Three-particle equations for the coupled Nπ-Nππ system

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An analysis of the coupled Nπ-Nππ system is carried out using the Hamiltonian of the Chew-Low theory. This Hamiltonian describes the interaction of pions with a static nucleon through the virtual process $N \rightleftarrows N+\pi$. A set of three-particle equations is obtained whose solutions satisfy two- and three-particle unitarity, as well as the discontinuity relations for the production amplitudes ($N+\pi \to N+2\pi$) in the subenergy variables, i.e., the energy of one of the final state pions. It is shown that in order to satisfy the subenergy discontinuity relation, it is necessary to include all four P-wave π-N amplitudes as input to the three-particle equations. In particular, even though the Δ is not an elementary particle in the model considered, its amplitude must be part of the input. The analysis presented shows how to modify the standard three-particle equations to account for single nucleon intermediate states. An expansion for the production amplitude similar to the isobar expansion emerges from the analysis. An approximation for the nucleon propagator is obtained which includes the effect of two meson states.

I. INTRODUCTION

For the last couple of years the author has been analyzing[1–3] model quantum field theories of the Lee model type in an effort to develop a tractable theory for the coupled Nπ-Nππ system. The Lee model[4–6] describes the interaction of two fermions, V and N, with a scalar boson θ through the virtual process $V \rightleftarrows N+\theta$. This model is tractable because of the conservation of charge and baryon number, and the lack of antiparticles in the theory. The amplitude for the process $N+\theta \to N+\theta$ was obtained in Lee's original work,[4] while the amplitudes for $V+\theta \to V+\theta$ and $V+\theta \to N+2\theta$ were first obtained by Amado.[7]

In Ref. 1 (hereafter referred to as L) it was shown that the amplitudes in the V-θ sector can be obtained from the solution of an Amado-Lovelace[8,9] type of three-particle equation. The technique used in L to derive this equation depends in an essential way on the restricted nature of the states in each sector of the Lee model, and therefore cannot be used to treat realistic field theories.

In Ref. 2 a crossing-symmetric extension of the Lee model,[10] which contains an antiparticle $\bar{\theta}$, was analyzed, and an Amado-Lovelace equation for V-θ scattering was obtained. The technique used for deriving this equation is based on a dispersion relation obtained from an exact formal expression for the $V+\theta \to N+2\theta$ amplitude. The dispersion relation is written in terms of the energy ω of one of the θ particles in the final state. The function dispersed, which is a part of the full production amplitude, has a branch cut for $\omega \ge \mu$ where μ is the θ mass. The discontinuity across the low energy end of the cut ($\mu \le \omega \le M_V - M_N + 2\mu$) is related linearly to the function itself. By assuming this discontinuity is valid for all $\omega \ge \mu$, a linear scattering integral equation is obtained. This technique is closely related to an approach used by other workers[11–13] to derive three-particle equations by imposing subenergy unitarity and analyticity on the isobar expansion for production amplitudes.

The dispersion relation approach[2] developed to treat V-θ scattering in the crossing-symmetric Lee model[10] is not general enough to treat a system such as the pion-nucleon system. This is because a process such as that shown in Fig. 1(a) does not occur in V-θ scattering, since conservation of charge prevents a V and a θ from combining to form a V or an N. The crossed process of Fig. 1(b) does occur and when properly dressed becomes the Born term for the Amado-Lovelace equations.[2] In Fig. 1(b) the solid line, the wiggly line, and the dashed line are to be identified with the V, the N, and the θ, respectively.

In order to see how to extend the methodology developed in Ref. 2 to allow for the presence of processes such as Fig. 1(a), an extension of the Lee model, considered some time ago by Bronzan[14] and Chen-Cheung,[15] has been analyzed.[3] In this extension there is an additional W field introduced so that the basic processes are $V \rightleftarrows N+\theta$ and $W \rightleftarrows V+\theta$. The lowest order diagrams for

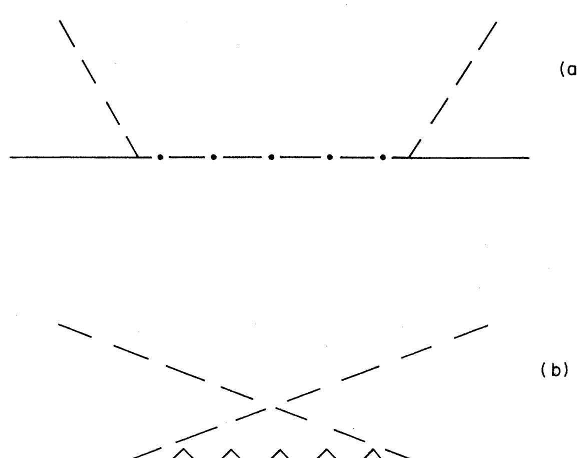

FIG. 1. Diagrams for Born terms. (a) Direct process, (b) crossed process.

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V-θ scattering are those of Fig. 1 with the dash-dot line, the W particle, and the other lines identified as above. The analysis of this model[3] shows that the physical processes $V+\theta \rightarrow V+\theta$ and $V+\theta \rightarrow \mathrm{N}+2\theta$ can be obtained from a single amplitude which is the sum of two parts. One part is the solution of a standard Amado-Lovelace equation[8,9] whose Born term is the dressed version of Fig. 1(b) and whose propagator is determined by the N-θ scattering amplitude. The other term involves the solution of the Amado-Lovelace equation and the W propagator. The techniques used to derive this result[3] are not peculiar to the model, since they rely mainly on unitarity and analyticity in the subenergy and total energy variables, and hence can be applied to models for real systems. Here we shall apply them to the Chew-Low model[16,5,6] for the pion-nucleon system.

The Chew-Low model describes the interaction of pions with a static nucleon by means of the virtual process $\mathrm{N} \rightleftarrows \mathrm{N}+\pi$. The N's and the π's are the only elementary particles in the theory. The lowest order diagrams are those of Fig. 1 with the dashed lines representing π's and all other lines representing N's.

The equations we shall derive for the coupled Nπ-N$\pi\pi$ system are similar to those proposed some time ago by Lovelace,[9] but with important differences in the treatment of processes such as Fig. 1(a), which give rise to the direct nucleon pole in the π-N amplitude. The development given here is somewhat more satisfying in that the structure of the three-particle equations is deduced from an underlying quantum field theory.

An interesting consequence of the author's analysis is that it is necessary to include all four π-N elastic scattering amplitudes (P_{11}, P_{13}, P_{31}, and P_{33}) as input to the three-particle equations in order to satisfy the subenergy dispersion relations obtained from the field theory. In particular, even though the Δ is not an elementary particle in the Chew-Low theory, its amplitude must be part of the input to the N$\pi\pi$ equations. A field theory with an elementary Δ will lead to three-particle equations which differ from those obtained here.

