Mass and range dependence in the binding energy of a three-body system

Humberto Garcilazo*
Physics Department, Texas A&M University, College Station, Texas 77843
and Institut für Theoretische Physik, Universität Graz, A-8010 Graz, Austria
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We have studied the bound-state solutions of a system of two particles of mass M which do not interact among themselves but which interact attractively with a third particle of mass μ as a function of the ratio M/μ. In the limit $M/\mu \rightarrow 0$, the solution of the system is well known since it corresponds to the problem of two independent particles in a well. We point out that in the limit $M/\mu \rightarrow \infty$ the solution is also very simple, and it corresponds to having the two particles of mass M fixed at the same point in space. We have found that the three-body binding energy tends to be proportional to the square of the range of the two-body interaction in momentum space, and that the sensitivity of the binding energy with respect to off-shell variations of the two-body amplitude increases when M/μ increases.

I. INTRODUCTION

Exact simple solutions of the three-body problem are important, since very often they can shed light on the nature of the solution of more general systems or serve as a natural starting point for a perturbative expansion. These solutions are normally obtained in some limiting cases when one of the interactions is zero and the masses of one or two of the particles are either very large or very small. Thus, for example, the system of three particles where two particles do not interact with one another but each of them interacts with a third particle of mass $\mu \rightarrow \infty$ is simply the problem of two independent particles in a well, so that the wave functions are products of single-particle wave functions and the energy eigenvalues the sum of single-particle energies.[1] Another well-known case which corresponds to the so-called "three-body model of the optical potential" is that in which a particle of mass μ interacts with a particle of mass m, while at the same time the particle of mass m interacts with a particle of mass M. The solution of this three-body problem in the limit case $\mu/m \rightarrow 0$ can be written in closed form,[2,3] and it has been applied to study the effect of the nucleon-nucleus well in the pion-nucleus optical potential.[3,4]

In this paper, we will study the ground-state solution of a system of two particles of mass M which do not interact with one another, but which both interact with a third particle of mass μ such that $M/\mu \rightarrow \infty$. As we will show, the ground-state solution in this case is determined by the configuration in which the two heavy particles are fixed at the same point in space. This result will be obtained in Sec. II first by solving numerically the Faddeev equations using two different families of separable potentials, and then proving it by using the Foldy-Walecka equations[5] for a system of a particle of mass μ interacting with two fixed centers of force. We will then use in Sec. III the solution of this system to study the dependence of the binding energy on the range of the two-body interaction and its sensitivity to off-shell variations of the two-body amplitudes, as a function of the mass ratio M/μ.

II. BOUND-STATE SOLUTIONS OF THE THREE-BODY SYSTEM

Let us consider two particles of mass M which do not interact with one another, but which interact with a third particle of mass μ, where $\mu < M$. We will consider two-body interactions in the $l=0$ and $l=1$ partial waves of the separable forms

$$V_l(p,p') = \frac{p^l}{\alpha^2+p^2}\gamma_l\frac{p'^l}{\alpha^2+p'^2} , \tag{1}$$

so that the two-body T matrices which are obtained by solving the Lippmann-Schwinger equation

$$t_l(p,p';E) = V_l(p,p') + \int_0^\infty p''^2 dp'' V_l(p,p'') \times \frac{1}{E-p''^2/2\eta+i\epsilon} \times t_l(p'',p';E) , \tag{2}$$

$$\eta = \frac{M\mu}{M+\mu} , \tag{3}$$

are

$$t_l(p,p';E) = \frac{p^l}{\alpha^2+p^2}\tau_l(E)\frac{p'^l}{\alpha^2+p'^2} , \tag{4}$$

with

$$\tau_0(E) = \frac{1}{\dfrac{1}{\gamma_0} + \dfrac{\pi\eta}{2\alpha}\dfrac{1}{(\alpha-iq)^2}} , \tag{5}$$

$$\tau_1(E) = \frac{1}{\dfrac{1}{\gamma_1} + \dfrac{\pi\eta}{2}\dfrac{\alpha-2iq}{(\alpha-iq)^2}} , \tag{6}$$

