to m)\frac{\Gamma^{\downarrow}_{m\to n}}{\Gamma_m}\ , \tag{3.43a}$$

$$\mathscr{Q}(n\to f)=\frac{\Gamma_{nf}}{\Gamma_n}+\sum_{0<m<n}\frac{\Gamma^{\uparrow}_{n\to m}}{\Gamma_n}\mathscr{Q}(m\to f)\ , \tag{3.43b}$$

with the total multistep cross section again defined by Eq. (3.41). Equations (3.42) and (3.43) are intuitively expected results.

IV. DISCUSSION

In this section we try to establish relations between the present formulation and those already existing in the literature, discuss their relative merits, and finally, establish the conditions, if any, under which the present formulation reduces to the other formulations. Specifically, we consider the formulations of Agassi, Weidenmüller, and Mantzouranis;[9] of Feshbach, Kerman, and Koonin;[5] and of Friedman, Hussein, McVoy, and Mello.[8] The last of these formulations exploits and elaborates the approach first presented by Hussein and McVoy.[7] The first two approaches (AWM and FKK) employ multiclass models and sort out levels of a particular model Hamiltonian into classes which then define the various states of the reaction. Both these approaches employ properties of statistical fluctuation of matrix elements of this Hamiltonian, assume overlapping resonances in each class, and have a large number of open channels. Over and above this FKK makes the chaining hypothesis which is physically expected to be violated in the limit of many open channels and needs a condition stronger than (1.1) to have the self-averaging property of their amplitudes. The present approach, like that of HM, is not based on a model Hamiltonian and is based on the nested averaging sequence (3.1) and (3.2). These two latter approaches, however, deal with overlapping resonances in each class and a large number of open channels.

First, we would like to relate the present approach and that of AWM. A connection can be established between the present simple result given by Eqs. (3.42) and (3.43) and the result of AWM. For establishing this connection we rewrite Eqs. (3.42) and (3.43) as

$$\sigma_n^{\mathrm{fl}} = \sum_{\substack{m \\ n>j,n>m}} \sum_{j} \frac{2\pi\Gamma_{mi}}{D_m} \overline{\mathscr{P}}(m \to n) \frac{D_n}{2\pi\Gamma_n} \overline{\mathscr{Q}}(n \to j) \frac{2\pi\Gamma_{jf}}{D_j} , \tag{3.44a}$$

where $\overline{\mathscr{P}}$ is defined by

$$\overline{\mathscr{P}}(k \to s) = \frac{\Gamma_{k\to s}^{\downarrow}}{\Gamma_k} + \sum_{k<j<s} \frac{\Gamma_{k\to j}^{\downarrow}}{\Gamma_k} \overline{\mathscr{P}}(j \to s) \tag{3.44b}$$

and

$$\overline{\mathscr{Q}}(m \to j) = \overline{\mathscr{P}}(j \to m) . \tag{3.44c}$$

Equation (3.44) can be easily verified using Eqs. (3.42) and (3.43). Summing Eq. (3.44a) over n we arrive at the final result of Agassi *et al.*, e.g.,

$$\sigma^{\mathrm{fl}} = \sum_{m=1}^{N} \sum_{j=1}^{N} \tau_{mi} \Pi_{m,j} \tau_{jf} , \tag{3.45}$$

where

$$\tau_{mc} = 2\pi D_m^{-1} \Gamma_{mc} , \quad c = i,f \tag{3.46a}$$

and

$$\Pi_{m,j} = \sum_{\substack{n \\ n>m \\ n>j}} \overline{\mathscr{P}}(m \to n) \frac{D_n}{2\pi\Gamma_n} \overline{\mathscr{Q}}(n \to j) . \tag{3.46b}$$

In Eq. (3.45) τ_{mc} gives the sticking probability of reaching the mth space from channel c, and $\Pi_{m,j}$ represents the propagation from stage m to j via all possible routes explicitly shown by Eq. (3.46b). Of course, one should remember that this simple connection between the present approach and that of Agassi *et al.* is approximate and valid when the Γ's and D's of the present approach are independent of the chain to which they belong.