It is worth noting that an expansion for the production amplitude that is similar to the isobar expansion emerges from the present analysis. It does not need to be assumed as was done in an earlier derivation[17] of equations for the N$\pi\pi$ system based on the Blankenbecler-Sugar[18] approach.

The outline of the paper is as follows. In Sec. II the Hamiltonian for the field theory is given and the formally exact expressions for the amplitudes for $\mathrm{N}+\pi \rightarrow \mathrm{N}+\pi$ and $\mathrm{N}+\pi \rightarrow \mathrm{N}+2\pi$ are written. This section is essentially an application to the Chew-Low model of the general relations given in Sec. II of L for static model Hamiltonians. The invariance of the field theory under spatial rotations and rotations in isospin space is used in Sec. III to reduce the number of amplitudes to a minimum. The basic discontinuity relations are derived in Sec. IV. These are the discontinuity of the elastic amplitude across its right-hand cut in the total energy and the discontinuity of the terms in the production amplitude across their right-hand cuts in the subenergy variable. In Sec. V the structure of the three-particle equations is deduced from the discontinuity relations in the subenergy variable. It is shown in Sec. VI that the solutions of the three-particle equations satisfy the discontinuity relation in the total energy variable if the driving terms in these equations are properly constrained. In Sec. VII the modification of the Amado-Lovelace[8,9] equations that is necessary to account for processes such as Fig. 1(a) is obtained. This modification arises only in the nucleon or P_{11} channel, and leads to an amplitude which is the sum of two parts, one of which is the solution of an Amado-Lovelace equation, while the other part involves this solution and the nucleon propagator. The analysis leads thereby to an approximation for this propagator which includes the effect of two meson states. A brief analysis of the form factors and propagators which are the input for the three-particle equations is given in Sec. VIII. Finally, Sec. IX gives a discussion of the results, a comparison with other relevant work, and suggestions for the future.

II. THE CHEW-LOW MODEL

We take for the Hamiltonian of the system

$$H=M_0+\sum_{\nu}\int_0^{\infty}dk\,k^2a_\nu^\dagger(k)a_\nu(k)\omega_k$$
$$+\sum_{\nu}\int_0^{\infty}dk\,k^2[a_\nu(k)J_\nu(k)+a_\nu^\dagger(k)J_\nu^\dagger(k)]\ , \quad (1)$$

where M_0 is the bare nucleon mass, and $a_\nu^\dagger(k)$ and $a_\nu(k)$ create and annihilate mesons with energy $\omega_k=(k^2+\mu^2)^{1/2}$ and index ν. Here ν is a cover index for m and n, the z components of the mesons orbital angular momentum and isospin, respectively, and

$$J_\nu(k)=i\left[\frac{f_0}{\mu}\right]\frac{4\pi k v(k)}{[3(2\pi)^3 2\omega_k]^{1/2}}\sigma_m\tau_n\ , \quad (2)$$

where f_0 is the bare coupling constant and $v(k)$ is a cutoff function normalized to one at $\omega_k=0$. The σ_m and τ_n are standard components of irreducible tensor operators of order one, e.g.,

$$\sigma_1=-\sigma_{-1}^\dagger=-\frac{1}{\sqrt{2}}(\sigma_x+i\sigma_y)\ ,$$
$$\sigma_0=\sigma_z\ . \quad (3)$$

The nonzero commutation relations for the meson operators are

$$[a_\mu(k),a_\nu^\dagger(p)]=\frac{\delta(k-p)}{k^2}\delta_{\mu\nu}\ . \quad (4)$$

The physical one-nucleon states $|\,r\,\rangle_\pm$ satisfy

$$H\,|\,r\,\rangle_\pm=M\,|\,r\,\rangle_\pm\ , \quad (5)$$

where M is the physical mass and r is a cover index for the z components of the nucleon's spin and isospin. According to Eqs. (2), (6), (7), and (11) of L, the one meson states which solve

$$H\,|\,k\mu r\,\rangle_\pm=(M+\omega_k)\,|\,k\mu r\,\rangle_\pm \quad (6)$$

can be written in the form

$$| k\mu r \rangle_{\pm} = \left[a_{\mu}^{\dagger}(k) + \frac{1}{M+\omega_k \pm i\epsilon - H} J_{\mu}(k) \right] \left| r \right\rangle_{+} , \quad (7)$$

and satisfy the orthogonality relation

$$_{\pm}\langle p\nu s \mid k\mu r \rangle_{\pm} = \frac{\delta(p-k)}{k^2} \delta_{\nu\mu} \delta_{sr} . \quad (8)$$

The plus and minus subscripts distinguish in and out states, respectively.

From Eq. (12) of L the on-shell amplitude for one-meson elastic scattering is given by

$$T_{sr}(p\nu, k\mu; \omega_k \pm i\epsilon) = {}_{+}\langle s \mid J_{\nu}^{\dagger}(p) \mid k\mu r \rangle_{\pm}, \omega_p = \omega_k , \quad (9)$$

where the upper sign gives the physical amplitude. Combining (7) and (9), we see that an off-shell extension of this amplitude is given by

$$T_{sr}(p\nu, q\mu; z) = {}_{+}\left\langle s \left| J_{\nu}^{\dagger}(p) \frac{1}{M+z-H} J_{\mu}(q) + J_{\mu}(q) \frac{1}{M-z-H} J_{\nu}^{\dagger}(p) \right| r \right\rangle_{+} . \quad (10)$$

This extension is the one that is usually studied in analyzing one-meson elastic scattering in the Chew-Low model.[5,6,16] With this form the dependence on p and q is determined by the cutoff function $v(k)$, and the analytic structure in z is easily determined by choosing physical states for the intermediate states. The first term on the right-hand side of (10) contains the direct pole at $z=0$ and the right-hand unitarity cut beginning at $z=\mu$, while the second term contains the crossed pole at $z=0$ and the left-hand crossing cut beginning at $z=-\mu$. It is important to realize that the off-shell extension given by (10) is not unique; two other possibilities have been discussed in Ref. 19.

According to Eqs. (13) and (18) of L the two meson states satisfy

$$H \mid p\nu q\mu s \rangle_{\pm} = (M+\omega_p+\omega_q) \mid p\nu q\mu s \rangle_{\pm} , \quad (11)$$

and have the orthogonality relations

$$_{\pm}\langle p\nu q\mu s \mid k\rho l\lambda r \rangle_{\pm} = [\delta(p-k)\delta_{\nu\rho}\delta(q-l)\delta_{\mu\lambda} + \delta(p-l)\delta_{\nu\lambda}\delta(q-k)\delta_{\mu\rho}]\delta_{sr} . \quad (12)$$

Explicit expressions for these states are given by Eqs. (14) and (15) of L; however, we will not need them.