$$E = \frac{q^2}{2\eta} . \tag{7}$$

We will use the Faddeev equations to calculate the binding energy of the three-body system. We will assume that the three particles are spinless, and take the mass of the light particle equal to that of the pion, while the mass of the heavy particle will be varied so as to approach smoothly the limit $M \to \infty$. We will consider families of two-body interactions such that at a given energy they produce the same phase shift. Thus, in the case of the S-wave potential, we will require that the two-body subsystems have a bound state at zero energy, while for the P-wave potential we will require that the two-body subsystems have a resonance at a relative momentum p_0. Using Eqs. (5) and (6) and the above conditions, we can obtain the strengths γ_0 and γ_1 in terms of the range of the interaction in momentum space α as

$$\gamma_0 = -\frac{2(M+\mu)}{\pi M\mu}\alpha^3 ; \tag{8}$$

$$\gamma_1 = -\frac{2(M+\mu)}{\pi M\mu}\alpha\beta, \quad \alpha > \beta ; \tag{9}$$

$$\beta = \frac{\left[1+\frac{p_0^2}{\alpha^2}\right]^2}{1+\frac{3p_0^2}{\alpha^2}} . \tag{10}$$

As a first example, let us calculate the binding energy of the system when the two-body interactions are S wave as given by Eqs. (1) and (8), so that the ground state has total angular momentum $L=0$. We show the results of these calculations in Fig. 1 as a function of the range of the interaction α. The curve labeled $M/\mu \to \infty$ was obtained by solving the one-body problem that results when the two particles of mass M are at the same point, and we

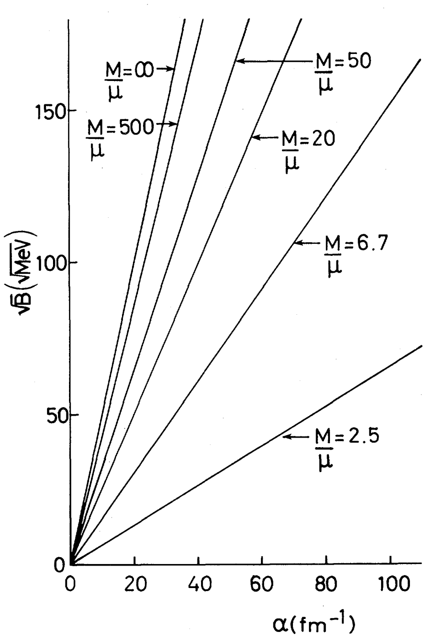

FIG. 1. Square root of the three-body binding energy when the two-body interaction is the S-wave potential defined by Eqs. (1) and (8), as a function of the range of the interaction in momentum space α.

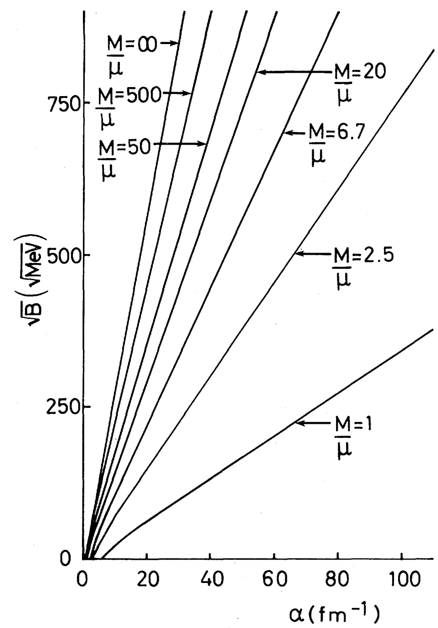

FIG. 2. Square root of the three-body binding energy when the two-body interaction is the P-wave potential defined by Eqs. (1) and (9), as a function of the range of the interaction in momentum space α.

see that it is the limit to which the binding energy converges when the mass of the heavy particles tends to infinity.

Next, let us assume that the two-body interaction is a P wave as given by Eqs. (1) and (9) and (10), and take the position of the resonance at $p_0 = 1.1$ fm^{-1} so that it resembles the pion-nucleon P_{33} resonance. In this case, the ground state of the system has total angular momentum $L=1$, so that the relative angular momentum of the third particle with respect to a pair can be either 0 or 2. We show the results of these calculations in Fig. 2, where again we see that the binding energy of the system converges as we increase the mass M, to the results labeled $M/\mu \to \infty$, which are obtained by solving the one-body problem that results when the two particles of mass M are at the same point.