Next let us compare the present approach with that of Feshbach, Kerman, and Koonin. If we impose the chaining hypothesis on our result given by Eqs. (3.38)—(3.41) and also assume that all the Γ's and D's of the present approach are independent of the chain to which they belong then the present result reduces to a form reminiscent of the FKK result. At this point it should be recalled that the chaining hypothesis should only be valid at extreme low energies in the absence of exoergic reaction channels. However, if we insist on the chaining hypothesis and demand the Γ's and D's be independent of the chain to which they belong then Eqs. (3.38)—(3.41) yield

$$\sigma^{\mathrm{fl}} = \sum_{n=1}^{N} \frac{2\pi\Gamma_{1i}}{D_1} \left[\prod_{k=1}^{(n-1)} \frac{\Gamma_k^{\downarrow}}{\Gamma_k} \right] \frac{\Gamma_n^{(f)}}{\Gamma_n} , \tag{3.47a}$$

where

$$\frac{\Gamma_n^{(f)}}{\Gamma_n} = \left[\prod_{j=n}^{1} \frac{\Gamma_j^{\uparrow}}{\Gamma_j} \right] \frac{\Gamma_{1f}}{\Gamma_1} , \tag{3.47b}$$

and $\Gamma_k^{\downarrow} = \Gamma_{k\to(k+1)}^{\downarrow}$ and $\Gamma_j^{\uparrow} = \Gamma_{j\to(j-1)}^{\uparrow}$. Equation (3.47a) has the familiar FKK form, where the factor $2\pi\Gamma_{1i}/D_1$ represents the sticking probability for the first stage of doorway states from the entrance channel i; then the product in the parentheses represents propagation to stage n and the last term shown explicitly in Eq. (3.47b) represents the escape to the final channel.

Apart from this apparent similarity between the two approaches there are striking differences between the two. First, as noted earlier the present approach is manifestly time reversal symmetric and the Nth class contribution to the cross section σ_N has the symmetric Hauser-Feshbach form. The same property does not hold good for the manifestly asymmetric FKK formulation. Secondly, the present approach is more exact and general than the FKK one and does not possess the limitations of the FKK approach as commented in the Introduction. Attempts were made in Refs. 13 and 21 to eliminate some of the limitations of the FKK formulation.

Now we would like to compare the present formulation with that of Friedman, Hussein, McVoy, and Mello.[8] While by construction the multistep amplitudes of the present approach are identical to those of HM, the multistep cross sections of the two formulations have different forms because of the different paths followed and different approximations made in calculating them. Their σ_n^{fl} has the approximate form given by Eq. (3.24) and the correlation width Γ_n of the two formulations are expected to be equal as has been emphasized by Hussein, Kerman, and McVoy[22] (in a different context). But the functional forms of the function X of the two approaches are different. The function X of HM is especially complicated because of the complicated square root factors in the

Hamiltonian [see Eqs. (2.22) and (2.23)] and because of this the formulation of HM resisted all attempts to yield the simple physical chaining property, such as given by Eqs. (3.38)—(3.41) of the present work. Otherwise these two approaches are very similar and both are time reversal symmetric and yield the symmetric Hauser-Feshbach form of compound nuclear cross section under appropriate conditions.

It is interesting to recall that the discussion related to the autocorrelation function presented in Sec. II is also valid in the case of general n and that the study of the autocorrelation function provides us with a simple and easy way of detecting the presence of various classes of doorway resonances in a certain reaction.[8,20]

In conclusion, we have proposed a new statistical theory of multistep compound reactions and have established simple connections between the present approach and those of AWM, FKK, and HM. We also indicate the advantages of the present approach over these approaches. We hope that the present approach will be useful for further understanding of the subject.

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