There are several equivalent expressions for the amplitude for the production process $\mathrm{N}+\pi \rightarrow \mathrm{N}+2\pi$. The ones that will be of use to us are given by

$$_{+}\left\langle s \left| J_{\lambda}^{\dagger}(q) \frac{1}{M+\omega_q+i\epsilon-H} J_{\nu}^{\dagger}(p) \right| k\mu r \right\rangle_{+} + (p\nu \leftrightarrow q\lambda)$$
$$= {}_{-}\langle p\nu s \mid J_{\lambda}^{\dagger}(q) \mid k\mu r \rangle_{+}$$
$$= {}_{-}\langle p\nu q\lambda s \mid J_{\mu}(k) \mid r \rangle_{+}, \quad \omega_p+\omega_q=\omega_k . \quad (13)$$

Here the initial meson is labeled by $(k\mu)$ and the final mesons by $(p\nu)$ and $(q\lambda)$, and it should be stressed that the above expressions are only equal on shell, i.e., when $\omega_p+\omega_q=\omega_k$. The first form follows from Eqs. (19) and (21) of L, while the second form can be shown to be equivalent to the first by writing $\mid p\nu s \rangle_{-}$ as in (7) and using Eq. (10) of L. The third form is obtained by writing the S-matrix element as

$$_{-}\langle p\nu q\lambda s \mid k\mu r \rangle_{+} = {}_{-}\langle p\nu q\lambda s \mid (\mid k\mu r \rangle_{+} - \mid k\mu r \rangle_{-}) ,$$

and using (7) to determine the quantity in parentheses.

III. ANGULAR MOMENTUM AND ISOSPIN ANALYSIS

As is well known, the Hamiltonian given by (1) is invariant under spatial rotations and rotations in isospin space. The key observation for exploiting this is that $J_{\nu}(k)=J_{mn}(k)$ is an irreducible tensor operator of order one in both spaces, therefore an expression of the form

$$\sum_{\substack{M'm \\ N'n}} J_{mn}(k) \mid M'N' \rangle_{+} \langle \tfrac{1}{2} 1 M'm \mid JM \rangle \langle \tfrac{1}{2} 1 N'n \mid TN \rangle ,$$

where $\mid M'N' \rangle_{+}$ is the nucleon state, is an unnormalized eigenstate of total angular momentum and isospin with eigenvalues (JM) and (TN), respectively.[20,21]

In order to compress the equations it is convenient to introduce the following shorthand notation:

$$\langle \beta b\mu \mid \alpha a \rangle = \langle J_{\beta} 1 M'm \mid J_{\alpha} M \rangle \langle T_{\beta} 1 N'n \mid T_{\alpha} N \rangle , \quad (14)$$

where α and β go from 1 to 4 and

$$T_1=T_2=J_1=J_3=\tfrac{1}{2}; \quad T_3=T_4=J_2=J_4=\tfrac{3}{2} , \quad (15)$$

while μ is a cover index for (mn), and a and b are cover indices for (MN) and $(M'N')$, respectively. From the properties of the Clebsch-Gordan coefficients, we have the orthogonality relations

$$\sum_{b\mu} \langle \beta b\mu \mid \alpha a \rangle \langle \beta b\mu \mid \alpha' a' \rangle = \delta_{\alpha\alpha'} \delta_{aa'} ,$$
$$\sum_{\alpha a} \langle \beta b\mu \mid \alpha a \rangle \langle \beta b'\mu' \mid \alpha a \rangle = \delta_{bb'} \delta_{\mu\mu'} . \quad (16)$$

From (10) and the tensor character of the J_{ν}, it follows that we can write

$$\sum_{\substack{s\nu \\ r\mu}} \langle 1 s\nu \mid \alpha' a' \rangle T_{sr}(p\nu, q\mu; z) \langle 1 r\mu \mid \alpha a \rangle = \delta_{\alpha'\alpha} \delta_{a'a} T_{\alpha}(p,q;z) . \quad (17)$$

Similarly, we have

$$\sum_{\substack{s\lambda \\ b\nu}} \langle \beta b\nu \mid \alpha' a' \rangle \langle 1 s\lambda \mid \beta b \rangle \, {}_{+}\left\langle s \left| J_{\lambda}^{\dagger}(q) \frac{1}{M+z-H} J_{\nu}^{\dagger}(p) \right| k\alpha a \right\rangle_{+} = \delta_{\alpha'\alpha} \delta_{a'a} F_{\beta}^{\alpha}(p,z,k), \quad \omega_p+\omega_q=\omega_k , \quad (18)$$

where we have introduced a one-meson eigenstate of total angular momentum and isospin by

$$| k\alpha a \rangle_\pm = \sum_{\mu s} | k\mu s \rangle_\pm \langle 1s\mu | \alpha a \rangle \ . \tag{19}$$

We see from (13) that $F^\alpha_\beta(p,z,k)$ is a part of the production amplitude when $z=\omega_q+i\epsilon$. Here we are introducing a particular off-shell extension of this amplitude which is convenient for our purposes.

By using the orthogonality relations (16), we can invert (18) to obtain

$${}_+\left\langle s \left| J^\dagger_\lambda(q)\frac{1}{M+z-H}J^\dagger_\nu(p)\right| k\alpha a\right\rangle_+ = \sum_{\beta b} \langle 1s\lambda | \beta b\rangle\langle \beta b\nu | \alpha a\rangle F^\alpha_\beta(p,z,k), \quad \omega_p+\omega_q=\omega_k \ . \tag{20}$$

From (13), (19), (18), and (20), we find

$$\begin{aligned}\sum_{\substack{s\lambda\\ b\nu}} \langle \beta b\nu | \alpha' a'\rangle\langle 1s\lambda | \beta b\rangle {}_-\langle q\lambda s | J^\dagger_\nu(p) | k\alpha a\rangle_+ &= \sum_{r\mu} {}_-\langle pq\beta\alpha' a' | J_\mu(k) | r\rangle_+ \langle 1r\mu | \alpha a\rangle \\ &= \delta_{\alpha'\alpha}\delta_{a'a}\Big[F^\alpha_\beta(p,\omega_q+i\epsilon,k)+\sum_\gamma C^\alpha_{\beta\gamma}F^\alpha_\gamma(q,\omega_p+i\epsilon,k)\Big], \quad \omega_p+\omega_q=\omega_k \ ,\end{aligned} \tag{21}$$

where

$$|pq\beta\alpha a\rangle_\pm = \sum_{\substack{s\lambda\\ b\nu}} |p\nu q\lambda s\rangle_\pm \langle 1s\lambda | \beta b\rangle\langle \beta b\nu | \alpha a\rangle \ , \tag{22}$$

is a two-meson eigenstate of total angular momentum and isospin labeled by αa, with the nucleon and a pion first coupled to βb. The coefficients $C^\alpha_{\beta\gamma}$ arise because of the two possible choices for the pion that is first coupled to the nucleon, and are given in terms of Wigner $6j$ symbols by

$$C^\alpha_{\beta\gamma} = (-)^{T_\beta+J_\beta+T_\gamma+J_\gamma}[(2T_\beta+1)(2J_\beta+1)(2T_\gamma+1)(2J_\gamma+1)]^{1/2}\begin{Bmatrix} T_\beta & 1 & T_\alpha \\ T_\gamma & 1 & \frac{1}{2}\end{Bmatrix}\begin{Bmatrix} J_\beta & 1 & J_\alpha \\ J_\gamma & 1 & \frac{1}{2}\end{Bmatrix} . \tag{23}$$