The proof that the binding energy in the limit $M/\mu \to \infty$ is determined by the configuration where the two heavy particles are at the same point is as follows: If the two particles of mass M become very heavy (we know from Heisenberg's uncertainty principle that they will tend to be localized in space, and therefore in the limit $M \to \infty$), the ground state of the system will be determined by the configuration of the two fixed particles for which the binding energy is maximum. In the case of separable potentials, as we show in Appendix A, this configuration corresponds to the two fixed particles being at the same point.

III. MASS AND RANGE DEPENDENCE OF THE THREE-BODY BINDING ENERGY

In order to understand the linear dependence of $\sqrt{B}$ with respect to α observed in Figs. 1 and 2, let us consider the bound-state solution of the Lippmann-Schwinger equation for the separable potentials (1) when the two particles of mass M are fixed at the same point in space, that is, when the strength of the potential is $2\gamma_l$ instead of γ_l. In the case of the S-wave potential (1), we get using Eq. (5) that

$$\frac{1}{2\gamma_0}+\frac{\pi\mu}{2\alpha}\frac{1}{(\alpha+k)^2}=0\ , \tag{11}$$

where the binding energy B is related to k by

$$B=k^2/2\mu\ . \tag{12}$$

If we substitute Eq. (8) (with $M=\infty$) into Eq. (11) and use Eq. (12), we get

$$B=\alpha^2(\sqrt{2}-1)^2/2\mu\ , \tag{13}$$

that is, the binding energy is proportional to the square of the range in momentum space α, which explains the linear dependence of $\sqrt{B}$ with respect to α that is observed in Fig. 1 for the case $M/\mu\rightarrow\infty$, although this behavior is also followed when M is finite.

In the case of the P-wave potential (1), we get using Eq. (6) that

$$\frac{1}{2\gamma_1}+\frac{\pi\mu}{2}\frac{\alpha+2k}{(\alpha+k)^2}=0\ , \tag{14}$$

so that substituting Eq. (9) (with $M=\infty$) into Eq. (14) and using Eq. (12), we get

$$B=\alpha^2(2\beta-1)[(2\beta-1)^{1/2}+(2\beta)^{1/2}]^2/2\mu\ . \tag{15}$$

We see from Eq. (10) that for $\alpha\gg p_0$, $\beta\rightarrow1$, so that for large values of α the binding energy is proportional to α^2, which again explains the linear behavior of $\sqrt{B}$ for large values of α that is observed in Fig. 2 for the case $M/\mu\rightarrow\infty$, although this behavior is also followed when M is finite.

The behavior of the three-body binding energy with respect to the mass ratio M/μ is also apparent in Figs. 1 and 2, where we see that the slope of $\sqrt{B}$ with respect to α increases when M/μ increases. In order to interpret this result, we notice first of all that one can rewrite the off-shell T matrix $t_l(p,p';E)$, given by Eq. (4), as

$$t_l(p,p';E)=\left[\frac{p}{q}\right]^l\frac{\alpha^2+q^2}{\alpha^2+p^2}t_l(q,q;E)\left[\frac{p'}{q}\right]^l\frac{\alpha^2+q^2}{\alpha^2+p'^2}\ , \tag{16}$$

where $t_l(q,q;E)$ is the on-shell T matrix. Thus, we see that the different values of the parameter α really correspond to different off-shell extrapolations of the two-body T matrices, and therefore the curves with larger slopes in Figs. 1 and 2 correspond to cases that have large sensitivity to off-shell effects. This interpretation would be rigorously true if the two-body interactions that we use were on-shell equivalent; however, as mentioned before, the interactions are only required to reproduce the same phase shift at one energy.

In order to show that indeed the sensitivity of the binding energy to off-shell effects increases when M/μ increases, we will now go to the opposite limit $M/\mu\rightarrow0$, and use two-body interactions that are on-shell equivalent.