In obtaining (23), we have used

$$\sum_{\substack{s\mu\\ b\nu\\ g}} \langle \beta b\nu | \alpha' a'\rangle\langle 1s\mu | \beta b\rangle\langle 1s\nu | \gamma g\rangle\langle \gamma g\mu | \alpha a\rangle = \delta_{\alpha'\alpha}\delta_{a'a}C^\alpha_{\beta\gamma} \ , \tag{24}$$

which follows from the standard expression for the $6j$ symbols[22] in terms of a sum of products of four Clebsch-Gordan coefficients. The matrix C^α is a real, symmetric matrix and satisfies

$$(C^\alpha)^2 = I \ , \tag{25}$$

which follows from the properties of the $6j$ symbols.

IV. THE DISCONTINUITY RELATIONS

The amplitudes $T_\alpha(p,q;z)$ and $F^\alpha_\beta(p,z;k)$ defined by (17) and (18), respectively, have right-hand cuts in z for $z \geq \mu$. The discontinuity across these cuts can be determined by using the identity

$$\frac{1}{M+\omega_k+i\epsilon-H}-\frac{1}{M+\omega_k-i\epsilon-H} = -2\pi i\,\delta(M+\omega_k-H) \ , \tag{26}$$

and expanding the delta function according to

$$\begin{aligned}\delta(M+\omega_k-H) &= \int_\mu^\infty \sum_{\alpha a} | x\alpha a\rangle_\pm x\omega_x d\omega_x \delta(\omega_k-\omega_x)_\pm\langle x\alpha a | \\ &\quad + \tfrac{1}{2}\int_\mu^\infty \sum_{\beta\alpha a} | pq\beta\alpha a\rangle_\pm p\omega_p d\omega_p q\omega_q d\omega_q \delta(\omega_k-\omega_p-\omega_q)_\pm\langle pq\beta\alpha a | + \cdots \ ,\end{aligned} \tag{27}$$

where we have used (8), (12), (19), and (22). With the help of (17), (10), (26), (27), (9), (19), (21), and (25), it is straightforward to show that

$$T_\alpha(k,k;\omega_k+i\epsilon)-T_\alpha(k,k;\omega_k-i\epsilon)=-2\pi i\left\{k\omega_k\,|\,T_\alpha(k,k;\omega_k+i\epsilon)\,|^2+\int p\omega_p d\omega_p q\omega_q d\omega_q \delta(\omega_k-\omega_p-\omega_q)\right.$$

$$\left.\times\left[\sum_\beta |F_\beta^\alpha(p,\omega_q+i\epsilon,k)|^2+\sum_{\beta,\gamma}F_\beta^{\alpha *}(p,\omega_q+i\epsilon,k)C_{\beta\gamma}^\alpha F_\gamma^\alpha(q,\omega_p,+i\epsilon,k)\right]\right\},$$

$$\mu\le\omega_k\le 3\mu\ . \tag{28}$$

This gives the discontinuity across the right-hand cut of the elastic scattering amplitude for energies below the three pion threshold.

From (18), (26), (27), (19), (9), (17), and (21) it follows that

$$F_\beta^\alpha(p,\omega_q+i\epsilon,k)-F_\beta^\alpha(p,\omega_q-i\epsilon,k)=-2\pi i q\omega_q T_\beta(q,q;\omega_q-i\epsilon)[F_\beta^\alpha(p,\omega_q+i\epsilon,k)+\sum_\gamma C_{\beta\gamma}^\alpha F_\gamma^\alpha(q,\omega_p+i\epsilon,k)]\ , \tag{29}$$

$$\mu\le\omega_q\le 2\mu,\quad \omega_p+\omega_q=\omega_k\ .$$

It should be emphasized that this relation is exact, moreover, the values of ω_q indicated are those that arise when $\omega_k\le 3\mu$. Using (28) and the fact that $T_\alpha(p,q;z)$ is a real, analytic function of z for p and q real, we can write

$$T_\alpha(k,k;\omega_k\pm i\epsilon)=-\eta_\alpha(k)\frac{e^{\pm i\delta_\alpha(\omega_k)}\sin\delta_\alpha(\omega_k)}{\pi k\omega_k},\quad \omega_k\ge\mu\ , \tag{30}$$

where below the inelastic threshold η_α is one and δ_α becomes the usual phase shift. The parameter η_α is the ratio of the elastic to the total cross section in channel α. Putting (30) into (29), we immediately obtain

$$F_\beta^\alpha(p,\omega_q+i\epsilon,k)-e^{2i\delta_\beta(\omega_q)}F_\beta^\alpha(p,\omega_q-i\epsilon,k)=-2\pi i q\omega_q T_\beta(q,q;\omega_q+i\epsilon)\sum_\gamma C_{\beta\gamma}^\alpha F_\gamma^\alpha(q,\omega_p+i\epsilon,k)\ ,$$

$$\mu\le\omega_q\le 2\mu,\quad \omega_p+\omega_q=\omega_k\ . \tag{31}$$

V. THE THREE PARTICLE EQUATIONS

We assume that the on-shell π-N amplitude can be written in the form

$$T_\beta(q,q;\omega_q+i\epsilon)=\frac{g_\beta^2(q)}{d_\beta(\omega_q+i\epsilon)},\quad \mu\le\omega_q\le 2\mu\ , \tag{32}$$

where $d_\beta(z)$ is a real, analytic function whose only singularity in the finite z plane is a right-hand cut for $z\ge\mu$, and whose only zero is a simple zero at $z=0$ for $\alpha=1$, the nucleon channel. Justification for this is given in Ref. 19. We normalize $d_1(z)$ so that

$$\lim_{z\to 0} d_1(z)/z=1\ . \tag{33}$$

It follows from (30) that

$$d_\beta(\omega_q-i\epsilon)/d_\beta(\omega_q+i\epsilon)=e^{2i\delta_\beta(\omega_q)},\quad \mu\le\omega_q\le 2\mu\ . \tag{34}$$

If we define A_β^α by

$$F_\beta^\alpha(p,z,k)=g_\beta(q)\frac{A_\beta^\alpha(p,z,k)}{d_\beta(z)}\ ,\quad \omega_p+\omega_q=\omega_k\ , \tag{35}$$

we find from (31), (32), and (34) that

$$A_\beta^\alpha(p,\omega_q+i\epsilon,k)-A_\beta^\alpha(p,\omega_q-i\epsilon,k)=-2\pi i q\omega_q\sum_\gamma g_\beta(q)C_{\beta\gamma}^\alpha g_\gamma(p)\frac{A_\gamma^\alpha(q,\omega_p+i\epsilon,k)}{d_\gamma(\omega_p+i\epsilon)}\ ,$$

$$\mu\le\omega_q\le 2\mu,\quad \omega_p+\omega_q=\omega_k\ . \tag{36}$$

It is easy to see that we can satisfy this discontinuity relation with

$$A^{\alpha}_{\beta}(p,z,k)=\sum_{\gamma}\int_{\mu}^{\infty}d\omega_x x\omega_x \frac{g_\beta(x)C^{\alpha}_{\beta\gamma}g_\gamma(p)}{z-\omega_x} \times \frac{A^{\alpha}_{\gamma}(x,\omega_k+i\epsilon-\omega_x,k)}{d_\gamma(\omega_k+i\epsilon-\omega_x)}\cdots , \quad (37)$$

where the ellipses indicate some function that does not have a cut in z for $\mu \le z \le 2\mu$.

If we let

$$A^{\alpha}_{\beta}(p,\omega_k+i\epsilon-\omega_p,k)=X^{\alpha}_{\beta 1}(p,k;\omega_k+i\epsilon) \quad (38)$$

we find

$$X^{\alpha}_{\beta 1}(p,k;\omega_k+i\epsilon)=\sum_{\gamma}\int_{\mu}^{\infty}d\omega_x x\omega_x B^{\alpha}_{\beta\gamma}(p,x;\omega_k+i\epsilon) \times \frac{X^{\alpha}_{\gamma 1}(x,k;\omega_k+i\epsilon)}{d_\gamma(\omega_k+i\epsilon-\omega_x)}\cdots , \quad (39)$$

where

$$B^{\alpha}_{\beta\gamma}(p,q;z)=\frac{g_\beta(q)C^{\alpha}_{\beta\gamma}g_\gamma(p)}{z-\omega_p-\omega_q} . \quad (40)$$

This suggests the existence of a linear integral equation of the form

$$X^{\alpha}_{\beta\gamma}(p,q;z)=V^{\alpha}_{\beta\gamma}(p,q;z) + \sum_{\lambda}\int_{\mu}^{\infty}V^{\alpha}_{\beta\lambda}(p,x;z)\frac{x\omega_x d\omega_x}{d_\lambda(z-\omega_x)} \times X^{\alpha}_{\lambda\gamma}(x,q;z) , \quad (41)$$

with

$$V^{\alpha}_{\beta\gamma}(p,q;z)=B^{\alpha}_{\beta\gamma}(p,q;z)+W^{\alpha}_{\beta\gamma}(p,q;z) , \quad (42)$$

where $W^{\alpha}_{\beta\gamma}$ is some function that does not have the singularity associated with the denominator in (40). We see that $B^{\alpha}_{\beta\gamma}(p,q;z)$ has a right-hand cut in z for $z \ge 2\mu$. We assume that this is the only right-hand cut in $V^{\alpha}_{\beta\gamma}$, i.e.,

$$V^{\alpha}_{\beta\gamma}(p,q;\omega_k+i\epsilon)-V^{\alpha}_{\beta\gamma}(p,q;\omega_k-i\epsilon) = -2\pi i g_\beta(q)C^{\alpha}_{\beta\gamma}g_\gamma(p)\delta(\omega_k-\omega_p-\omega_q), \quad \omega_k \ge \mu , \quad (43)$$

and that

$$V^{\alpha}_{\beta\gamma}(p,q;z^*)=V^{\alpha}_{\gamma\beta}{}^*(q,p;z) , \quad (44)$$

which is obviously true for $B^{\alpha}_{\beta\gamma}$. It is easy to show that (44) and (41) imply that

$$X^{\alpha}_{\beta\gamma}(p,q;z^*)=X^{\alpha}_{\gamma\beta}{}^*(q,p;z) . \quad (45)$$

We shall show in the next section that these assumptions about $V^{\alpha}_{\beta\gamma}$ guarantee that the solutions of (41) satisfy the unitarity relation (28).

VI. THREE PARTICLE UNITARITY

In order to make the development of this section as transparent as possible, it is convenient to introduce a set of states $|\beta p\rangle$ which satisfy

$$\langle \beta p \mid \alpha q\rangle=\delta_{\beta\alpha}\frac{\delta(\omega_p-\omega_q)}{p\omega_q} , \quad (46)$$

and a set of operators $X^{\alpha}(z)$, $V^{\alpha}(z)$, and $t(z)$ which are defined by

$$\langle \beta p \mid X^{\alpha}(z) \mid \gamma q\rangle = X^{\alpha}_{\beta\gamma}(p,q;z) ,$$
$$\langle \beta p \mid V^{\alpha}(z) \mid \gamma q\rangle = V^{\alpha}_{\beta\gamma}(p,q;z) , \quad (47)$$
$$\langle \beta p \mid t(z) \mid \gamma q\rangle = \delta_{\beta\gamma}\frac{\delta(\omega_p-\omega_q)}{p\omega_p d_\gamma(z-\omega_q)} .$$

In this notation (41) becomes

$$X^{\alpha}(z)=V^{\alpha}(z)+V^{\alpha}(z)t(z)X^{\alpha}(z) ,$$
$$=V^{\alpha}(z)+X^{\alpha}(z)t(z)V^{\alpha}(z) , \quad (48)$$

where the second line can be shown to be equivalent to the first by comparing iterations of the two equations.