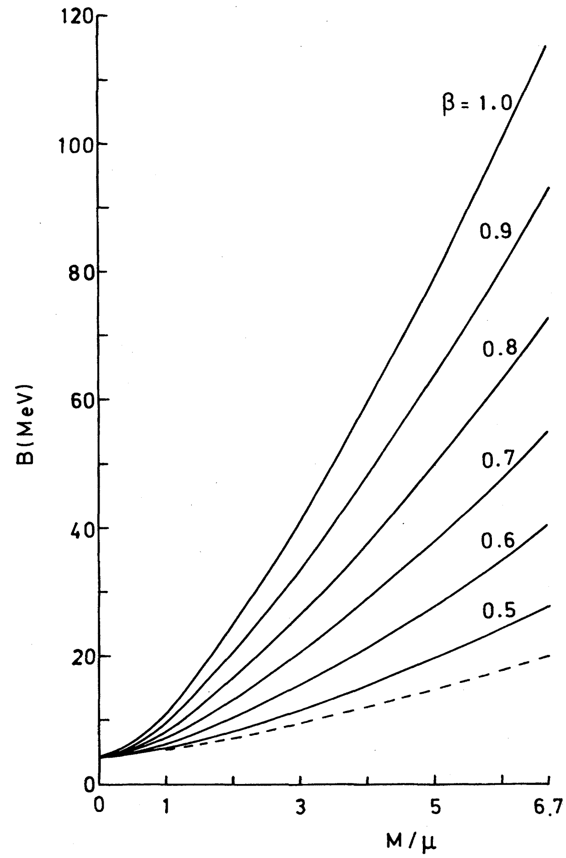

FIG. 3. Three-body binding energy when the two-body interaction is an S-wave potential that possesses a bound state of 2.225 MeV as a function of the mass ratio M/μ. The dashed line is the result of a two-body potential with range $\alpha=1.449$ fm^{-1}, while the solid lines are the results of the on-shell equivalent potentials generated by the transformation (17)—(18) with the parameter β given in fm^{-1} for each curve.

Thus, we will consider again the system with S-wave two-body interactions where now the mass μ will be varied and the mass M will be kept constant and equal to the mass of the nucleon, so that in the case $M/\mu=1$ our system resembles the three-nucleon problem and in the case $M/\mu=6.7$ it resembles the πNN problem. We will consider a two-body interaction with a bound state of energy 2.225 MeV which is the binding energy of the deuteron, and a range $\alpha=1.449$ fm^{-1} so that it corresponds to the separable potential of Yamaguchi[6] in the case $M/\mu=1$. In order to generate on-shell equivalent interactions, we have applied to the two-body Hamiltonian $H=K+V$ a unitary transformation[7] $U=1-2\Lambda$ where $\Lambda+\Lambda^\dagger=2\Lambda\Lambda^\dagger$, so that the resulting two-body potential is

$$\tilde{V}=V-2\Lambda(K+V)-2(K+V)\Lambda^\dagger+4\Lambda(K+V)\Lambda^\dagger\ , \tag{17}$$

and have taken the operator Λ to be of rank one as

$$\langle p\,|\,\Lambda\,|\,p'\rangle=\frac{32}{\pi\beta^3}\frac{1}{(\beta^2+p^2)^2}\frac{1}{(\beta^2+p'^2)^2}\ . \tag{18}$$

We show in Fig. 3 the results for the three-body binding energy as a function of the mass ratio M/μ considering five different values of the parameter β in Eqs. (17) and (18). We see that indeed the sensitivity of the binding energy with respect to off-shell effects increases when M/μ increases. In the limit $M/\mu \to 0$, the binding energy tends to the value 4.45 MeV independently of β, since in this limit the system corresponds to that of two independent particles in a well and the binding energy is equal to twice the single-particle energy.

It is interesting to point out that the effect that we have just discussed may provide an explanation for the observed sensitivity to off-shell effects of some well-known three-body systems. Thus, for example, the three-nucleon bound-state problem is rather insensitive to off-shell effects,[7–9] which corresponds approximately to the behavior of the case $M/\mu = 1$ in Fig. 3. The πNN system, on the other hand, which corresponds to the case $M/\mu = 6.7$ in Fig. 3, is very sensitive to the off-shell behavior of the pion-nucleon T matrix,[10–12] and it has been shown for the case of a negative pion and two neutrons that it may even become bound if the pion-nucleon interaction in the resonant P_{33} channel has a very long range in momentum space,[10–12] which is also consistent with the results of Fig. 2 for the case $M/\mu = 6.7$.