From (48) it follows that

$$[1-V^{\alpha}(+)t(+)][X^{\alpha}(+)-X^{\alpha}(-)] = V^{\alpha}(+)-V^{\alpha}(-)+[V^{\alpha}(+)t(+) - V^{\alpha}(-)t(-)]X^{\alpha}(-) , \quad (49)$$

and

$$[1+X^{\alpha}(z)t(z)][1-V^{\alpha}(z)t(z)]=1 , \quad (50)$$

where in (49) $(\pm)=(\omega_k \pm i\epsilon)$ with $\omega_k \ge \mu$. Using (50) to solve (49) for the discontinuity in X^{α}, and rearranging the result with the help of (48), leads to

$$X^{\alpha}(+)-X^{\alpha}(-)=V^{\alpha}(+)-V^{\alpha}(-)+X^{\alpha}(+)t(+)[V^{\alpha}(+)-V^{\alpha}(-)]+[V^{\alpha}(+)-V^{\alpha}(-)]t(-)X^{\alpha}(-) + X^{\alpha}(+)\{t(+)-t(-)+t(+)[V^{\alpha}(+)-V^{\alpha}(-)]t(-)\}X^{\alpha}(-) . \quad (51)$$

From (33), (30), and (32), it follows that

$$\frac{1}{d_\alpha(\omega_k+i\epsilon)}-\frac{1}{d_\alpha(\omega_k-i\epsilon)}=-2\pi i\left[\delta_{\alpha 1}\delta(\omega_k)+\frac{k\omega_k g^2_\alpha(k)}{|d_\alpha(\omega_k+i\epsilon)|^2}\theta(\omega_k-\mu)\right], \quad \omega_k \le 2\mu . \quad (52)$$

If we take on-shell matrix elements of (51) and use (43), (52), (45), (35), and (38), we find that $X^{\alpha}_{11}(k,k;z)$ satisfies the discontinuity relation (28) with

$$X^{\alpha}_{11}(k,k;\omega_k \pm i\epsilon)=T_\alpha(k,k;\omega_k \pm i\epsilon) . \quad (53)$$

It should be kept in mind that as long as V^{α} satisfies

(43) and (44) and d_α satisfies (52), $X_{11}^{\alpha}(k,k;z)$ will satisfy the unitarity relation (28). The importance of this will be seen in the next section where we shall show how to include the direct nucleon pole in the three-particle equations.

VII. THE DIRECT NUCLEON POLE

From (23), it follows that

$$C_{11}^{\alpha}=A_{\alpha 1} , \tag{54}$$

where A is the matrix that appears in the crossing relation[5]

$$T_\alpha(k,k;-z)=\sum_\beta A_{\alpha\beta}T_\beta(k,k;z) , \tag{55}$$

and satisfies

$$A^2=I . \tag{56}$$

In the Chew-Low theory the Born term satisfies (55). If in (42), $W_{\beta\gamma}^{\alpha}$ is zero, then the Born term for (41) becomes $B_{11}^{\alpha}(k,k;\omega_k)$, which according to (40), (54), and (56) does not satisfy (55). This inadequacy of the standard three-particle equations was pointed out some time ago by Lovelace.[9] The problem is that the standard equations do not allow for intermediate states in which only the nucleon is present. In Ref. 3 it is shown how to choose $W_{\beta\gamma}^{\alpha}$ in (42), so as to account for these processes.

We take

$$W_{\beta\gamma}^{\alpha}(p,q;z)=\delta_{\beta 1}g_1(p)\frac{\lambda}{M+z-M_0}g_1(q)\delta_{\gamma 1},\quad \alpha=1 ,$$
$$=0,\quad \alpha=2,3,4 , \tag{57}$$

where M_0 is the bare nucleon mass and λ is a dimensionless constant, which we shall subsequently determine. Reverting to our operator notation, we can write (42) as

$$V^1(z)=B^1(z)+|g\rangle\frac{\lambda}{M+z-M_0}\langle g| , \tag{58}$$

where

$$\langle\beta p|g\rangle=\delta_{\beta 1}g_1(p) . \tag{59}$$

From (48) and (58), we have

$$X^1(z)=B^1(z)+B^1(z)t(z)X^1(z)+|g\rangle\langle S(z)| , \tag{60}$$

with

$$\langle S(z)|=\frac{\lambda}{M+z-M_0}\langle g|[1+t(z)X^1(z)] . \tag{61}$$

We introduce $Y(z)$ and $|R(z)\rangle$ as solutions of the equations

$$Y(z)=B^1(z)+B^1(z)t(z)Y(z) \tag{62}$$

and

$$|R(z)\rangle=|g\rangle+B^1(z)t(z)|R(z)\rangle , \tag{63}$$

which allows us to express the solution of (60) in the form

$$X^1(z)=Y(z)+|R(z)\rangle\langle S(z)| . \tag{64}$$

Substituting this into (61), we find

$$\langle S(z)|=\lambda G(z)\langle g|[1+t(z)Y(z)] , \tag{65}$$

where

$$G^{-1}(z)=M+z-M_0-\lambda\langle g|t(z)|R(z)\rangle . \tag{66}$$

Using an equation like (50), it is easy to show that the solution of (63) is

$$|R(z)\rangle=[1+Y(z)t(z)]|g\rangle . \tag{67}$$

Replacing z by z^*, taking the adjoint, and using the fact that $Y(z)$ and $t(z)$ have the property (45), we find

$$\langle R(z^*)|=\langle g|[1+Y(z)t(z)] , \tag{68}$$

which when combined with (65) and (64) leads to

$$X^1(z)=Y(z)+|R(z)\rangle\lambda G(z)\langle R(z^*)| . \tag{69}$$

Writing out (62), (63), and (69) explicitly, we see that $Y_{\beta\gamma}(p,q;z)$ satisfies (41) with $\alpha=1$ and $W_{\beta\gamma}^{1}=0$, $R_\beta(p;z)$ is the solution of

$$R_\beta(p;z)=\delta_{\beta 1}g_1(p)+\sum_\gamma\int B_{\beta\gamma}^{1}(p,q;z)\frac{q\omega_q d\omega_q}{d_\gamma(z-\omega_q)}R_\gamma(q;z)$$
$$=R_\beta^*(p;z^*) , \tag{70}$$

and (69) becomes

$$X_{\beta\gamma}^{1}(p,q;z)=Y_{\beta\gamma}(p,q;z)+R_\beta(p;z)\lambda G(z)R_\gamma(q;z) . \tag{71}$$

It is clear that $G(z)$ is an approximation for the nucleon propagator defined by

$$G_N(z)=\left\langle r\left|\frac{1}{M+z-H}\right|r\right\rangle , \tag{72}$$

where $|r\rangle$ is a bare nucleon state. Accordingly we require that $G(z)$ have a simple pole at $z=0$ whose residue is the nucleon's wave function renormalization constant, i.e.,

$$\lim_{z\to 0}zG(z)=|\langle r|r\rangle_+|^2=Z_N . \tag{73}$$

It is straightforward to show that this allows us to write

$$\lambda G(z)=\rho z\left[1-\rho\frac{J(z)-J(0)-zJ'(0)}{z}\right]^{-1} , \tag{74}$$

where

$$\rho=Z_N\lambda , \tag{75}$$

$$J(z)=\int_\mu^\infty g_1(x)\frac{x\omega_x d\omega_x}{d_1(z-\omega_x)}R_1(x;z) , \tag{76}$$

and

$$Z_N=1+\rho J'(0) . \tag{77}$$

We can determine ρ by demanding that the elastic scattering amplitude have the correct residue at the direct nucleon pole. The exact amplitude has the property[5,19]

$$T_\alpha(k,k;\omega_k+i\epsilon) \underset{\omega_k\to 0}{\longrightarrow} \frac{3k^2}{\pi\omega_k}\left[\frac{f}{\mu}\right]^2 \frac{\delta_{\alpha 1}-A_{\alpha 1}}{\omega_k}, \tag{78}$$

where f is the renormalization coupling constant, and the $\delta_{\alpha 1}$ term gives the direct pole. Using (53), (71), (74), and (78), we obtain

$$\rho = \frac{3}{\pi}\left[\frac{f}{\mu}\right]^2 \lim_{\omega_k\to 0}\frac{k^2}{\omega_k R_1^2(k;\omega_k)} . \tag{79}$$

The crossed pole, i.e., the $A_{\alpha 1}$ term in (78), resides in the Born terms $B_{11}^{\alpha}(k,k;\omega_k)$. This is not completely obvious, but it is possible to see that this is so by studying V-θ scattering in the Lee model.[1] Upon comparing the Born term in the Amado-Lovelace equations for V-θ scattering [see Eqs. (42) and (43) of L] with the representation of the elastic V-θ scattering amplitude analogous to (10), it is found that the Born term carries the full crossed pole. Here, this places a restriction on the form factor $g_1(k)$. Comparing (40) on shell with (78), and using (54), we see that we must require

$$\lim_{\omega_k\to 0}\frac{\omega_k g_1^2(k)}{k^2} = \frac{3}{\pi}\left[\frac{f}{\mu}\right]^2 . \tag{80}$$

VIII. INPUT FOR THE THREE-PARTICLE EQUATIONS

We conclude our analysis with some comments on the form factors $g_\alpha(q)$ and propagators $d_\alpha^{-1}(z)$ which supply the input for the three-particle equations. Recall that $d_\alpha(z)$ is a real, analytic function of z whose only singularity in the finite z plane is a right-hand cut beginning at $z=\mu$, and whose only zero is a simple one at $z=0$ for $\alpha=1$. As long as the $d_\alpha(z)$ have these properties and Eqs. (32)–(34), (79), and (80) are valid, the solutions of (41) will satisfy the basic discontinuity relations (28) and (29), and will have the correct residues at the direct and crossed nucleon poles.

It is not difficult to determine the form of the $d_\alpha(z)$ from their analytic structure. From Cauchy's theorem, it follows that

$$d_\alpha(z) = P_\alpha(z) + \int_\mu^\infty d\omega \frac{\sigma_\alpha(\omega)}{\omega - z} , \tag{81}$$

where $P_\alpha(z)$ is determined by the behavior of $d_\alpha(z)$ at infinity, and

$$\begin{aligned}\sigma_\alpha(\omega_q) &= \frac{1}{\pi}\mathrm{Im}\, d_\alpha(\omega_q+i\epsilon), \quad \omega_q \ge \mu \\ &= q\omega_q g_\beta^2(q), \quad \mu \le \omega_q \le 2\mu .\end{aligned} \tag{82}$$

We have used (30) and (32) in getting the second line of (82).

We can obtain some information on $d_1(z)$ for large $|z|$ by considering our approximate result for the nucleon propagator. According to (72)

$$\lim_{|z|\to\infty} zG_N(z) = 1 . \tag{83}$$

If we require $G(z)$ to have this property then it follows from (66) and (70) that $d_1(z)$ must diverge for large $|z|$. Using this and invoking the constraint of simplicity, we choose

$$P_\alpha(z) = a_\alpha z\delta_{\alpha 1} + b_\alpha . \tag{84}$$

Imposing (33) on $d_1(z)$ and taking the other $d_\alpha(z)$ to be normalized to one at $z=0$, we obtain

$$\begin{aligned} d_1(z) &= z\left[1 + z\int_\mu^\infty \frac{d\omega\, \sigma_1(\omega)}{\omega^2(\omega-z)}\right], \\ d_\alpha(z) &= 1 + z\int_\mu^\infty \frac{d\omega\, \sigma_\alpha(\omega)}{\omega(\omega-z)}, \quad \alpha = 2,3,4 .\end{aligned} \tag{85}$$

Another representation for the $d_\alpha(z)$ can be obtained in terms of their phases along the right-hand cut. We write

$$d_\alpha(\omega-i\epsilon)/d_\alpha(\omega+i\epsilon) = e^{2i\Delta_\alpha(\omega)}, \quad \omega \ge \mu , \tag{86}$$

where, according to (34),

$$\Delta_\alpha(\omega) = \delta_\alpha(\omega), \quad \mu \le \omega \le 2\mu . \tag{87}$$

The functions $\ln[d_\alpha(z)/d_\alpha(\infty)]$ $(\alpha=2,3,4)$ are analytic except for the cut given by (86), and vanish at infinity. The same is true for $\alpha=1$ if we replace $d_\alpha(z)$ by $d_1(z)/z$. It follows almost immediately that

$$\begin{aligned} d_1(z) &= z\exp\left[-\frac{z}{\pi}\int_\mu^\infty \frac{d\omega\, \Delta_1(\omega)}{\omega(\omega-z)}\right], \\ d_\alpha(z) &= \exp\left[-\frac{z}{\pi}\int_\mu^\infty \frac{d\omega\, \Delta_\alpha(\omega)}{\omega(\omega-z)}\right], \quad \alpha=2,3,4 .\end{aligned} \tag{88}$$

It is worth emphasizing that the $\sigma_\alpha(\omega)$ in (85) and the $\Delta_\alpha(\omega)$ in (88) are to a large extent arbitrary for $\omega > 2\mu$. It should be possible to exploit this arbitrariness so as to force the π-N phase shift δ_α obtained from the solution of the three-particle equations [see Eqs. (53) and (30)] to be the same as the phase $\Delta_\alpha(\omega)$ in the elastic range, as required by (87). This self-consistency requirement implies that it is somewhat misleading to refer to the three-particle equations discussed here as linear.