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APPENDIX: THE FOLDY-WALECKA BOUND-STATE PROBLEM

The problem of a particle of mass μ interacting by means of separable potential with N fixed centers of force can be solved exactly, as it has been shown by Foldy and Walecka.[5] Let us consider the case of two identical particles of infinite mass which are fixed at positions $\vec{x}_1$ and $\vec{x}_2$ and which interact attractively with a particle of mass μ by means of a separable potential of the form

$$V_l(p,p') = -g_l(p) g_l(p') \ . \tag{A1}$$

The Foldy-Walecka equations for the bound-state problem are[5]

$$\chi^i_{lm_i}(\vec{x}_1,\vec{x}_2;B) = \sum_{j=1}^{2} \sum_{m_j=-l}^{l} G^{ij}_{lm_i m_j}(x_{ij};B) \chi^j_{lm_j}(\vec{x}_1,\vec{x}_2;B), \quad i=1,2 \ , \tag{A2}$$

where B is the (positive) binding energy, and

$$\vec{x}_{ij} = \vec{x}_i - \vec{x}_j \tag{A3}$$

is the separation between the two fixed centers, while

$$G^{ij}_{lm_i m_j}(x_{ij};B) = \int d\vec{p} \frac{g_l^2(p)}{B+p^2/2\eta} Y_{lm_i}(\hat{p}) Y^*_{lm_j}(\hat{p}) e^{i\vec{p}\cdot\vec{x}_{ij}} = \delta_{m_i m_j} \int_0^\infty p^2 dp \frac{g_l^2(p)}{B+p^2/2\eta} F^{ij}_{lm_i}(px_{ij}) \ , \tag{A4}$$

with

$$F^{ij}_{lm}(px_{ij}) = \frac{2l+1}{2} \int_{-1}^{1} d\cos\theta P^2_{lm}(\cos\theta) \cos(px_{ij}\cos\theta) \ . \tag{A5}$$

In the case $i=j$, we have from Eq. (A3) that $\vec{x}_{ii}=0$, so that from Eq. (A5) we see that

$$F^{ii}_{lm}(0) = \frac{2l+1}{2} \int_{-1}^{1} d\cos\theta P^2_{lm}(\cos\theta) = 1 \ , \tag{A6}$$

and therefore we get from Eq. (A4) that

$$G^{ii}_{lm_i m_j}(x_{ij};B) = \delta_{m_i m_j} G_l(B) \ , \tag{A7}$$

where

$$G_l(B) = \int_0^\infty p^2 dp \frac{g_l^2(p)}{B+p^2/2\eta} \ . \tag{A8}$$

Using Eqs. (A4)–(A8) into Eq. (A2), we find that the condition for the existence of a bound state of binding energy B is

$$1 - G_l(B) - G^{ij}_{lmm}(x_{ij};B) = 0 \ . \tag{A9}$$

If the two fixed centers are at the same point, then $\vec{x}_{ij}=0$, so that repeating again the arguments of Eqs. (A6)–(A8) we get that the bound-state condition is in this case

$$1 - 2G_l(B) = 0 \ . \tag{A10}$$

We will now show that the solution for the binding energy B obtained from Eq. (A10) is larger than that obtained from Eq. (A9). First of all, we notice from Eqs. (A5) and (A6) that

$$F^{ij}_{lm}(px_{ij}) < 1 \quad \text{if } x_{ij} \neq 0 \ , \tag{A11}$$

so that using this result in Eq. (A4) and comparing with Eq. (A8), we see that

$$G^{ij}_{lmm}(x_{ij};B) < G_l(B) \text{ if } x_{ij} \neq 0 \ . \tag{A12}$$

Finally, since both $G_l(B)$ and $G^{ij}_{lmm}(x_{ij};B)$ are monotonically decreasing functions of B, this means that the solution for B obtained from Eq. (A9) is always smaller than that obtained from Eq. (A10). Thus, we have shown that the binding energy is maximum when the two centers of force are at the same point, and therefore this is the configuration that corresponds to the ground-state solution of the three-body system of Sec. II in the limit when the mass M tends to infinity.

*On leave from Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional, México 14 D.F., México.

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