It is interesting to note that if we choose the crudest approximations for the $\alpha=1$ form factor and propagator consistent with (80) and (33), i.e.,

$$\begin{aligned} g_1^2(k) &\simeq \frac{3}{\pi}\left[\frac{f}{\mu}\right]^2 \frac{k^2 v^2(k)}{\omega_k}, \\ d_1(z) &\simeq z ,\end{aligned} \tag{89}$$

and use these in (76) with $R_1(x;z)$ replaced by $g_1(z)$ [see Eq. (70)], we then find from (66) and (77) that to lowest order in the coupling constant

$$\begin{aligned} M &= M_0 - \frac{3}{\pi}\left[\frac{f_0}{\mu}\right]^2 \int_0^\infty dx \frac{x^4 v^2(x)}{\omega_x^2} + \cdots , \\ Z_N &= 1 - \frac{3}{\pi}\left[\frac{f_0}{\mu}\right]^2 \int_0^\infty dx \frac{x^4 v^2(x)}{\omega_x^3} + \cdots ,\end{aligned} \tag{90}$$

which agrees with the lowest order perturbation theory calculations of these quantities.[5,6,16] This is a rather convincing check of a fairly complicated analysis.

IX. DISCUSSION

We have obtained a set of equations for the coupled Nπ-N$\pi\pi$ system whose solutions satisfy two and three-particle unitarity, as well as the discontinuity relations for the production amplitudes in the subenergy variable, which is the energy of one of the final state pions. In order to satisfy the subenergy discontinuity relation it is necessary to include all four P-wave π-N amplitudes as input to the three-particle equations. The three-particle equations obtained here for the P_{13}, P_{31}, and P_{33} channels are similar to Lovelace's[9] except for the inclusion of the P_{13} and P_{31} π-N amplitudes in the input. In Lovelace's[9] approach, it would appear unnatural to include those amplitudes in the input as he argues that separable T matrices should only be used in channels which are dominated by bound states or resonances.

The treatment of the direct nucleon pole given in Sec. VII is quite different from Lovelace's.[9] As Eq. (71) shows, the P_{11} amplitude consists of two parts; the first part $Y_{\beta\gamma}$ is obtained by solving Eq. (41) with $V^{\alpha}_{\beta\gamma}=B^{1}_{\beta\gamma}$, which is a standard three-particle equation,[8,9] while the second part is obtained by solving Eq. (70) for the vertex function R_{β} and using this function to construct the nucleon propagator $G(z)$ from Eqs. (74) and (76). The form obtained here for the P_{11} π-N elastic scattering amplitude [Eq. (71) with $p=q=k$, $z=\omega_k+i\epsilon$, and $\beta=\gamma=1$] is similar in structure to that found by other authors,[23] with the important difference that it includes the effect of two-pion states. Thus the present work has the important consequence of suggesting a way of including two-pion effects in existing models of the π-N amplitude.

Somewhat beside the point, but still of interest, is the fact that we have obtained a fairly sophisticated approximation for the nucleon propagator which can be used to study the relation between the bare and physical nucleon mass in the Chew-Low model, as well as the nucleon's wave function renormalization constant. As pointed out at the end of the last section the lowest order results for these are reproduced when reasonable approximations are made in the equations obtained here.

It is tempting to believe that equations for the Nπ-N$\pi\pi$ system that contain the P_{33} π-N amplitude as input describe a situation in which an elementary Δ is present. As the above remarks should make clear, this is not the case since the Chew-Low model does not include an elementary Δ but still its amplitude must appear in the three-particle equations. Numerical results indicate the importance of this.

Some time ago Aaron[24] obtained exact numerical solutions of Lovelace's equations for π-N scattering, and found that the Δ resonance is not obtained from the three-particle equations if the P_{11} amplitude is the only input. In order to get the Δ resonance out it must be put in. Aaron's[24] results also indicate the importance of treating single-nucleon intermediate states carefully. As a result of the fact that his P_{11} amplitude does not include the second term on the right-hand side of Eq. (71), his P_{11} phase shifts have the wrong sign at low energies. It is interesting to note that his P_{13}-P_{31} phase shifts, which are identical, change from negative to positive as the energy increases. This contradicts solutions of the Low equation for the Chew-Low model.[25] Following Lovelace,[9] Aaron[24] has not included the P_{13} and P_{31} amplitudes in his input, and it is possible that this accounts for the unwanted sign change. The author is currently studying this possibility.

The role of crossing symmetry in the equations obtained here is somewhat unclear. Since the elastic π-N amplitude has the correct residues at the direct and crossed nucleon poles [see Sec. VII, following Eq. (77)], it will satisfy the crossing relation [Eq. (55)] for pion energies ω_k close to zero. How well this relation will be satisfied for energies $\omega_k \geq \mu$ is difficult to ascertain. This question deserves further study. In any case, it is the opinion of this author as well as others[9,17,24] that it is more important to treat three-particle unitarity carefully than crossing. In fact it has been shown[26] in the charged scalar static model that for strong coupling, and particularly at higher energies, production has a decidedly greater effect on scattering than does crossing.

It should be possible to extend the analysis given here to more realistic field theories for the π-N system. In particular, it is of some interest to see what modifications of the three-particle equations arise when the underlying field theory contains an elementary Δ. The cloudy bag model[27] provides a natural setting for such an investigation, as the second quantized Hamiltonian obtained in this model is essentially a combination of the Chew-Low model and the Lee model, and describes[27-29] the interaction of π's with a static N, Δ, and R (the Roper resonance). It is clear that one modification of the equations obtained here will be the addition of a term to the P_{33} amplitude which will account for processes such as Fig. 1(a) where the intermediate particle is a Δ. This term will be similar to the direct pole term in the P_{11} amplitude, and will obviously involve the Δ propagator. Clearly there will also be additional Born terms in order to account for diagrams such as Fig. 1(b) with the solid line and wiggly line representing different fermions.

Another important extension of the work presented here will be the inclusion of recoil effects. The existing literature on π-N scattering with nonstatic nucleons presents a somewhat confusing picture. In the relativistic Aaron, Amado, and Young (AAY) (Ref. 17) model for the N$\pi\pi$ system it is found that the Δ resonance can be obtained for a reasonable choice of the parameters; however, in the static limit the resonance disappears. This agrees with Aaron's static model results,[24] since the AAY model[17] only contains a P_{11} input. The AAY results[17] suggest that the recoil effect makes the "force" in the P_{33} channel more attractive. There are other calculations,[30,31] however, which indicate that the inclusion of recoil effects makes the force in the P_{33} channel less attractive. These calculations[30,31] are based on relativistic generalizations of the Chew-Low model. In light of the earlier remarks on the importance of including all π-N channels as input to the three-particle equations, it is possible that extending the number of input channels in the AAY model[17] will bring the two sets of calculations into agreement on the effect of recoil on the Δ resonance. This is of some importance, since it is claimed in Ref. 30 that once recoil effects are taken into account it is not possible to generate the Δ resonance with the basic process of the Chew-Low theory, i.e., $N \rightleftarrows N+\pi$. This suggests that it is necessary to have the Δ occur as an elementary particle in the underlying field theory, as in the cloudy bag model.[27] It will be very interesting to see if this turns out to be true, as it would provide further support for the quark picture, according to which the Δ is just as "elementary" a particle as the N.

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